行政院國家科學委員會補助專題研究計畫 ■ 成果報告 □期中進度報告

The classification of Willmore spheres and tori in the three dimension sphere

計畫類別：■ 個別型計畫 □ 整合型計畫
計畫編號：NSC 96-2115-M-009-0064
執行期間：96 年 8 月 1 日至 97 年 7 月 31 日

計畫主持人： 許義容
共同主持人：
計畫參與人員：

成果報告類型(依經費核定清單規定繳交)：■精簡報告 □完整報告

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中 華 民 國 97 年 7 月 31 日

行政院國家科學委員會專題研究計畫成果報告

三維球中 Willmore 球面與環面之分類

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摘要

假設 M 為 3 維球面上之緊緻 Willmore 曲面。此報告旨在利用 Minkowski 空間之保角幾何建構出球面之對應的四次微分形與保角變換。

關鍵詞 Willmore 曲面、解析四次微分形

Abstract

In this report, we will construct the holomorpic quartic differential form and the conformal transformation of a Willmore surface in the three dimensional sphere S^3 Which were constructed by Bryant in the Minkowski space, and study surfaces with constant conformal transformation which is given in this report.

Keywords: Willmore surface, holomorpic quartic differential form

1. Introduction

Let $x: M^2 \to S^3$ be a compact immersed surface in the 3-dimensional unit sphere S^3. Let h_{ij} be the components of the second fundamental form of M^2, and $H=\sum h_{ii}$ the mean curvature. The Willmore functional of x is given by

$$W(X) = \int_{M^2} \Phi,$$

where $\phi_{ij} = h_{ij} - \frac{H}{2}\delta_{ij}$ and $\Phi = \sum(\phi_{ij})^2$.

The critical points of the Willmore functional are called Willmore surfaces, they satisfy the Euler-Lagrange equation $\Delta H + \Phi H = 0$ (see [W]). Willmore sphere was classified by Bryant, so call Bryant's sphere (see [B1] and [B2]).

Since minimal surfaces are Willlmore surfaces, it is nature that certain results about minimal surfaces also worked for Willmore surfaces. Indeed if M^2 is a compact immersed Willmore surface in the 3-dimensional unit sphere, and $0 \le \Phi \le 2 + \frac{\mathrm{H}^2}{4}$, then M^2 is either totally umbilical or the Clifford torus (see [CH1]). On the other hand, if $2 + c\mathrm{H}^2 \ge \Phi \ge 0$, for some $\frac{1}{2} > c$, on M^2, then M^2 is either a Bryant's sphere with nonnegative Gaussian curvature or the Clifford torus. For the case of pinching constant $c = \frac{1}{2}$, we know that M^2 is either a Bryant's sphere with nonnegative Gaussian curvature or a flat Willmore torus.

In section 2 of this report we will find the quartic differential form of x in S^3 that corresponds to the holomorpic quartic differential of x in the Minkowski space constructed by Bryant. In section 3 we relate conformal differential geometry of the Minkowski space L^5 with geometry of the three dimensional sphere S^3, and construct the Willmore dual of x in S^3. In section 4 we study the surfaces with constant conformal transformation which is given in section 3

2. The holomorpic quartic differential

Let $x: M^2 \to S^3$ be an immersed surface. In this section we will find the quartic differential of x in S^3 that corresponds to the holomorpic quartic differential of x in the Minkowski space constructed by Bryant ([B1]). Palme presented it in the Euclidean space ([P]).

Let (x_1, x_2) be an isothermal coordinate of M^2, $e_j = e^u \frac{\partial}{\partial x_j}, j = 1,2,$ be an orthonormal frame field, and $\theta_j = e^{-u} dx_j, j = 1,2,$ the corresponding dual coframe. Then the Codazzi's equation is given by

$$e^u \frac{\partial H}{\partial x_1} = \frac{\partial}{\partial x_1}(e^{2u}(h_{11} - h_{22})) + 2\frac{\partial}{\partial x_2}(e^{2u} h_{12}),$$

$$e^u \frac{\partial H}{\partial x_2} = -\frac{\partial}{\partial x_2}(e^{2u}(h_{11} - h_{22})) + 2\frac{\partial}{\partial x_1}(e^{2u} h_{12}),$$

and the Gauss equation is given by

$$u_{z\bar{z}} = \frac{1}{16}(e^{-2u}|\varphi|^2 - e^{2u}(4 + H^2)),$$

where $\varphi = e^{2u}\phi$, $\phi = (h_{11} - h_{22}) - 2ih_{12}$ is the Hopf's dfferential. On the set of M^2 – umbilic locus, let

$$q = \frac{1}{4}\varphi^2(\frac{1}{4}H^2 + \Delta \log \varphi)$$
$$= \frac{1}{16}\varphi^2(H^2 + 4) + e^{-2u}\varphi\varphi_{z\bar{z}} - e^{-2u}\varphi_z\varphi_{\bar{z}}.$$

It follows from the equations of Codazzi and Gauss that

$$q_{\bar{z}} = e^{4u}(\frac{1}{4}\varphi(\Delta H + \Phi H)_z + (\varphi u_z - \frac{1}{4}\varphi_z)(\Delta H + \Phi H)).$$

Thus if $x: M^2 \to S^3$ is a Willmore surface then q is holomophic, and hence

$$\vartheta = q dz^4 = \frac{1}{4}((1 + \frac{H^2}{4} - 2K)\phi^2 + \phi\Delta\phi - \phi_k^2)(\omega_1 + i\omega_2)^4$$

is a holomorpic quartic differential. When ϑ vanishes identically, combining with the Willmore equation, the second derivatives of the mean curvature can be presented in terms of lower order derivatives of the second fundamental form.

3. The Willmore dual surfaces

Let $x: M^2 \to S^3$ be a Willmore surface. In this section we relate conformal differential geometry of the Minkowski space L^5 with geometry of the three dimensional sphere S^3, and construct the Willmore dual of x in S^3.

Here we use the notions of the moving frame used by Bryant ([B1]) and Chern et al ([CDK]) respectively. We choose a local orthonormal frame field E_1, E_2, and E_3 in S^3 so that when restricted to $x(M^2)$ the vectors E_1, E_2 are tangent to $x(M^2)$, and E_3

is a local field in the normal bundle of $x(M^2)$. Let $\theta_1, \theta_2, \theta_3$ be its dual coframes in S^3.

Let L^5 denote the Minkowski space, R^5 together with the standard Minkowski inner product, orientation and time orientation, and ℓ^+ denote the space of positive null vectors in L^5. Let

$$e_0 = (1, x), e_1 = (0, E_1), e_2 = (0, E_2), e_3 = (0, E_3), e_4 = \frac{1}{2}(1, -x).$$

Then e_0, e_1, e_2, e_3, e_4 is a frame field in the first order frame bundle of x, a positively oriented basis, $x = [e_0]$, $e_4 \in \ell^+$, $\langle e_a, e_b \rangle = B_b^a$, where $\langle\, , \rangle$ is the Minkowski inner product, and

$$B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

It follows that there exist 1-forms ω_b^a satisfying

$$de_a = e_b \omega_a^b, \quad d\omega_b^a = -\omega_c^a \Lambda \omega_b^c.$$

Since $\omega_0^3 = 0$, Cartan's Lemma implies that there are h_{ij} such that $\omega_i^3 = h_{ij}\omega_0^j$, $h_{ij} = h_{ji}$.

Compare the structure equations of the three dimensional sphere S^3 with that of the Minkowski space L^5, we get

$$\begin{bmatrix} \omega_0^0 & \cdots & \omega_4^0 \\ \vdots & \vdots & \vdots \\ \omega_0^4 & \cdots & \omega_4^4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}\theta_1 & -\frac{1}{2}\theta_2 & 0 & 0 \\ \theta_1 & 0 & \theta_{21} & -\sum h_{1j}\theta_j & -\frac{1}{2}\theta_1 \\ \theta_2 & \theta_{12} & 0 & -\sum h_{2j}\theta_j & -\frac{1}{2}\theta_2 \\ 0 & \sum h_{1j}\theta_j & \sum h_{2j}\theta_j & 0 & \\ 0 & \theta_1 & \theta_2 & 0 & 0 \end{bmatrix}..$$

We note that the second fundamental forms of M^2 in S^3 coincide with these h_{ij} in L^5.

Since on the first order frame bundle, $\omega_i^3 = h_{ij}\omega_0^j$, we may define the covariant derivatives of h_{ij} by

$$dh_{ij} + h_{ij}\omega_0^0 - h_{kij}\omega_i^k - h_{ik}\omega_j^k + \delta_{ij}\omega_3^0 = h_{ijk}\omega_0^k,$$

$h_{ijk} = h_{ikj}$. It follows that $dH = h_j \omega_0^j$, where $h_j = h_{iij} = H_j$.

The first order frame bundle of x is a right principal G-bundle with fibers of the form

$$g = \begin{bmatrix} \frac{1}{r} & p^t C & \frac{r}{2}|p|^2 \\ 0 & C & rp \\ 0 & 0 & r \end{bmatrix},$$

where

$$r > 0, \mathrm{p} = \begin{bmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \\ \mathrm{c}_3 \end{bmatrix}, C = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, c^2 + s^2 = 1.$$

We notice that if

$$g_i = \begin{bmatrix} \frac{1}{r_i} & p_i^t C_i & \frac{r_i}{2}|p_i|^2 \\ 0 & C_i & r_i p_i \\ 0 & 0 & r_i \end{bmatrix}, \; i = 1,2,$$

then $$g_1 g_2 = \begin{bmatrix} \frac{1}{r} & p^t C & \frac{r}{2}|p|^2 \\ 0 & C & rp \\ 0 & 0 & r \end{bmatrix}, \qquad (*)$$

where

$$r = r_1 r_2, \mathrm{A} = \mathrm{A}_1 \mathrm{A}_2, \mathrm{p} = \frac{1}{r_1} C_1 p_2 + p_1.$$

To construct the conformal transformation, we follows the procedures of Bryant ([B1]),

(1) $r = 1,\ p = \begin{bmatrix} 0 \\ 0 \\ \frac{H}{2} \end{bmatrix}, A = I$.

(2) $r = 1,\ p = 0, A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$.

(3) $r = \dfrac{2}{\sqrt{(h_{11} - h_{22})^2 + 4h_{12}^2}},\ p = 0, A = I$

(4) $r = 1,\ p = \begin{bmatrix} \frac{H_1}{2} \\ -\frac{H_2}{2} \\ 0 \end{bmatrix}, A = I$.

Where

$$c = \frac{1}{\sqrt{2}}\sqrt{1+\frac{h_{11}-h_{22}}{\sqrt{(h_{11}-h_{22})^2+4h_{12}^2}}},$$

$$s = \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{h_{11}-h_{22}}{\sqrt{(h_{11}-h_{22})^2+4h_{12}^2}}}\ (\pm \text{ depends on the sign of } h_{12}).$$

Applying (*) to the procedures, we find

$$\tilde{e}_4 = r(\frac{1}{2}p^t p e_0 + (e_1,e_2,e_3)p + e_4)$$
$$= r(\frac{1}{2}(p^t p+1), \frac{1}{2}(p^t p-1)x + c_1E_1 + c_2E_2 + c_3E_3).$$

Thus the conformal transformation $\hat{x}$ in the sense of S^3 is given by

$$\hat{x} = \frac{2}{\frac{|\nabla H|^2}{4r^2}+\frac{H^2}{4}+1}(\frac{1}{2}(\frac{|\nabla H|^2}{4r^2}+\frac{H^2}{4}-1)x$$
$$-\frac{1}{r}(\frac{H_1}{2}(cE_1+sE_2)-\frac{H_2}{2}(sE_1-cE_2))+\frac{H}{2}E_3),$$

where

$$r = \frac{2}{\sqrt{(h_{11}-h_{22})^2+4h_{12}^2}},$$

$$c = \frac{1}{\sqrt{2}}\sqrt{1+\frac{h_{11}-h_{22}}{\sqrt{(h_{11}-h_{22})^2+4h_{12}^2}}}, s = \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{h_{11}-h_{22}}{\sqrt{(h_{11}-h_{22})^2+4h_{12}^2}}}.$$

The conformal transformation $\hat{x}$ is constant if M^2 is not totally umbilic and the holomorpic quartic differential ϑ vanishes on M^2. In particular that if M^2 is a topological sphere, then ϑ vanishes identically. Thus M^2 is either totally umbilic or $\hat{x}$ is constant.

4. Constant conformal transformation

In this section we characterize the surface with constant conformal transformation which is constructed in section 3. Suppose that M^2 is not totally umbilic. Let $\hat{x}$ be the constant unit vector a, and $f = |\phi|^2|\nabla H|^2 + 4H^2$, then

$$(x,a) = 1-\frac{32}{f+16}, (E_3,a) = \frac{16H}{f+16}.$$

From the structure equations, then we have

$$(E_j,a) = \frac{32f_j}{(f+16)^2} \quad \text{for j} = 1,2, \quad \text{and}$$

$$\mathrm{H}(E_3,a) - 2(x,a) = \frac{32}{(f+16)^2}\Delta f - \frac{64}{(f+16)^3}|\nabla f|^2.$$

It follows that

$$1-(x,a)^2-(E_3,a)^2 = \frac{1024}{(f+16)^4}|\nabla f|^2 \quad \text{and}$$

$$8(f+16)^2(\mathrm{H}^2+4)-(f+16)^3 = 16(f+16)\Delta f - 32|\nabla f|^2.$$

Since

$$|\nabla f|^2 = \frac{1}{16}((f+16)^3 - 4(f+16)^2(4+H^2)),$$

we have $\Delta f = \frac{1}{16}(f+16)^2 \geq 0.$

Since M^2 is compact, f is constant, and x is a hypersphere which is totally umbilic, a contradiction. Thus M^2 must be totally umbilic.

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