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The Geometry of Rectangular Matrices and Their Characterizations
--- searching for *cospectral mates* and *distance regular mates* of the bilinear forms graphs

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Abstract:
A non-distance-regular cospectral mate and a non-vertex-transitive distance- regular mates for *the Grassmann graph* $J_q(n,d)$ with $n \geq d+3 \geq 6$ or with $n = d+1 \geq 3$ were recently given by E.R. van Dam and J.H.Koolen, and by E.R. van Dam, W.H. Haemer, J.H. Koolen and E. Spence respectively. From the view point that $J_q(n,d)$ and *the bilinear form graph* $H_q(n,d)$ are the point graphs and their dual of the close related *projective incidence structures* and *attenuated space*, we may wonder whether similar situations hold for the bilinear forms graphs $H_q(n,d)$?

E.R. van Dam and J.H. Koolen, A new family of distance-regular graphs with unbounded diameter, Invent. Math. 162, 189-193 (2005)

E.R. van Dam, W.H. Haemer, J.H. Koolen and E. Spence, Characterizing distance-regularity of graphs by the spectrum, JCT A 113 (2006) 1805-1820.

Contents:

I Preliminary and background

Association schemes were first introduced by statisticians R.C. Bose et.al. around 1950 in connection with the design of experiments in statistics, and independently by some group theorists. In addition to its applications in the design of experiments, Ph. Delsarte recognized and fully used association schemes as a basic underlying structure of coding theory and design theory around 1970. A project of classifying association schemes in term of their geometric structures, their parameters or even their spectra was proposed by E. Bannai around 1980. The study of the geometry of matrices by Z. Wan provided an abundant source for association schemes around 1960.

1-1 association schemes, Bose Mesner algebra and distance regular graphs

Definition: (commutative, symmetric) *d-class association schemes*

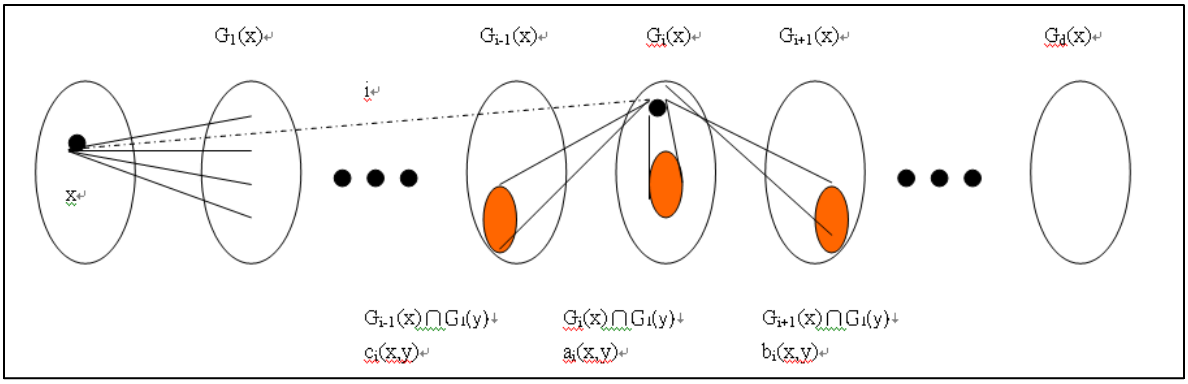
1. A (commutative, symmetric) d-class ***association scheme*** on X is a $(d+1)$ - tuple $\mathrm{A}=(A_0, A_1, A_2, \cdots, A_d)$ of $(0,1)$-matrices of order $|X|\times|X|$ satisfying
   $a.$ $A_0 = I$, $A_i \circ A_j = \delta_{ij} A_i$ for $i, j \le d$, $\sum_{i=0}^{d} A_i = J$,
   $b.$ for every i , there exist i' with ${}^tA_i = A_{i'}$ (or ${}^tA_i = A_i$ respectively), and
   $c.$ there exist nonnegative integers $p_{i,j}^k$ for all $i, j, k \in \{0,1,2,\cdots,d\}$ such that
   $A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{i,j}^k A_k$.
2. The linear span of the matrices $\{A_i \mid i = 0,1,\cdots,d\}$ is called the ***Bose-Mesner algebra*** of the association scheme $\mathrm{A}=(A_0, A_1, A_2, \cdots, A_d)$.
3. A symmetric scheme $(\mathrm{X}, \{A_i\}_{0\le i\le d})$ is called a ***distance regular graph*** G if $p_{1,j}^k = 0$ except $j = k-1, k,$ or $k+1$; usually, $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$ and $c_i = p_{1,i-1}^i$.

Let

$G = (V(G), E(G))$ be a connected graph of diameter d,

$G_i(x) = \{y \mid y \in V(G)$ is at distance i from x $\}$;

for $x \in V(G)$, consider partitions of $V(G)$:



Definition The graph G is called *distance-regular* if $c_i(x,y)=c_i$, $a_i(x,y)=a_i$, $b_i(x,y)=b_i$ are constants whenever $x,y\in V(G)$ are at distance $i\le d$. The array of parameters

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_d \\ a_0 & a_1 & a_2 & \dots & a_d \\ b_0 & b_1 & b_2 & \dots & b_d \end{bmatrix}$$

or $\{b_0,b_1,b_2,\dots,b_{d-1};c_1,c_2,c_3,\dots,c_d\}$ is called the *intersection array* of G. Moreover

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_d \\ a_0 & a_1 & a_2 & \dots & a_d \\ b_0 & b_1 & b_2 & \dots & b_d \end{bmatrix} \quad \longleftrightarrow \quad Spec(G)=(b_0(=k),\theta_1,\theta_1,\dots,\theta_d)$$

in a systematic way.

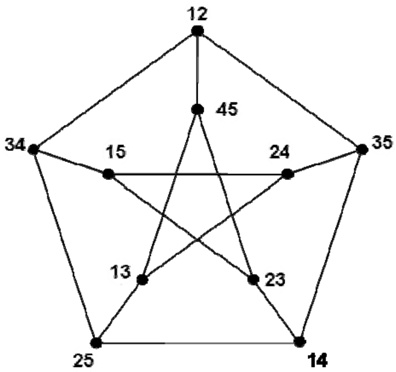
Some families of examples:

1. ***Hammning graph*** $H(n,q)$
2. ***Johnson graph*** $J(n,d)$
3. The ***Grassmann graph*** $J_q(n,d)$ (also called the ***q-analog of the Johnson graph***) is defined on the set of all *d*-dimensional subspaces of an *n*-dimensional vector space over $GF(q)$. Two vertices A and B are adjacent whenever $\dim(A\cap B)=d-1$.
4. The ***bilinear forms graph*** $H_q(n,d)$ (also called ***the q-analog of the Humming graph***) is defined on the vertex set $M_{d\times n}(q)$ of all $d\times n$ matrices over GF(q), $A,B\in M_{d\times n}(q)$ adjacent if and only if the rank of $A - B$ is 1. Then $H_q(n,d)$ is a distance-regular graph with parameters

$c_i = q^{i-1}(q^i - 1)/(q-1)$ (independent of *n*, *d*), and

$b_i = q^{2i}(q^{d-1} - 1)(q^{n-i} - 1)/(q-1)$.

5. The ***Petersen graph***, the ***Odd graph*** O_k , and the ***Generalized odd graphs*** *of* diameter d ,



The distance-regular graph with *classical parameters* (d, q, α, β)

Regular graphs related to classical graphs and groups of Lie type have an intersection array the parameters of which can be expressed in terms of the diameter d and three other parameters q , α and β , called the *classical parameters*, as follows:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right), \qquad \mathrm{c_i} = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad \text{(1a, b)}$$

$(i = 0,1,...,d)$, where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the Gaussian coefficient with basis q (for $q = 1$, it is the ordinary binomial coefficient). Clearly,

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(\beta - 1 + \alpha \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \qquad \text{(1c)}$$

$(i = 0,1,...,d)$. Furthermore, the corresponding eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$ can be calculated in terms of the intersection array as follows:

$$\theta_i = \begin{bmatrix} d-i \\ 1 \end{bmatrix} \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) - \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \text{(2)}$$

$(i = 0,1,...,d)$. Refer to [4, Chapters 6 and 8] for more details.

Some examples of distance regular graphs with classical parameters:

1. the Johnson graph $J(n,d)$ with classical parameters $(d,1,1,n-d)$ $(n \geq 2d \geq 4)$,

2. the Hamming graph $H_q(d,n)$ with classical parameters $(d,1,0,n-1)$ $(n,d \geq 2)$.

3. the Grassmann graphs $J_q(n,d)$ (with parameters $(\alpha,\beta)=(q,[{n-d+1 \atop 1}]-1)$),

4. the bilinear forms graphs $H_q(n,d)$ (with parameters $(\alpha,\beta)=(q-1,q^n-1)$),

As a consequence, the Grassmann graphs and the bilinear forms graphs are characterized simultaneously among distance-regular graphs with classical parameters, together with some extra geometric conditions. (HFu37) Some remarks on distance-regular graphs with classical parameters (d,q,α,β):

1. The four families of distance-regular graphs mentioned above have the property that β can be arbitrary large with respect to the other three parameters d, q, and α. The following result shows that there are no other graphs with this property. (M24)

   Corollary: Suppose that the distance-regular graph Γ has classical parameters (d,q,α,β) with $d \geq 3$ and that Γ is not a Grassmann graph, a bilinear forms graph, a Hamming graph, or a Johnson graph. Then β is bounded in terms of d, q and α.

2. Suppose Γ is a strongly regular graph. It is easy to see that Γ has classical parameters $(2,q,\alpha,\beta)$ and these can be uniquely chosen in such a way that $\beta > 0$. If the parameters are integers and if β is sufficiently large with respect to α and q, then Bose [1] (see also [8]) showed that the graph is the collinearity graph of a partial geometry with parameters $(q+1,\beta+1,\alpha+1)$. This is the analogue to Corollary 1.3 in the case $d=2$. (M24)

3. Distance-regular graphs with classical parameters (d,q,α,β) and $q=1$ have been characterized by Neumaier and Terwilliger [4, Theorem 6.1.1]. (HFu 37)

1-2 Association Schemes, Bose Mesner Algebras and Distance Regular Graphs in the western world

1950 Bose
1973 Delsarte [D73]
1980 Bannai (坂内英一)

geometrical characterization

$\Rightarrow$ parametric characterization (with or without additional constraints)

$\Rightarrow$ spectral characterization

1984 Bannai, Ito, *Algebraic Cocmbinatorics: Association Schemes*, Cumminghams

1989 Brouwer, Cohen and Neumaier, *Distance Regular Graphs*, Springer

1998 Delsarte [DL98]

2004 Bailey, Association Schemes, Design Experiments, Algebra and Combinatorics [B04]

The notion of *distance-regularity* for graphs goes back to the platonic solids of antiquity, which has deep connection to many topics of the present-day theory and applications of geometrics, groups, codes and designs. E. Bannai [1, Chapter 3] has compiled a list of distance-regular graphs with large diameters, believing that has an essentially complete list around 1980. The characterization problems of known important classes of distance-regular graphs by their parameters have a long history in combinatorics. (H3) The classification of all infinite families of distance-regular graphs is a major problem.

It seems that there are several different stages in the characterization of association schemes. Bannai propose to distinguish the following:

1. geometric characterization;
2. parametric characterization, assuming various structures of the neighborbood of a point; or the right sizes of maximal cliques;
3. parametric characterization, assuming arithmetic restrictions on the parameters;
4. complete characterization by the parameters. (Bannai 367)

A way to check whether an algebra of symmetric matrices is a Bose-Mesner algebra and hence an association schems.

Theorem (BCN page. 57)

Let *A* be a vector space of symmentric $n \times n$ matrices.

1. *A* has a basis of mutually disjoint (0,1) matrices if and only if *A* is closed under Hadamard multiplication.
2. *A* has a basis of mutually orthogonal idempotent if and only if *A* is closed under ordinary multiplication.
3. *A* is the Bose Mesner algebra of an association scheme if and only if *A* is closed under both ordinary and Hadamard multiplication, and $I, J \in A$

Referrences:

[BI84] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes,

Benjamin/Cummings 1984.
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[D73]. Philippe Delsarte, An Algebraic Approach to the Association Schemes of Coding Theory, *Philips Research Reports Supple*. 10 (1973).
[DL98]. Philippe Delsarte and Vladimir I. Levenhtein, Association Schemes and Coding Theory, *IEEE Transactions on Information Theory* Vol.44, No.6 October 1998.
[B04] R. A. Bailey, Association Schemes, Design Experiments, Algebra and Combinatorics Cambridge Studies in Advanced Mathematics 84, Cambridge University Press 2004 Ch. 13 History and References

1-3. a state of the art survey of *the geometry of matrices*

Relating to his study of the theory of functions of several complex variables, L.K. Hua（華羅庚）initiated the work in the geometry of matrices in the middle forties. In this geometry, the point of the space are a certain kind of matrices of a given size, the four types of matrices under studied <u>*rectangular matrices*</u>, *alternate matrices*, *symmetric matrices*, and *Hermitian matrices*; including their fundamental theorems and the characterization of motion groups in terms of their invariants (papers were published in Transaction of AMS around 1945-1946)

… to each such space there is associated a group of motions, and the aim of the study is then to characterize the group of motions in the space by as few geometric invariants as possible.

… studying the geometry of matrices of various types over the complex field, discovered that the invariant “adjacency” alone is sufficient to characterize the group of motion of space…

… the complex field was replaced by any field or division ring around 1950 (see Annals of Math. 1949, Chinese J. of Mathematics, Vol. 1 (1951))
… Hua’s pioneer work has been followed by many mathematicians, and more general results have been obtained. The study of the geometry of matrices was then succeeded by many mathematicians, and it has also been applied to graph theory in recent years.

(Geometry of Matrices, Zhe-Xian Wan 萬哲先, World Scientific 1996)

Let D be a division ring, $m, n \geq 2$, ***the geometry of rectangular matrices*** is defined on the space $M_{m\times n}(D) = \{X \mid m\times n \text{ over } D\}$, with the group

$$\{X \mapsto PXQ + R \mid P,\ Q \text{ and } R \text{ are } m\times m \text{ invertible matrices}\},$$

and the arithmetic distance between X and Y $\in M_{m\times n}(D)$ is $\text{rank}(X-Y)$, and X and Y $\in M_{m\times n}(D)$ are adjacent if $\text{rank}(X-Y)=1$.

Theorem (Fundamental Theorem, Wan 1965)

Let A be a bijection from $M_{m\times n}(D)$ to itself. Assume that both A and A^{-1} reserve the adjacency of any two points of $M_{m\times n}(D)$. Then,

1. when $m \neq n$, A is of the form

   $$A(X) = PX^{\sigma}Q + R \quad (*)$$

   where $P, Q \in GL_m(D)$, $R \in M_{m\times n}(D)$, and σ is an automorphism of D;
2. when $m = n$, in addition to $(*)$, A can also be of the form $A(X) = P^t(X^{\tau})Q + R$ where P, Q, R are as above, and τ is an anti-automorphism of D.

In the fundamental theorem of the geometry of rectangular matrices, all bijective mappings of $M_{m\times n}(D)$ are determined such that both φ and φ^{-1} preserve adjacency. Wan showed that if a bijective map φ of $M_{m\times n}(D)$ preserves the adjacency, then so φ^{-1} preserve the adjacency. Thus the supposition that φ^{-1} preserves adjacency may be omitted in the fundamental theorem. It is also shown in [ ] that this is also possible in the case of symmetric and hermitian matrices

Theorem (Wan et. al., 2004, 2006)

Let D be a divison ring and let $m, n \geq 2$ be integers.

1. If a bijective map φ from $\Gamma(M_{m\times n}(D))$ to itself preserves the adjacency in $M_{m\times n}(D)$, then also φ^{-1} preserves the adjacency.
2. If a bijective map φ is from $\Gamma(M_{m\times n}(D))$ to itself for which any two adjacent vertices A, B of $\Gamma(M_{m\times n}(D))$ implies adjacent A^{φ}, B^{φ}, then φ is a graph automorphism of $\Gamma(M_{m\times n}(D))$.

Geometry of rectangular matrices

Let D be a division ring, $m, n \geq 2$, and let $M_{m\times n}(D) = \{X \mid m \times n \text{ over } D\}$

space: $M_{m\times n}(D)$

group: $\{X \mapsto PXQ + R \mid P,\ Q \text{ and } R$ are $m \times m$ invertible matrices$\}$

invariants: $\text{rank}(X - Y)$ the arithmetic distance between X and Y ,

adjacency: $\text{rank}(X - Y)$ =1.

Definition: maximal sets in $M_{m\times n}(D)$

a *maximal set* in $M_{m\times n}(D)$ is a maximal set of points (matrices) such that any two of them are adjacent. The normal forms of maximal sets are

$$\left\{\begin{pmatrix} x_{11} & x_{12} & \cdots x_{1n} \\ 0 & 0 & \cdots 0 \\ \cdots & \cdots & \\ 0 & 0 & \cdots 0 \end{pmatrix} : x_{1i} \in D\right\}, \text{ or } \left\{\begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ \mathrm{x}_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \mathrm{x}_{m1} & 0 & \cdots & 0 \end{pmatrix} : x_{i1} \in D\right\}$$

Maximal sets are called *maximal cliques* in graph theory in the western world around 1980, and the geometric structures, called ***the attenuated spaces*** (Ray-Chaudhuri an A. Sprague 1980) in term of *lines* defined by the intersection of two maximal sets containing two adjacent points in common have been studied.

Theorem: There are two types of maximal cliques with sizes q^n, q^d respectively, and each with the numbers $q^{n(d-1)}\begin{bmatrix} d \\ 1 \end{bmatrix}$, $q^{d(n-1)}\begin{bmatrix} n \\ 1 \end{bmatrix}$ respectively. (Bannai 372)

[H51] Hua, L.K. A theorem on matrices over a field and it sapplications. Acta Math. Sinica (1951), 109-163

[W96] Zhe-Xian Wan, Geometry of Matrices: In Memory of Professor L.K. Hua (1910-1985), 443-453 in: Progress in Algebraic Combinatorics, Advanced Studied in Pure Mathematics 24, 1996.

[W96] Zhe-Xian Wan, Geometry of Matrices, World Scientific, Singapore 1996

[HW04] Wen-ling Huang, Ronald Hofer and Zhe-Xian Wan, Adjacency preserving mappings of symmetric and hermitian matrices, Aequationes Math. 67 (2004) 132-139

[HW04] Wen-ling Huang and Zhe-Xian Wan, Adjacency Preserving Mappings of Rectangular Matrices, Beitrage zur Algebra und Geometrie, Contribution to Algebra and Geometry Volume 45(2004) No.2, 435-446

1-4. The bilinear forms graphs $H_q(n,d)$ and their characterizations

When the field is finite, say *GF*(*q*), ***the bilinear forms graph*** $H_q(n,d)$ is defined over $M_{d\times n}(GF(q))$, which is a *distance regular graph* with the *intersection array*

$$c_i = q^{i-1}(q^i - 1)/q - 1 \quad \text{for} \quad i \le d-1, \text{ and}$$

$$b_i = q^{2i}(q^{d-i} - 1)/q - 1 \quad \text{for} \quad i \le d-1.$$

The weak 4-vertex condition was used by Sprague and Huang respectively in the characterizations of the bilinear forms graphs $H_q(n,d)$ for d = 3 or general d. Sprague [ ]

characterized $H_q(n,3)$ by assuming the parameters and the weak 4-vertex condition where $n \ge 6, q \ge 2$ and $(q,n) \ne (2,6)$. Later Ray-Chaudhuri and Sprague [13] gave a combinatorial characterization of 3-attenuated spaces by using the arguments similar to those used in [14]. (H 2) Finally, Sprague [16] characterized *d*-attenuated spaces in terms of the structure of their 2-spaces:

Huang (1987)

Let Γ be a distance-regular graph of diameter d with intersection array $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$ such that

1. The weak 4-vertex conditions holds in Γ,
   the number of edges of the induced subgraph on the common neighborhood of vertices *x* and *y* depends only on the distance between them. (H1)
2. $c_2 = q(q+1),\ b_i = q^{2i}(q^{d-i}-1)/q-1,\ \ 0 \le i \le d-1$,
3. $n \ge 2d \ge 4,\ \ q \ge 4$.

Then q is a prime power, and the graph Γ is isomorphic to the bilinear form graph $H_q(n,d)$.

… the restriction $q \ge 4$ in section 3 (Prop 3.7) of [7] was used by Huang to apply a well-known theorem of Buekenhout. (M33) However, it was noticed by Cuypers that the use of Buekenhout result can be avoided, one uses a result of Thas and DeClerck [12], which also holds for $q = 2$ and $q = 3$. A shorter proof of the results of Section 4. (C18)

Cuypers (1992) improved the conditions:

1. $(1-q^{d-1})/(q-1)$ is an eigenvalue of the adjacency matrix of Γ; and
2. $c_2 = q(q+1)$, $b_i = q^{2i}(q^{d-i}-1)/q-1$, $0 \le i \le d-1$,
3. $n \ge 2d \ge 6$ and $q \ge 4$.

i.e., the bilinear form graph $H_q(n,d)$ is characterized by its intersection array

whenever $n \ge 2d \ge 6$ and $q \ge 4$.

… the conditions $n \ge 2d \ge 6$ and $q \ge 4$ are used by Huang to derive an incidence structure from Γ, and $q \ge 4$ is needed to apply a theorem by Buekenhout (see [2]). Cuyper showed that the use of Buekenhout's result can be avoided by quoting a result of Thas and Declerck [5] which is also valid for $q = 2,3$. So Cuyper generalize Huang's theorem to the following: (M…)

… however, Huang needs it only to show that lines have size $\beta+1$ and that the incidence structure consisting of the vertices and lines satisfies the dual of *Pasch's Axiom*, which we have already shown. Thus the arguments of Huang show that q is a prime power, $\beta = q^n - 1$ for some integer $n \geq d$ and Γ is the bilinear forms graph $H_q(n,d)$. (M34)

… the proof can be completed using the techniques of Huang … Thus the arguments of Huang show that q is a prime power, $\beta = q^n - 1$ for some integer $n \geq d$ and is the bilinear forms graph as required. (M34)

Metsch (1999) further improved the conditions:

the bilinear form graph $H_q(n,d)$ is characterized by its intersection array whenever either $n \geq d+3, q \geq 3$ or $n \geq d+4, q = 2$.

Theorem (Huang 1987 + Cuyper 1992 + Metsch 1999)

Let Γ be a distance-regular graph of diameter d with intersection array $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$ such that

1. $b_i = q^{2i}(q^{d-i}-1)/q-1,\ 0 \leq i \leq d-1,\ c_i = q^{i-1}\begin{bmatrix} i \\ 1 \end{bmatrix},\ 1 \leq i \leq d.$
2. $n \geq d+3, q \geq 3$; $n \geq d+4, q = 2$.

Then q is a prime power, and $\Gamma \cong H_q(n,d)$.

1-5. Joint Characterizations of Grassmann graphs and Bilinear forms graphs

Because the bilinear forms graph $H_q(n,d)$ is an induced subgraph of the Grassman graph $J_q(n,d)$, joint characterizations of Grassmann graphs and Bilinear forms graphs are surveyed in the following.

1. a joint characterization of Grassmann graphs and bilinear forms graphs among distance regular graphs with classical parameters is given by Fu and Huang (1994); and by Metsch (1999) respectively;

2. a joint characterization of Grassmann spaces and attenuated spaces over amply regular $(0,\alpha)$-geometries satisfying the dual of Veblen-Young's axiom was given by by G.Bonoli, N. Melone (2003).
3. a joint characterization of Grassmann graphs and bilinear forms graphs over distance-regular $(0,\alpha)$ - Reguli by F. de Clerck, S. De Winter E. Kujiken and C. Tonesi (2006). Moreover,

[FH94] T.S. Fu and T. Huang, A Unified Approach to a Characterization of Grassamn graphs and bilinear forms graphs, Europ. J. Combinatorics (1994)15, 363-373.

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[BM03] G. Bonoli, N. Melone, A Characterization of Grassman and Atteenuated Spaces as $(0,\alpha)$-Geometries, Europ. J. Combinatorics (2003)24, 489-498.

[CW06] F. de Clerck, S. De Winter E. Kujiken and C. Tonesi, Distance-Regular $(0,\alpha)$-Reguli, Designs, Codes and Cryptography 38 (2006) 179-194.

Definition:

For an integer $\alpha \geq 1$, a connected semilinear space $\Pi = (P, B)$ is called a **finite** $(0,\alpha)$ **-geometry** with parameters (s,t) if it satisfies the following axioms:

1. each block contains $s+1$ points;
2. each point belongs to $t+1$ blocks; and
3. for every anti-flag (x,B), $\alpha(x,B) = 0, \alpha$.

Any (0, α)-geometry with parameters (*s, t*) is called an **amply** regular (0, α)-geometry with parameters (*s, t,* α , μ) if the number of points adjacent to any pair of points at distance two is a constant μ > 0.

Amply regular $(0,\alpha)$-geometries satisfying (VY*) for which $\mu = \alpha^2$ or $\alpha(\alpha+1)$ was studied by G.Bonoli, N. Melone in [ ]. This result generalizes those of Debroey [ ] and of Ray-Chaudhuri and Sprague [ ]. It gives a common characterization of the Grassmann and attenuated spaces as amply regular $(0,\alpha)$-geometries and, under weaker assumptions, we obtain the result due to Fu-Huang [9].

An important role in incidence structures is played by the so-called *Pasch's axiom* (Veblen-Young's axiom (VY)) and the *diagonal axiom* (the dual VY*), resp. (see for example [5, 8, 16, 17]). (P52~53)

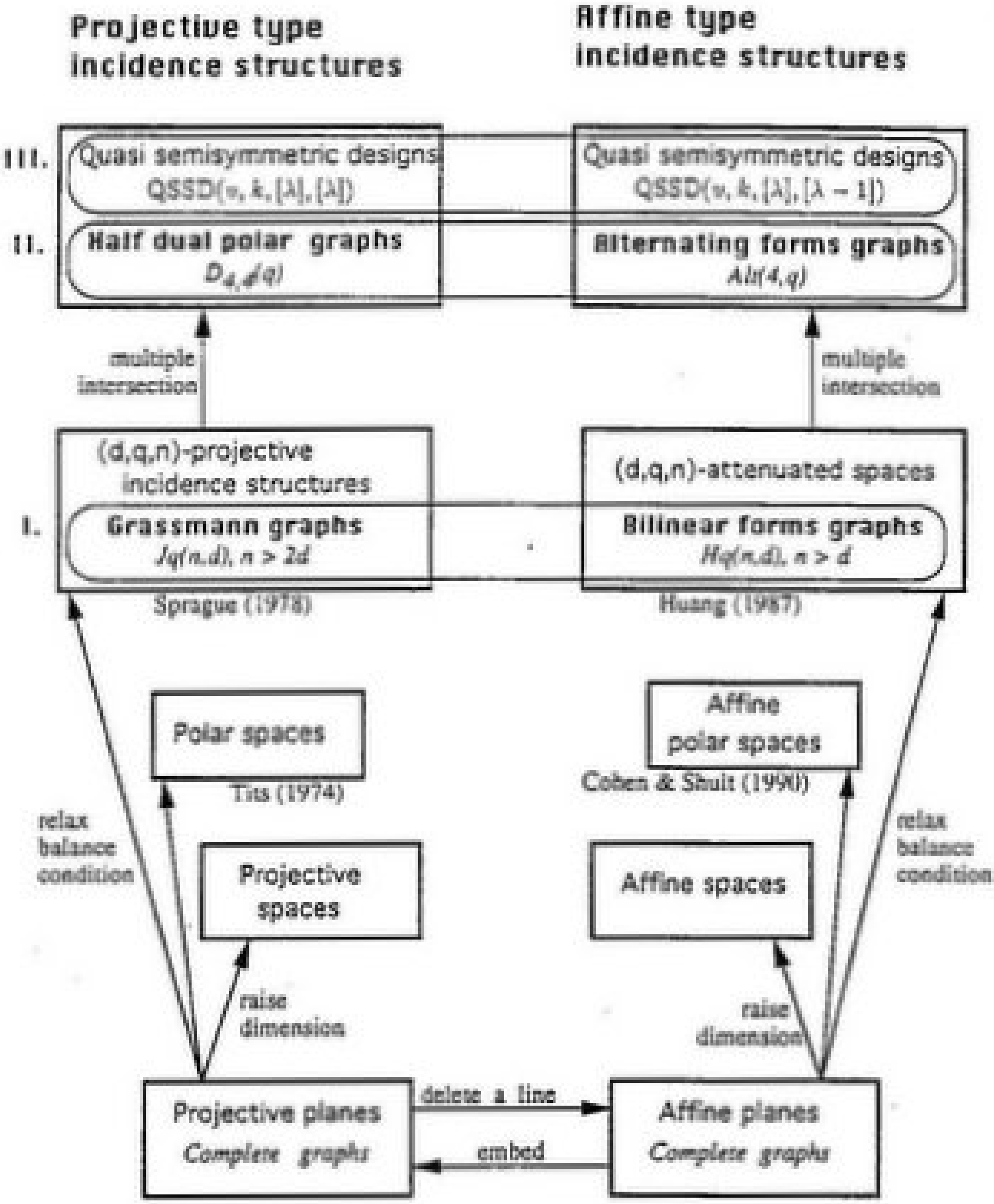
1. Two distinct transversals of two intersecting blocks are incident. (*Pasch's axiom*)
2. Two distinct points *x, y* not on a block *B* and adjacent to two distinct common points on *B*, are adjacent. (*the diagonal axiom*)

Note that both Johnson and Grassmann geometries satisfy *Pasch's axiom* and the diagonal axiom (see [14]), while the attenuated spaces satisfy the diagonal axiom but not *Pasch's axiom*.

Definition A *distance-regular geometry* is an incidence structure *S* satisfying the following axioms: (P60~61)

1. S is a partial linear space of order (s, t).
2. The point diameter (i.e. the largest possible distance between two vertices corresponding to points) of the incidence graph Φ of S is $2d \geq 4$.
3. There exist integers α_{2i-1}, $2 \leq i \leq d$, such that for any point p and any line L of S which are at distance 2i-1 in Φ there are precisely α_{2i-1} points incident with L and at distance 2i-2 from p.
4. There exist integers t_{2i}, $2 \leq i \leq d$, such that for any two points p and q of S which are at distance 2i in Φ there are precisely $t_{2i}+1$ lines incident with q and at distance 2i-1 from p.

Using results proved by Huang in [H87], Cuypers has proved in [Cu92] that a distance-regular graph with the same intersection array as $H_q(m,d)$, with $m \geq 2d \geq 6$ and $q \geq 4$, is the point graph of a (0,α)-geometry satisfying the so-called *diagonal axiom*, which is the dual of the Pasch (or Veblen-Young) axiom. Form arguments similar as in [15] (see also [2]) it follows that the geometry is uniquely defined and hence that Γ is isomorphic to $H_q(m,d)$. (P73~74) The geometry constructed in Subsection 4.1, seen as a geometry on the bilinear forms graph, is also known as an *attenuated space* [13]. (P73~74)


Projective type
incidence structures
Affine type
incidence structures
III.
Quasi semisymmetric designs
QSSD(v, k, [λ], [λ])
Quasi semisymmetric designs
QSSD(v, k, [λ], [λ − 1])
II.
Half dual polar graphs
Alternating forms graphs
Alt(4,q)
multiple
intersection
multiple
intersection
(d,q,n)-projective
incidence structures
(d,q,n)-attenuated spaces
I.
Grassmann graphs
Jq(n,d), n > 2d
Bilinear forms graphs
Hq(n,d), n > d
Sprague (1978)
Huang (1987)
Polar spaces
Tits (1974)
Affine
polar spaces
Cohen & Shult (1990)
relax
balance
condition
relax
balance
condition
Projective
spaces
Affine spaces
raise
dimension
raise
dimension
delete a line
embed
Projective planes
Complete graphs
Affine planes
Complete graphs

2. The geometry of rectangular matrices, the attenuated spaces and *d*-nets

In addition to the *intersectional arrays* and the *spectra* of distance regular graphs, the *geometric structures* associated with distance regular graphs are also play vital roles.

2-1 The structure of nets and *d*-nets

1. A ***net*** (of dimension 2) is a semilinear space $D=(P,B)$ satisfying the following condition:
   1. B is partitioned into at least three non-empty classes such that
   2. the blocks of each class partition P,
   3. blocks of different classes intersect.

2. A d ***-net*** or ***net of dimension*** d is a semilinear space $D=(P,B)$ of dimension $d \geq 3$ satisfying the following three conditions.
   1. Each *plane* of D is a net.
   2. The intersection of two *subspaces* is a subspace.
   3. Two planes in a 3-*space* are disjoint or intersect in a block.

Theorem (Sprague 1983)
Every finite d-net, where $d \geq 3$ is an integer, is an (n, q, d)-attenuated space for some prime power q and positive integer n. (The original theorem covers the infinite case also.)

2-2 The (d,q,n) - **projective incidence structures** and the (d,q,n) - ***attenuated space***

Definition

1. **The** (d,q,n) ***-projective incidence structure***
   the collection of subspaces of the n-dimensional vector space over $GF(q)$ where subspaces of dimension d are called *points*, those of dimension $d-1$ are called *lines*, and incidence is the usual containment. (HF38)

2. **The** (d,q,n) **-attenuated space**
   Let V be an $(n+d)$-dimensional vector space over $GF(q)$ and let W be a given n-dimensional subspace of V. The (d,q,n)-attenuated space is the collection of subspaces U of V with $U \cap W = 0$, where subspaces U of dimension d are called *points*, those of dimension $d-1$ are called *lines*, and incidence is the usual containment. (HF38)

The bilinear forms graph $H_q(n,d)$ is therefore the induced subgraph of the Grassman graph $J_q(n+d,d)$ on the vertices that meet *W* trivially. Indeed, $J_q(n,d)$ and $H_q(n,d)$ are the ***collinearity graphs*** of the (d,q,n)- projective incidence structures [11] and the (d,q,n)-attenuated spaces [10, 12], respectively (HFu38). The bilinear forms graph $H_q(n,d)$ can be viewed not only as a subgraph but also as a geometric hyperplane of the Grassmann graph $J_q(n+d,d)$. (HF38)

2-3 Attenuated spaces and the bilinear forms graphs

Let *V* be a vector space of dimension *n* + *d* over *GF*(*q*) and $W=\langle w_1,w_2,...,w_n\rangle\subseteq V$ a fixed subspace of dimension *n*. Let further $\{w_1, w_2, \ldots, w_n; u_1, u_2, \ldots, u_d\}$ be a basis for *V*, where $U=\langle u_1,...,u_d\rangle$. Let

$$\Im_i=\left\{ A \;\middle|\; A\in\begin{bmatrix}V\\ i\end{bmatrix} \text{ and } \dim(A\cap U)=0\right\},$$

then $A=\langle u_1+v_1,u_2+v_2,...,u_d+v_d\rangle$ for unique choices of $v_1,v_2,...,v_d\in W$. Let $v_i=\sum_{j=1}^{n}a_{ij}w_j$,

and $M_A=\left[a_{ij}\right]_{d\times n}$, then each *A* corresponds to the unique matrix M_A of order $d\times n$. Moreover If A, B correspond to M_A, M_B respectively as given, then $d-\dim(A\cap B)=rank(M_A-M_B)$.

2-4 How to associate geometric structures to distance-regular graphs?

Two techniques dealing with the *geometric structures* associated with distance regular graphs in terms of of *maximal cliques*

How to treat the maximal cliques of distance regular graphs ?

1. The Bose and Laskar argument

One of the crucial steps to characterize distance regular graphs is to show the existence of maximal cliques of the right size. The characterization of the Hamming graph $H(n,2)$ and the Johnson graph $J(v,2)$ by Bose and Lasker (1967) is one of the original papers towards this goal; the technique used by them is called *Bose-Laskar argument*. This argument provide essential information on intersections of maximal cliques of a distance-regular graph simply in terms of the parameters of the graphs. In order to apply this technique, we usually need some restriction on the parameters; for example, *q* must be sufficiently large compare with *n* in $H(n,q)$, and *v* must be sufficiently large compare with *k* in $J(v,k)$.

Theorem (Bose and Laskar 1967) Let *G* be a graph satisfying the following conditions:

1. $\deg(x) = r(k-1)$ for all $x \in V(G)$,
2. $|G_1(x) \cap G_1(y)| = k - 2 + \alpha$ if x and $y \in V(G)$ are adjacent,
3. $|G_1(x) \cap G_1(y)| \leq 1 + \beta$ if *x* and $y \in V(G)$ are not adjacent, where $r \geq 1$, $k \geq 2$, $\alpha \geq 0$ and $\beta \geq 0$ are fixed integers.

A maximal clique with at least $k - (r-1)\alpha$ vertices is called a *grand clique*.
Suppose $k > p(\alpha,\beta,\gamma), \rho(\alpha,\beta,\gamma)$, where

$p(\alpha,\beta,\gamma) = 1 + ((\gamma+1)(\gamma\beta - 2\alpha)/2)$, and $\rho(\alpha,\beta,\gamma) = 1 + \beta + (2\gamma - 1)\alpha$.

Then

1. each vertex of *G* is contained in exactly γ grand cliques, and
2. each pair of adjacent vertices is contained in exactly one grand clique.

2. A technique in terms of graph representation

In addition to t*he Bose-Laskar argument*, the following theorem obtained by the technique of graph representations provides another mechanism to deal with *the structures of maximal cliques*. The above two theorems provide essential information on intersections of maximal cliques of a distance-regular graph simply in terms of the parameters of the graphs.

Theorem (BCN, p.160)
Let Γ be a distance-regular graph of diameter *d* with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$.

1. If $d \geq 3$, $\theta_1 < b_1 - 1$ and suppose that every singular line of Γ has size at least *s*+1;

   a. if $s \geq 3$, $b_1/(\theta_1 + 1) \leq s^2 - s + 1$, or if *s* = 2, $b_1/(\theta_1 + 1) \leq 2$, then distinct maximal cliques intersect in a singular line, a point or the empty set.

   b. if $s \geq 4$ and $b_1/(\theta_1 + 1) \leq s + 1$, then every edge is in at most two maximal cliques.

2. If $d \geq 2$, then the size of a clique *C* in Γ is bounded by $|C| \leq 1 - k/\theta_d$. If equality holds then every vertex $x \notin C$ is adjacent to either 0 or $b_1/(\theta_d + 1) + 1 - k/\theta_d$ vertices of *C*. No vertex of Γ has distance *d* to C.

Theorem (**Hoffman bound**). Let Γ be a distance-regular graph with valency k and an eigenvalue $\theta < 0$, then the size of a clique C in Γ is bounded by $|C| \leq 1 + |k/\theta|$. (C20)

2-4 The structure of maximal cliques

T*he Bose-Laskar argument* provide essential information on intersections of maximal cliques of a distance-regular graph simply in terms of the parameters of the graphs. (HFu38). In order to apply this technique, we usually need some restriction on the parameters; for example,

1. q must be sufficiently large compare with n in $H(n,q)$, and
2. v must be sufficiently large compare with k in $J(v,k)$.

For the case of the bilinear form graph with intersection parameters $b_0 = (q^n - 1)(q^d - 1)/(q-1)$, $a_1 = q^n + q^d - q - 2$ and $c_2 = q(q+1)$, $(k,\gamma,\alpha,\beta) = (q^n, (q^d - 1)/(q-1), q^d - q, q^2 + q + 1)$. In order that k is larger than the maximum $p(\alpha,\beta,r)$ and $\rho(\alpha,\beta,\gamma)$, this causes some constraints over n, d, and q. Moreover each grand clique contains at least

$$k - (\gamma - 1)\alpha = q^n - ((q^d - q)^2/(q-1))$$

vertices.

In addition to t*he Bose-Laskar argument*, the following theorem (BCN 160) obtained by the technique of graph representations provides another mechanism to deal with the tructures of maximal cliques. (HFu38)

… tthe above two numerical constraints on n and d, interpreted as $\beta \ge \alpha[{}^{2d}_{1}]$ with $d \ge 3$ in terms of the classical parameters, were needed in both cases because both Sprague and Huang used the Bose-Laskar argument.

… instead of *the Bose-Laskar argument*, the following two theorems in terms of the technique of graph representations were used by Huang and Fu [ ] to improve the bound from $\beta \ge \alpha[{}^{2d}_{1}]$ to $\beta > \alpha[{}^{d}_{1}]$, i.e. $n \ge 2d+1$ for $J_q(n,d)$ and $n \ge d+1$ for $H_q(n,d)$. However, the assumption $\alpha + 1 \ge \max\{5, q\}$ is still required. (Theorem A(ii)) [HFu 38]

… the Grassmann graphs $J_q(n,d)$ with $n \ge 3d \ge 9, q \ge 3$, or with $n \ge 3d+1 \ge 4$ whenever $q = 2$, and the bilinear forms graphs $H_q(n,d)$ with $n \ge 2d \ge 6, q \ge 4$ have been characterized by Sprague [13], Huang [10] respectively.

… regarding the classes of maximal cliques as their lines, two incidence structures are derived from the distance-regular graph considered in the Main Theorem; note that any two points at distance 2 have $(q+1)(\alpha+1)$, i.e. c_2 in the theorem, common neighbors in both incidence structures. (HFu38)

2-5 A pairs of incidence structures derived from Γ

A maximal clique in Γ with at least q^n vertices is called a *grand clique*. Let $x, y \in V(\Gamma)$ be adjacent and $A_{x,y} = S_1 \bigcup S_2$, where

$S_1 = (\Gamma_1(x) \bigcap \Gamma_1(y)) - <x, y>$, and
$S_2 = \{z \mid z \in <x, y> \text{ is adjacent to each point of } S_1\}$

Then $A_{x,y}$, called the *assembly* determined by the adjacent pair *x*, *y*, is a clique. Let

L = the set of all grand clique of Γ, and
A = the set of all assemblies of Γ.

A pair of semilinear incidence structures $\Pi = (V(\Gamma), L, \in)$ and $(V(\Gamma), A, \in)$ are considered by Huang and Cuyper. The *weak* 4-vertex condition is used by Huang in the proof of 2.2 and 2.4 of [4]. **(**C20**)** However, the role played by the *weak* 4-vertex condition can be replaced by Hoffman bound and a result of Brouwer and Wilbrink. Indeed,

1. Proposition 2.2 of [4] is a direct consequence of the fact that $(1-q^{d-1})/(q-1)$ is an eigenvalue of the adjacency matrix (i.e., condition (1) of Theorem 1.3) and the following Hoffman bound (see [3]). **(**C20**)**

2. It remains to prove 2.4 of [4] without using the weak 4-vertex condition.

Proposition 2.4 $(\Gamma_1(x) \bigcap \Gamma_1(y)) - <x, y>$ is a clique for any adjacent pair *x* and *y*.

Proposition 2.4 is a first step in attaching an affine structure to a subset of the form $(\Gamma_1(x) \bigcap \Gamma_1(y)) \bigcup \{x, y\}$ for adjacent pair *x* and *y* which is essential in determining the structure of the subspaces of $\prod$.

The above mentioned incidence structures have been considered by Brouwer and Wilbrink [1], even under weaker conditions. We then conclude that the incidence structure $\prod$ satisfies the dual of *Pasch's axiom*, and hence Prop 2.4 follows. The proof of Theorem 1.3 as in [4] continues. Wilbrink and Brouwer [18] proved that certain semi-partial geometries with some weak restrictions on parameters satisfy *the dual of Pasch's axiom*. (HFu37)

Theorem The semi-linear incidence structure $\prod = (V(\Gamma), L, \in)$ satisfies the following properties;

1. each point of Γ is contained in exactly $\gamma = (q^d - 1)/(q-1)$ lines, and
2. each pair of adjacent points *x*, *y* is contained in exactly one line, denoted by $\langle x, y \rangle$.
3. every line contains q^n points;
4. a point *x* is adjacent to 0 or *q* points on a line not containing *x*;
5. for any two non-adjacent points *x* and *y* we have $|\Gamma_1(x) \cap \Gamma_1(y)| = 0$ or $q(q+1)$.

Moreover, Γ is the adjacency graph of $\prod$.

We then show that the restriction of $\prod$ to $A_{x,y}$ is an affine space of dimension d over $GF(q)$, which is the first in determining the structure of the subspaces of $\prod$. (H6)

Theorem The incidence structure $(V(\Gamma), A, \in)$ is a semi-linear incidence structure with Γ as its collinearity graph, having many properties in common with $(V(\Gamma), A, \in)$.

The crucial observation in the proof of Theorem 1.2 (C). With the lines of $\prod$ playing the role of the assemblies for $(V(\Gamma), A, \in)$, we can prove similar results. (C19)

We recall that the 2-spaces of any assembly are affine plane of order *q*. To prove that the assembly is indeed isomorphic to a *d*-dimensional affine space over $GF(q)$. (C18) The structure of 2-spaces of $\prod$ is studied in Section 4 in terms of the structure of assemblies obtained in the last section. (H10), we show that any 2-space of $\prod$ is a net. Further properties about parallel lines, which are essential to subsequent development of the structure of 2-spaces of $\prod$. (H10)

Any 3-space of $\prod$ is a $(n, q, 3)$-attenuated space is proved in Proposition 5.2 , which provides a starting point of the induction argument. We finally show that $\prod = (V(\Gamma), L)$ is a *d*-net. (H13)

Sections 4 and 5 of Huang [12] provide the rest of the proof; precisely,

- the definition of parallelism between blocks, and the characterization of planes as net are exactly the same of Section 4 of [12];
- the intersection of two subspaces is a subspace (D2) follows from the assumption of connectedness of intersection between subspaces.
- two planes in a 3-space are disjoint or intersect in a block (D3) is exactly the same as Proposition 5.2 of [12]. (B53)

3. Searching for cospectral mates and distance regular mates of bilinear forms graphs

Recently, a non-distance-regular cospectral mate and a non-vertex-transitive distance-regular mates for the Grassmann graph $J_q(n,d)$ with $n \ge d+3 \ge 6$ or with $n = d+1 \ge 3$ respectively were given by E.R. van Dam and J.H.Koolen, and by E.R. van Dam, W.H. Haemer, J.H. Koolen and E. Spence respectively. From the view point that $J_q(n,d)$ is the point graph of *the projective incidence structures* $\left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d-1 \end{bmatrix}; \supseteq\right)$, $\left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d+1 \end{bmatrix}; \subseteq\right)$, and the bilinear form graph $H_q(n,d)$ is the collinearity graph of *the attenuated space* $\left(\Im_d(V,W), \Im_{d-1}(V,W); \supseteq\right)$, we may wonder whether similar situations hold for the bilinear forms graphs $H_q(n,d)$ due to the close relationship between the projective incidence structures and the attenuated spaces,.

Some necessary backgrounds and some proposal are collected for references.

[DK05] E.R. van Dam and J.H.Koolen, A new family of distance-regular graphs with unbounded diameter, Invent. Math. 162, 189-193 (2005)

[DH 06] E.R. van Dam, W.H. Haemer, J.H. Koolen and E. Spence, Characterizing distance-regularity of graphs by the spectrum, JCT A 113 (2006) 1805-1820.

3-1 some known facts about the Grassmann graph $J_q(n,d)$

Let $V = F_q^{\,n}$, consider the semilinear incidence structures

$$\pi_1 = \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d-1 \end{bmatrix}; \supseteq\right), \quad \pi_2 = \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d+1 \end{bmatrix}; \subseteq\right)$$

1\. . for the incidence structure $\pi_1 = \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d-1 \end{bmatrix}; \supseteq\right)$

a. there are $\begin{bmatrix} n \\ d \end{bmatrix}$ points and there are $\begin{bmatrix} n \\ d-1 \end{bmatrix}$ lines.

b. each line is incident to $\begin{bmatrix} n-(d-1) \\ d-(d-1) \end{bmatrix} = \begin{bmatrix} n-d+1 \\ 1 \end{bmatrix}$ points, and each point is incident to $\begin{bmatrix} d \\ d-1 \end{bmatrix} = \begin{bmatrix} d \\ 1 \end{bmatrix}$ lines.

c. two points are collinear if and only if they meet in an $(d-1)$-dimensional subspace.

d. the point graph is the Grassmann graph $J_q(n,d)$ with the adjacency matrix of $NN^t - \begin{bmatrix} d \\ 1 \end{bmatrix} I$, where N is the point-line incidence matrix of π_1.

2. for the incidence structure $\pi_2 = \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d+1 \end{bmatrix}; \subseteq\right)$

a. there are $\begin{bmatrix} n \\ d \end{bmatrix}$ points, and there are $\begin{bmatrix} n \\ d+1 \end{bmatrix}$ lines.

b. each line is incident to $\begin{bmatrix} d+1 \\ d \end{bmatrix} = \begin{bmatrix} d+1 \\ 1 \end{bmatrix}$ points, and each point is incident to $\begin{bmatrix} n-d \\ d-(d-1) \end{bmatrix} = \begin{bmatrix} n-d \\ 1 \end{bmatrix}$ lines.

c. two points are collinear if and only if they together generate a *d*+1 subspace, and hence they meet in an $(d-1)$-dimensional subspace.

d. the point graph is the Grassmann graph $J_q(n,d)$ with the adjacency matrix of $NN^t - \begin{bmatrix} n-d \\ 1 \end{bmatrix} I$, where *N* is the point-line incidence matrix of π_2.

3. The lines in π_1, π_2 correspond to the two types of maximal cliques of the Grassmann graph $J_q(n,d)$ respectively.

3-2 cospectral mates and distance-regular mate of the Grassmann graph $J_q(n,d)$

3-2 a. A *non-distance-regular cospectral mate* of $J_q(n,d)$

in terms of the *line graphs* of some incidence structures:

[DH 06] E.R. van Dam, W.H. Haemer, J.H. Koolen and E. Spence, Characterizing distance-regularity of graphs by the spectrum, JCT A 113 (2006) 1805-1820.

1. the Grassmann graph $J_q(n,d)$ is the line graph of the incidence structure $I_q(n,d)$:

Let $V = F_q^{\,n}$, consider the semilinear incidence structure

$$I_q(n,d) = \left(\begin{bmatrix} V \\ d-1 \end{bmatrix}\right), \begin{bmatrix} V \\ d \end{bmatrix})$$

with the point-line incidence matrix *N*,

1. the *point graph* of $I_q(n,d)$ with adjacent matrix $NN^t - \begin{bmatrix} n-d+1 \\ 1 \end{bmatrix} I$ is isomorphic to $J_q(n,d-1)$

2. the *line graph* of $I_q(n,d)$ with an adjacent matrix $N^t N - \begin{bmatrix} d \\ 1 \end{bmatrix} I$ is isomorphic to $J_q(n,d)$.

Note that NN^t and $N^t N$ have the same nonzero eigenvalues;

2. ***Adjusting*** the partial linear space $I_q(n,d)$ to the partial linear space $C_q(n,d)$ while

a. the point graph remains the same, and

b. the number of lines, and the sizes of lines

remain the same, then the line graph of $C_q(n,d)$ is *cospectral* with that of $I_q(n,d)$, *i.e.*, the Grassmann graph $J_q(n,d)$.

First we define the partial linear space $C_q(n,d)$ for $t \ge 1$ in general, and then reduce to the special case that $t = 1$. For $n \ge 2d-1, d \ge 3$, let $V = F_q^{\ n}$, and $H_1, ..., H_t$ be $(2d-2)$-dimensional subspaces of V such that $\dim(H_i \cap H_j) \le d-1$ for $i \ne j$. Consider the semilinear incidence structure

$$C_q(n,d) = \left(\begin{bmatrix} V \\ d-1 \end{bmatrix}, L_1 \cup L_2 \right)$$

where

$L_1 = \{(S,i) \mid S \text{ is a } d-2 \text{ dimensional subspace of } H_i\}$

the line (S,i) consists of all $d-1$ dimensional subspaces of H_i containing S;

$L_2 = \{T \mid T \text{ is a } d \text{ dimensional subspace not contained in any } H_i\}$

the line T consists of all its $d-1$ subspaces.

More explicitly,

lines $(S,i), (T,j) \in L_1$ are adjacent in the line graph,

if $i = j$ and $S \cup T$ spans a $(d-1)$ dimensional subspace, or

if $H_i \cap H_j$ is a $(d-1)$ dimensional subspace containing $S \cup T$.

lines $(S,i) \in L_1$ and $T \in L_2$ are adjacent

if $S \subseteq T$ and moreover T intersect H_i in a $(d-1)$ dimensional subspace;

lines $T, T^{'} \in L_2$ are adjacent if they intersect in a $(d-1)$ dimensional subspace. Q.E.D.

Theorem The line graph of the semilinear space $C_q(n,d)$ is cospectral with $J_q(n,d)$, which is not distance-regular.

For $t = 1$, $(U,1)$ and W are not adjacent, and have at least $\begin{bmatrix} d \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ many common neighbors, while $c_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2$ for $J_q(n,d)$, the line graph of $C_q(n,d)$ is a non-distance-regular cospectral with the Grassmann graph $J_q(n,d)$.

Remark: If $n \ge 2d$, many (if not all) of the constructed cospectral graphs are not distance-regular, for example: there is a *d*-dimensional subspace W that intersects H_1 (say) in a $(d-2)$ dimensional subspace *U* and not contained in H_i for each *i*.

Question: Do similar arguments work for the *bilinear forms graphs* in terms of the *attenuated spaces*?

There is a correspondence between bipartite regular graphs with 5 eigenvalues and so called *partial geometric designs*. Examples of the latter are *transversal designs*, and these form the key to the construction of graphs cospectral with distance-regular antipodal covers of complete bipartite graphs.

The incidence structure between the two biparts of such a cover is a (square) ***resolvable transversal deigns*** (also called a *symmetric net*). A transversal design is a design of points and lines, such that all blocks have the same size, each point is in the same number of blocks, and such that the points can be partitioned into groups, such that each block intersects each group in one point, and such that two points from different groups meet in a constant number μ points.

Lemma [Lemma 9.3.2, BCN 269]
Let *V* be a vector space of dimension *n* over $GF(q)$, and $0 \le i, j \le n$, then

a. the number of *m*-subspaces is $\begin{bmatrix} n \\ m \end{bmatrix}$,

b. if *X* is a *j*-spaces of *V*, then there are precisely $q^{ij}\begin{bmatrix} n-j \\ i \end{bmatrix}$ *i*-spaces *Y* in *V* such that $X \cap Y = 0$.

c. if *X* is a *j*-spaces of *V*, then there are precisely $q^{(i-m)(j-m)}\begin{bmatrix} n-j \\ i-m \end{bmatrix}\begin{bmatrix} j \\ m \end{bmatrix}$ *i*-spaces *Y* in *V* such that $X \cap Y$ is an *m*-space.

3-2 b: a non-transitive distance-regular mate of the Grassmann graph $J_q(2e+1, e)$

[DK05] E.R. van Dam and J.H.Koolen, A new family of distance-regular graphs with unbounded diameter, Invent. Math. 162, 189-193 (2005)

The case $(n,d) = (2e+1, e)$

Let $V = F_q^{2e+1}$ with a fixed hyperplane *H*, consider the semilinear incidence structure

$$\pi_3 = \left(\begin{bmatrix} V \\ e \end{bmatrix}, L_1 \cup L_2 ; \# \right)$$

where

$L_1 = \{ A \mid A \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \text{ but } A \not\subset H \}$, the line $A \in L_1$ is incident to its e - dimensional subspaces;

$L_2 = \begin{bmatrix} H \\ e-1 \end{bmatrix}$, the line $B \in L_2$ is incident to the *e*-dimensional subspaces of *H* containing *B*.

Note that in the semilinear incidence structure $\pi_3 = \left(\begin{bmatrix} V \\ e \end{bmatrix}, L_1 \cup L_2; \# \right)$

1. there are $\begin{bmatrix} 2e+1 \\ e \end{bmatrix}$ points and $\begin{bmatrix} 2e+1 \\ e \end{bmatrix}$ lines; (check: $\begin{bmatrix} 2e+1 \\ e+1 \end{bmatrix} - \begin{bmatrix} 2e \\ e+1 \end{bmatrix} + \begin{bmatrix} 2e \\ e-1 \end{bmatrix} = \begin{bmatrix} 2e+1 \\ e \end{bmatrix}$?)
2. each line is incident to $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$ points, and each point is incident to $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$ lines;
3. two points are collinear if and only if they meet in an $(e-1)$-dimensional subspace.
4. its point graph is the Grassmann graph $J_q(2e+1, e)$ with $NN^t - \begin{bmatrix} e+1 \\ 1 \end{bmatrix} I$ as its adjacency matrix where *N* is the point-line incidence matrix of π_3.
6. its **the line graph** (or called the block graph), with $N^t N - \begin{bmatrix} e+1 \\ 1 \end{bmatrix} I$ as its adjacency matrix, has the same spectrum as that of the Grassmann graph $J_q(2e+1, e)$, because NN^t and $N^t N$ have the same nonzero eigenvalues.

Theorem [DK05]
Let *G* be the graph with vertex set
all $(e+1)$-dimensional subspaces of *V* not contained in *H*, together with
the $(e-1)$-dimensional subspaces of *H*, where

1. two vertices of the 1st kind are adjacent if they intersect in an *e*-dimensional subspace;
2. a vertex of the 1st kind is adjacent to a vertex of the 2nd kind if the first contains the second;
3. two vertices of the 2nd kind are adjacent if they intersect in an $(e-2)$ dimensional subspaces.

Then *G* is distance-regular with the same parameters as that of the Grassmann graph $J_q(2e+1, e)$, not vertex-transitive and hence not isomorphic to $J_q(2e+1, e)$.

1. *G* is distance - regular
   a. a graph *cospectral* with a *distance-regular graph* Γ with diameter *e* is itself distance regular if for every vertex the number of vertices at distance *e* is the same as in Γ;
   b. since k_e in *G* is indeed the same as in the Grassmann graph, and hence *G* is distance-regular; c. the parameters of a distance-regular graph follows from its spectrum, *G* has the same parameters as $J_q(2e+1, e)$.
2. *G* is not vertex-transitive:

Question: Do similar arguments work for $J_q(2e, e)$? $H_q(d+2, d)$, $H_q(d+1, d)$ or $H_q(d, d)$?

3-3 the bilinear forms graph $H_q(n,d)$

Let $V = F_q^{n+d}$, and $W \in \begin{bmatrix} V \\ n \end{bmatrix}$; moreover let

$$\Im_d(V,W) = \{A \mid A \in \begin{bmatrix} V \\ d \end{bmatrix}, \text{with } A \cap W = \{0\}\},$$

$$\Im_{d-1}(V,W) = \{A \mid A \in \begin{bmatrix} V \\ d-1 \end{bmatrix}, \text{with } A \cap W = \{0\}\}$$

consider the semilinear incidence structure $\pi_1 = \left(\Im_d(V,W), \Im_{d-1}(V,W); \supseteq\right)$, called attenuated spaces.

Note that

1. there are q^{nd} points and $\begin{bmatrix} d \\ 1 \end{bmatrix} q^{n(d-1)}$ lines in the semilinear incidence structure;
2. each line is incident to q^n points, and each point is incident to $\begin{bmatrix} d \\ d-1 \end{bmatrix} = \begin{bmatrix} d \\ 1 \end{bmatrix}$ lines;
3. two points are collinear if and only if they meet in an $(d-1)$-dimensional subspace.
4. the point graph of π_1 is the bilinear forms graph $H_q(n,d)$ with $NN^t - \begin{bmatrix} d \\ d-1 \end{bmatrix} I$ as its adjacency graph where N is the point-line incidence matrix of π_1.

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3-4 candidates for distance-regular mate of the bilinear forms graphs $H_q(n,d)$
(all need to be further checked)

3-4 a: Searching for cospectral mates of the the bilinear forms graphs

3-4 b: Searching for distance regular mates of the the bilinear forms graphs

The case $(n,d) = (d+1,d)$

Let $V = F_q^{2d+1}$, $W \in \begin{bmatrix} V \\ d+1 \end{bmatrix}$ and $H \in \begin{bmatrix} V \\ 2d \end{bmatrix}$ with $W \subseteq H$; moreover let

$$\Im_i(V,W) = \{A \mid A \in \begin{bmatrix} V \\ i \end{bmatrix}, \text{and } A \cap W = \{0\}\}, \text{ and}$$

$$\Re = \{B \mid B \in \begin{bmatrix} V \\ d+1 \end{bmatrix}, B \not\subset H\}$$ (the condition $B \not\subset H$ needs double checks!)

Consider the semilinear incidence structure

$$\pi_2 = \left(\Im_d(V,W), \Re \cup \Im_{d-1}(H,W); \#\right).$$

(The condition $B \not\subset H$ considered in $\Re$ needs double checks, one check point is that $|\Re| + |\Im_{d-1}(H,W)| = q^{nd}$.)

Note that

1. there are q^{nd} points and … lines in π_2;
2. each line is incident to q^n points, and each point is incident to $\begin{bmatrix} d \\ d-1 \end{bmatrix} = \begin{bmatrix} d \\ 1 \end{bmatrix}$ lines in π_2;
3. two points are collinear in π_2 if and only if they meet in an $(d-1)$-dimensional subspace.
4. the point graph of π_2 is the bilinear forms graph $H_q(n,d)$ with $NN^t - \begin{bmatrix} d \\ d-1 \end{bmatrix} I$ as its adjacency matrix where *N* is the point-line incidence matrix of π_2.
5. the line graph *G* of π_2 is defined on the vertex set consisting of all $(d+1)$-dimensional subspaces of *V* not contained in *H*, together with the $(d-1)$-dimensional subspaces of *H* meeting trivially with *W*, where
   1. two vertices of the 1st kind are adjacent if they intersect in an *e*-dimensional subspace;
   2. a vertex of the 1st kind is adjacent to a vertex of the 2nd kind if the first contains the 2nd;
   3. two vertices of the 2nd kind are adjacent if they intersect in an $(d-2)$ dimensional subspaces.

Claim: the line graph *G* is distance-regular with the same parameters as that of the bilinear form graph $H_q(d+1,d)$, not vertex-transitive and hence not isomorphic to $H_q(d+1,d)$.

3.5 A non-distance-regular *cospectral mate* of $J(n,d)$

E.R. van Dam, W.H. Haemers, J. H. Koolen, E. Spence,
Journal of Combinatorial Theory Series A 113 (2006) 1805-1820

A constructions of cospectral mates in terms of *switching tool* by Godsil and McKay:

Theorem [*Godsil switching* G82]Let *G* be a graph and let $\prod = \{D, C_1, C_2, ..., C_m\}$ be a partition of the vertex set of *G*. Suppose that

1. $\{C_1, C_2, ..., C_m\}$ is a *regular partition* of $V(G) - D$;
2. every vertex $x \in D$ and every $i \in \{1,2,...,m\}$, *x* has either 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in C_i.

Make a new graph *H* as follows:

for each $x \in D$ and $i \in \{1,2,...,m\}$ such that *x* has $\frac{1}{2}|C_i|$ neighbors in C_i, delete the corresponding $\frac{1}{2}|C_i|$ edges and join *x* instead to the $\frac{1}{2}|C_i|$ other vertices in $|C_i|$.

Then *G* and *H* have the same spectrum.

Theorem The Johnson graph $J(n,d)$ with $n \ge d+3 \ge 4$ has a non-distance regular cospetral mate.

Question: Does Godsil switching preserve the walk-regularity of graphs?

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無衍生研發成果推廣資料

98 年度專題研究計畫研究成果彙整表

計畫主持人：黃大原			計畫編號：98-2115-M-009-013-				
計畫名稱：關於圖形的值譜及其距離正則性的研究							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
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	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
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請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

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