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Kaluza–Klein higher-derivative induced gravity

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Abstract
The existence and stability analysis of an inflationary solution in a $(D+4)$-dimensional anisotropic-induced gravity is presented in this paper. Nontrivial conditions in the field equations are shown to be compatible with a cosmological model in which the 4-dimension external space evolves inflationary while the D-dimension internal one is static. In particular, only two additional constraints on the coupling constants are derived from the abundant field equations and perturbation equations. In addition, a compact formula for the non-redundant $(4+D)$-dimensional Friedmann equation is also derived for convenience. Possible implications are also discussed in this paper.

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1. Introduction

Higher-order corrections to the Einstein gravity [1, 2] can be derived from the quantum gravity and the string theory [3]. Applications to the study of the inflationary universe [4, 5] have been a focus of research interest. In particular, higher-derivative terms also arise as an effective theory for the quantum corrections of matter fields in curved space [3].

In addition, the Kaluza–Klein theory [6, 7] is also important in the study of the evolution of the early universe. Indeed, the dimensional-reduction process could affect the evolution of the inflationary universe significantly. Recently, the brane universe scenario has also become a focus of interest [8].

Induced-gravity models have been a focus of study for many reasons. Weyl was the first to propose that the scale invariant theory is a candidate for the unified theory of gauge field and gravitational field. In addition, Dirac's large number theory also asserts that all dimensionful parameters in the physical theory are in fact the dynamical functions of time. As a result, various interesting models have attracted researchers' interest for a long time.

Therefore, we intend to study an $N(=4+D)$-dimensional Kaluza–Klein higher-derivative induced-gravity model with all dimensionful coupling constants replaced by appropriate scale-dependent fields. We will show that a constant internal-space solution will lead to a nontrivial constraint to the field equations. In addition, there is also another constraint derived from the assumption of the constant internal scale-dependent scalar field ψ.

In fact, we will show in this paper that there are three constraints to be imposed on the choice of three different coupling constants coupled to the higher-curvature terms. Note that, in the 4-dimensional space, $E = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the integrand of the Gauss–Bonnet term [9]. In addition, the Weyl tensor [9] connects these fourth-order curvature terms in the 4-dimensional space. Hence, we only need to deal with an R^2 term in the 4-dimensional de Sitter space. These constraints do not hold in higher-dimensional spaces. Therefore, we must deal with all three different fourth-order terms in an N-dimension. Hence, the equations of motion for the higher-derivative Kaluza–Klein induced-gravity theory [10–13] are much more complicated than the 4-dimensional higher-derivative gravity. We will show, however, that the aforementioned abundant constraints are not only consistent with the already abundant stability constraints of the system but also lead to an interesting result, i.e. only the R^2 coupling term is consistent with the inflationary solution. Possible implications will also be discussed in this paper.

In order to reduce the complication of the derivation of field equations, we will also derive a simple expression for the Friedmann equation [14] in a $(4 + D)$-dimensional space in this paper. The redundancy of the associated field equations due to the Bianchi identity will also be analyzed. We will show that the quadratic terms do not affect the Friedmann equation in a constant flat internal space scale factor $d(t)$ and a flat de Sitter 4 space.

This paper will be organized as follows. In section 2, we will introduce the Kaluza–Klein higher-derivative induced-gravity model with all dimensionful parameters replaced by appropriate dynamical fields. The constraint derived from the constant ψ field will be obtained in this section. In section 3, we will derive a model-independent expression for the N-dimensional generalized Friedmann equation in the higher-dimensional higher-derivative theory. These formulae are derived from a reduced 1-dimensional theory. In section 4, we will discuss the stability conditions for an inflationary solution in the induced-gravity theory. Finally, the conclusions are presented in section 5.

2. Kaluza–Klein higher-derivative induced gravity

The $(4 + D)$-dimensional Kaluza–Klein higher-derivative induced-gravity theory can be described by the following Lagrangian:

$$L_N = \left(L - \tfrac{1}{2}\partial_A\psi\partial^A\psi\right)\psi^D, \tag{1}$$

with the 4-dimensional relevant higher-derivative induced-gravity Lagrangian L given by

$$L = L_1 + L_2 + L_\phi = -\frac{\epsilon}{2}\phi^2\mathbf{R} - c_1\left(\mathbf{R}^{AB}{}_{CE}\right)^2 - c_2\left(\mathbf{R}^A{}_B\right)^2 - c_3\mathbf{R}^2 - \frac{1}{2}\partial_A\phi\partial^A\phi - V(\phi). \tag{2}$$

Here, $L_1 \equiv -\epsilon\phi^2\mathbf{R}/2$, $L_2 \equiv -c_1\left(\mathbf{R}^{AB}{}_{CE}\right)^2 - c_2\left(\mathbf{R}^A{}_B\right)^2 - c_3\mathbf{R}^2$ and $L_\phi \equiv -(\partial\phi)^2/2 - V(\phi)$ denote, respectively, the induced Einstein–Hilbert Lagrangian, the higher-derivative terms and the scalar field Lagrangian.

Note that, throughout this paper, the curvature tensor $\mathbf{R}^E_{ABC}(\mathbf{g}_{AB})$ will be defined by the following equation:

$$[D_A, D_B]V_C = \mathbf{R}^E{}_{CBA}V_E. \tag{3}$$

Accordingly, $\mathbf{R}^E_{ABC} = -\partial_C\Gamma^E_{AB} - \Gamma^F_{AB}\Gamma^E_{CF} - (B \leftrightarrow C)$. Here, Γ^C_{AB} denotes the Christoffel symbol (or spin connection of the covariant derivative, i.e. $D_AV_B \equiv \partial_AV_B - \Gamma^C_{AB}V_C$). To be more specific, $\Gamma^C_{AB} = \frac{1}{2}\mathbf{g}^{CE}(\partial_A \mathbf{g}_{EB} + \partial_B \mathbf{g}_{EA} - \partial_E \mathbf{g}_{AB})$. Moreover, the Ricci tensor $\mathbf{R}_{AB}$ is defined as

$$\mathbf{R}_{AB} = \mathbf{R}^C{}_{ABC} \tag{4}$$

and the scalar curvature $\mathbf{R}$ is defined as the trace of the Ricci tensor $\mathbf{R} \equiv \mathbf{g}^{AB}\mathbf{R}_{AB}$. Also, the Einstein tensor is defined as $\mathbf{G}_{AB} \equiv \frac{1}{2}\mathbf{g}_{AB}\mathbf{R} - \mathbf{R}_{AB}$.

Note that we will use bold-faced notation (e.g. $\mathbf{R}$) to denote field variables in $N(= 4+D)$-dimensional space. In addition, normal notation will denote field variables evaluated in the 4- or D-dimensional spaces as the dimensional-reduction process $M^N \to M^4 \times M^D$ takes place. Here, M^4 is the 4-dimensional Friedmann–Robertson–Walker (FRW) space and M^D is the internal space. We will assume that M^D is the D-dimensional FRW space for simplicity. Note that the metric of the $(4 + D)$-dimensional FRW anisotropic space M^N is given by

$$\mathrm{d}s^2 \equiv \mathbf{g}_{AB}\,\mathrm{d}Z^A\,\mathrm{d}Z^B \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + f_{mn}\,\mathrm{d}z^m\,\mathrm{d}z^n \tag{5}$$

$$= -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-k_1 r^2} + r^2\,\mathrm{d}^3\Omega\right) + \mathrm{d}^2(t)\left(\frac{\mathrm{d}z^2}{1-k_2 z^2} + z^2\,\mathrm{d}^D\Omega\right). \tag{6}$$

Here, $\mathrm{d}^p\Omega$ is the solid angle, $\mathrm{d}^p\Omega = \mathrm{d}\theta_1{}^2 + \sin^2\theta_1\,\mathrm{d}\theta_2{}^2 + \cdots + \sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{p-3}\,\mathrm{d}\theta^2_{p-2}$, and $k_1, k_2 = 0, \pm 1$ stand for a flat, closed or open universe, respectively. Note that we will also write $g_{ij} = a^2 h_{ij}$ and $g_{mn} = \mathrm{d}^2 h_{mn}$ for convenience. Note that θ_i is the phase angle of the D-dimensional spherical coordinate. For example, we have

$$z_1 = z\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}. \tag{7}$$

Note that we will write the N-dimensional spacetime coordinate as $Z^A \to (x^\mu, z^m)$ with A (=0, 1, ..., $N-1$), μ (=0, 1, 2, 3) and m (=1, 2, ..., D) denoting the N-, 4- and D-dimensional spacetime indices, respectively. Specifically, capital Roman letters $A, B, C, \ldots$ will denote N-dimensional indices. In addition, Greek letters will denote 4-indices while the second-half of the Roman letters will denote D-dimensional spacetime indices. Here we have assumed that the internal space (z) is independent of the external space (x). The only t-dependence of the internal space is through the scale factor $d(t)$.

Induced gravity proposes that all dimensionful parameters are dynamical variables. Therefore, all coupling constants in this model, ϵ and c_i, are dimensionless. Indeed, the action

$$\int \mathrm{d}^4x\,\mathrm{d}^D z\sqrt{g}L_N \tag{8}$$

is invariant under the global scale transformation: $g'_{AB} = \Lambda^{-2}g_{AB}$, $\phi' = \Lambda\phi$, $\psi' = \Lambda\psi$ in the absence of the potential V unless $V(\phi) \sim \phi^4$. Here, Λ = constant denotes the global scale transformation parameter. For a practical application, one needs to introduce a symmetry-breaking potential so that a physical scale can be picked up dynamically. As a result, a physical inflationary solution can be managed. In addition, it is easy to observe from the scale transformation that ψ is introduced to compensate the transformation properties of the internal D-space. Indeed, $\psi^D\,\mathrm{d}^D z$ is made dimensionless by construction.

The variational equation of the ψ field gives

$$D\psi^{D-1}\big(L - \tfrac{1}{2}\partial_A\psi\partial^A\psi\big) + D_A(\psi^D\partial^A\psi) = 0. \tag{9}$$

After the dimensional-reduction process takes place, the ψ field is expected to be a function of the internal space coordinate z only. Consequently, this internal space scalar field will not affect the 4-dimensional physical universe thereafter. For simplicity, one will assume that a constant solution $\psi = \psi_0$ = constant is adopted so that the constant ψ field can be absorbed into the internal coordinate z by a proper re-scaling. In effect, the ψ = constant solution introduces a physical scale of the internal space. Note that the ψ = constant is not only a solution to equation (9), but also a solution consistent with the static internal-space solution. Indeed, the scalar field ψ is responsible for the internal-space dimension such that $\mathrm{d}z^D \psi^D$ remains dimensionless. As a comparison, the scalar field ϕ is introduced to take care of the dynamical dimension of all 4-space coupling constants. As a result, the ψ field will decouple from the 4-space completely after the dimensional-reduction process is completed. The remaining impact of this solution is an additional constraint $L = 0$ to be made compatible with the dimensionally-reduced 4-space of interest.

Therefore, we will focus on the $(4 + D)$-dimensional model given by the effective Lagrangian density L described by equation (2). Consequently, we will study model (2) in the presence of the constraint $L = 0$ to be imposed later. In fact, we will show that the constraint $L = 0$ is consistent with the inflationary de Sitter solution bounded by the abundant and interesting constraints to be imposed on the coupling constants c_i. We will also study the stability of this inflationary solution and discuss interesting implications of this Kaluza–Klein induced-gravity model.

The Euler–Lagrange equation of the system can be obtained from the variational equation of the $(4 + D)$-dimensional metric $\mathbf{g}_{AB}$. We will write it as

$$\mathbf{J}_{AB} = \mathbf{G}_{AB} - \mathbf{T}_{AB} = 0. \tag{10}$$

The derivation is very complicated and delicate. Fortunately, if we are only interested in the $(4 + D)$-dimensional FRW space, the dynamical variables reduce to a set of 1-dimensional variables $a(t)$, $d(t)$ and $\phi(t)$.

Effectively, we can write the Lagrangian $L(\mathbf{g}_{AB}(a(t), d(t)), \phi(t)) \rightarrow L_r(a(t), d(t), \phi(t))$. Hopefully, the final expression of the Euler–Lagrange equations can be reproduced from the variation of L_r with respect to the dynamical variables $a(t)$, $d(t)$ and $\phi(t)$.

If this is indeed applicable, the field equations can be derived more easily without involving the complicated tensor algebra. In particular, this will be easier to access when complicated interactions are introduced. We will show that there is a little problem with this approach for the non-redundant Friedmann equation. Fortunately, the non-redundant Friedmann equation can be reconstructed by restoring the $\mathbf{g}_{tt}$ variable. In a moment, we will derive a set of modified formulae for this reduced Lagrangian L_r. We will drop the subscript 'r' in L_r for simplicity and economics of notation.

Note also that the Bianchi identity $D_M G^{MN} = 0$ and the energy–momentum conservation law $D_M T^{MN} = 0$ implies that $D_M J^{MN} = 0$. In addition, the tt-component of this Bianchi identity can be brought to the following form:

$$(\partial_t + 3H + DI)J_{tt} + 3a^2 H J_3 + D\, d^2 I J_D = 0, \tag{11}$$

with $J_{ij} = J_3 h_{ij}$. Since we only need two field equations for the 1-dimensional system of $a(t)$ and $d(t)$. Therefore, one of the three field equations $J_{tt} = 0$, $J_3 = 0$ and $J_D = 0$ is presumably redundant. They are, however, not equally redundant. Let us first assume that $H \neq 0$, $I \equiv \dot{d}/d \neq 0$ for simplicity. Indeed, the Bianchi identity implies that the first equation $J_{tt} = 0$ is truly non-redundant: (i) $J_{tt} = 0$ implies that $3a^2 H J_3 + D\, d^2 I J_D = 0$ for all times. Hence, the constraint $J_3 = 0$ (or $J_D = 0$) implies the vanishing of the other equation $J_D = 0$ (or $J_3 = 0$). (ii) $J_3 = J_D = 0$ implies that $(\mathrm{d}/\mathrm{d}t + 3H + DI)J_{tt} = 0$. Hence, we have J_{tt} = constant $\exp[-a^3 d^D]$ which does not go to zero unless $a^3 d^D \rightarrow \infty$. Cases (i) and (ii) mean that the Friedmann equation $J_{tt} = 0$ is truly non-redundant while $J_3 = 0$ and $J_D = 0$ are equally redundant. Therefore, we can ignore either one of the equations without losing any information. This is, however, not true under the condition where $d =$ constant, or $I \equiv \dot{d}/d = 0$. As indicated in equation (11), we have instead

$$(\mathrm{d}/\mathrm{d}t + 3H)\bar{J}_{tt} = 3a^2 H \bar{J}_3 \tag{12}$$

under this condition. Here we have written $\bar{J}_{tt} \equiv J_{tt}|_{I=0}$ and similarly $\bar{J}_3 \equiv J_3|_{I=0}$. Therefore, the reduced Bianchi identity (12) only tells us that $\bar{J}_3 = 0$ is redundant as compared to the non-redundant Friedmann equation $\bar{J}_{tt} = 0$ under the constraint $I = 0$. In fact, the d-equation or the a-equation has to be retained for a consistent check in order to make sure the system does accommodate a constant d solution. This point is often overlooked and should be checked carefully in the analysis of the Kaluza–Klein theory under a constant internal-space solution.

Bianchi identity implies that the $N(N+1)/2$ Einstein equations $G_{AB} = T_{AB}$ are not all independent, but related by N constraint equations $D_A G^{AB} = 0$. By looking at the conservation law of the energy–momentum tensor $D_A T^{AB} = 0$, one may interpret that the conservation of the energy–momentum tensor implies the vanishing of $D_A G^{AB}$. The Bianchi identity has, however, a more intrinsic geometric meaning. It is in fact a geometric conservation law. Note that the Bianchi identity $D_A G^{AB} = 0$ has a simple geometric interpretation, namely, the boundary of a boundary is zero [15]. It implies that the energy–momentum is automatically conserved for a system coupled consistently to the geometry of spacetime. In practice, the Bianchi identity is helpful in providing an easier approach for studying the Einstein equation. For example, we can focus on, with the help of the Bianchi identity, the independent components of the field equations. Indeed, as shown above, the Bianchi identity in the FRW spacetime implies that the Friedmann equation, the a-equation and the d-equation are related by the differential equation (11). This implies that the Friedmann equation $G_{tt} = T_{tt}$ has a less differential order than the other components of the Einstein equation. As a result, the Friedmann equation may serve as a useful tool in solving the differential equations.

3. Generalized Friedmann equation in 1-dimensional formulation

In order to derive the non-redundant Friedmann equation from the reduced Lagrangian, we must restore the $\mathbf{g}_{tt}$-dependence of the reduced Lagrangian L. Indeed, $\mathbf{J}^{tt}$ comes from the variation of L with respect to $\mathbf{g}_{tt}$, $\delta L/\delta \mathbf{g}_{tt} \sim \epsilon\phi^2 J^{tt}/2$. Hence, the most convenient way to restore the $\mathbf{g}_{tt}$-dependence of the reduced Lagrangian L is to introduce the lapse function $b(t)$ connecting the $\mathbf{g}_{tt}$ metric component

$$\mathrm{d}s^2 \equiv \mathbf{g}_{AB}\,\mathrm{d}Z^A\,\mathrm{d}Z^B \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + f_{mn}\,\mathrm{d}z^m\,\mathrm{d}z^n \tag{13}$$

$$= -b(t)^2\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-k_1 r^2} + r^2\,\mathrm{d}^3\Omega\right) + \mathrm{d}^2(t)\left(\frac{\mathrm{d}z^2}{1-k_2 z^2} + z^2\,\mathrm{d}^D\Omega\right). \tag{14}$$

This metric will be named as the generalized FRW (GFRW) metric for the $4+D$ anisotropic space. Once the non-redundant Friedmann equation is derived from the variation of L with respect to b, one can set $b = 1$ and reconstruct the b-independent Friedmann equation.

The non-vanishing spin connections can be listed as follows:

$$\Gamma^\gamma_{\mu\nu} = \Gamma^\gamma_{\mu\nu}, \tag{15}$$

$$\Gamma^p_{mn} = \Gamma^p_{mn}, \tag{16}$$

$$\Gamma^\gamma_{mn} = -\partial^\gamma \beta g_{mn}, \tag{17}$$

$$\Gamma^p_{\mu m} = \partial_\mu \beta \delta^p_m. \tag{18}$$

Here, $\partial_\mu \beta \equiv \partial_\mu d/d$ comes with a non-vanishing t-component. In addition, we will write $I = \partial_t \beta$ for convenience from now on. Consequently, all non-vanishing Riemannian curvature components can be listed as follows:

$$\mathbf{R}^{ti}{}_{tj} = \frac{1}{2}[H\dot{B} + 2B(\dot{H} + H^2)]\delta^i_j, \tag{19}$$

$$\mathbf{R}^{ij}{}_{kl} = \left(H^2 B + \frac{k_1}{a^2}\right)\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right), \tag{20}$$

$$\mathbf{R}^{tm}{}_{tn} = \frac{1}{2}[I\dot{B} + 2B(\dot{I} + I^2)]\delta^m_n, \tag{21}$$

$$\mathbf{R}^{im}{}_{jn} = HI\delta^m_n \delta^i_j, \tag{22}$$

$$\mathbf{R}^{mn}{}_{pq} = \left(I^2 B + \frac{k_2}{d^2}\right)\left(\delta^m_p \delta^n_q - \delta^m_q \delta^n_p\right). \tag{23}$$

As a result, deriving all Ricci tensors $\mathbf{R}^A{}_B$ and scalar curvature tensors $\mathbf{R}$ is straightforward.

Note that there is also a t-dependent factor $b(t)$ in the square-root of the metric determinant $\sqrt{\mathbf{g}} \sim ba^3 d^D$. Hence, the variational equation of b can be shown to be [14]

$$L - H\frac{\delta L}{\delta H} - I\frac{\delta L}{\delta I} + \left[H\frac{\mathrm{d}}{\mathrm{d}t} + H(3H + DI) - \dot{H}\right]\frac{\delta L}{\delta \dot{H}} + \left[I\frac{\mathrm{d}}{\mathrm{d}t} + I(3H + DI) - \dot{I}\right]\frac{\delta L}{\delta \dot{I}} = \dot{\phi}^2. \tag{24}$$

Note that the last term $\dot{\phi}^2$ comes from the kinetic term of the scalar Lagrangian. Indeed, there is a kinetic coupling term for ϕ via $-g^{tt}(\dot{\phi})^2/2 = b^{-2}(\dot{\phi})^2/2$. This will bring us an additional $\dot{\phi}^2$ to the left-hand side of equation (24). In fact, equation (24) can generalize to all fields coupled to the Einstein–Hilbert action.

In addition, the variational equations of a and d also give

$$3L - H\frac{\delta L}{\delta H} + (H^2 - \dot{H})\frac{\delta L}{\delta \dot{H}} - \left(2H + DI + \frac{\mathrm{d}}{\mathrm{d}t}\right) \times \left[-\left(4H + DI + \frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\delta L}{\delta \dot{H}} + \frac{\delta L}{\delta H}\right] - 2k_1\frac{\delta L}{\delta k_1} = 0, \tag{25}$$

$$DL - I\frac{\delta L}{\delta I} + (I^2 - \dot{I})\frac{\delta L}{\delta \dot{I}} - \left(3H + (D-1)I + \frac{\mathrm{d}}{\mathrm{d}t}\right) \times \left[-\left(3H + (D+1)I + \frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\delta L}{\delta \dot{I}} + \frac{\delta L}{\delta I}\right] - 2k_2\frac{\delta L}{\delta k_2} = 0. \tag{26}$$

Note that the above equations also hold in the presence of the scalar coupling once we assume $\phi(Z) = \phi(t)$.

We will first study a simple model with a constant d solution. This is a physical solution since the internal space seems to be small from any physical observation up until now. As shown above, the Bianchi identity indicates that the second a-equation (25) is derived implicitly by the first Friedmann equation (24) for the constant internal-space model. Therefore, we will try to solve equations (24) and (26) for a complete analysis. As a result, the Friedmann equation (24) and the $d(t)$ equation (26) become

$$\bar{L} - H\frac{\delta\bar{L}}{\delta H} + \left[H\frac{\mathrm{d}}{\mathrm{d}t} + 3H^2 - \dot{H}\right]\frac{\delta\bar{L}}{\delta\dot{H}} = \dot{\phi}^2, \tag{27}$$

$$D\bar{L} = \left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)\left[-\left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\delta\bar{L}}{\delta\dot{I}} + \frac{\delta\bar{L}}{\delta I}\right] + 2k_2\frac{\delta\bar{L}}{\delta k_2} \tag{28}$$

under the $d(t) =$ constant background solution. Here, a bar notation of a variable L denotes a variable evaluated at $I = 0$. Explicitly, $\bar{L} \equiv L|_{I=0}$. In addition, we must also check whether the solution is consistent with the constraint $\bar{L} = 0$. Our results agree with the equations shown in [21]. Details will be provided in the appendix.

4. Stability of an inflationary external space

We need to find out whether the inflationary background de Sitter solution $H = H_0$ and $I = 0$ is a possible solution to the Kaluza–Klein induced-gravity model. As a result, a stability analysis is also needed to find out whether this background solution is stable or not. Furthermore, following the conventional approach, the constant internal-space solution is assumed and should be served as a reasonable ansatz. This is because the internal-space information appears to be minor as compared to the 4-space counterpart. Otherwise, we would be able to measure the impact of the internal-space physics once the internal space scale factor $d(t)$ grows to some appreciable size. In addition, we will also focus on the $k_1 = 0$ flat 4-space condition which appears to agree with the latest observations [16].

Therefore, we will study the existence and stability problem of the inflationary universe for the induced Kaluza–Klein model described above. The effective Lagrangian $\bar{L}$ for the model L under the $I = 0$ condition can be shown to be

$$\bar{L} = 3\epsilon\phi^2[\dot{H} + 2H^2] - 12(c_1 + c_2 + 3c_3)[(\dot{H} + H^2)^2 + H^4] - 12(c_2 + 6c_3)[(\dot{H} + H^2)H^2] + \tfrac{1}{2}\dot{\phi}^2 - V(\phi). \tag{29}$$

In addition, the variations of the I-equation are

$$\frac{\delta}{\delta I}\bar{L} = 6DH\left\{\frac{\epsilon}{2}\phi_0^2 - c_2[\dot{H} + 3H^2] - 12c_3[\dot{H} + 2H^2]\right\}, \tag{30}$$

$$\frac{\delta}{\delta\dot{I}}\bar{L} = 2D\left\{\frac{\epsilon}{2}\phi_0^2 - c_2[3\dot{H} + 3H^2] - 12c_3[\dot{H} + 2H^2]\right\}. \tag{31}$$

To summarize, there are two metric equations and one ψ equation for the system

$$\dot{\phi}^2 + H\frac{\delta\bar{L}}{\delta H} = \left[H\frac{\mathrm{d}}{\mathrm{d}t} + 3H^2 - \dot{H}\right]\frac{\delta\bar{L}}{\delta\dot{H}}, \tag{32}$$

$$\left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\delta\bar{L}}{\delta I} = \left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)^2\frac{\delta\bar{L}}{\delta\dot{I}}, \tag{33}$$

$$\bar{L} = 0 \tag{34}$$

under the flat space condition and the constant internal-space solution $I = 0$. The equation $\bar{L} = 0$ denotes the constraint from the constant ψ equation. Here, $\bar{L} \equiv L|_{I=0}$. The Friedmann equation reads

$$\tfrac{1}{2}\dot{\phi}^2 + V = 3\epsilon\phi^2H^2 + 6\epsilon H\phi\dot{\phi} + 12(c_1 + c_2 + 3c_3)(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}). \tag{35}$$

At the end of this section, we will show that the perturbation equation of the constraint $\bar{L} = 0$ is consistent with the inflationary de Sitter solution we are interested in. More specifically, the linear perturbation equations $\delta\bar{L}$ can be shown to vanish automatically without generating any further constraint on the system. Therefore, there is no need to worry about the perturbation effect of this constraint. Hence, we can add or remove the effect of this constraint anytime. This result also indicates that the constant ψ solution, the origin of this constraint, is a very reasonable ansatz. Furthermore, the ϕ-equation can be shown to be

$$6\epsilon\phi(\dot{H} + 2H^2) = \ddot{\phi} + 3H\dot{\phi} + V'. \tag{36}$$

Let $\phi = \phi_0 + \delta\phi$ and $H = H_0 + \delta H$ with ϕ_0 and H_0 be some constant initial states for the inflationary de Sitter background solution. Here, $\delta\phi$ and δH denote small perturbations against the background solutions. We will try to extract the linear solutions to the perturbation effect. As a result, the leading-order Friedmann equation (35) gives

$$V_0 = 3\epsilon\phi_0^2 H_0^2. \tag{37}$$

Similarly, the leading-order scalar equation (36) gives

$$12\epsilon\phi_0 H_0^2 = V_0'. \tag{38}$$

Here, $V_0' \equiv V'(\phi_0)$. Therefore, we have the constraints combined together as

$$\phi_0 V_0' = 4V_0 = 12\epsilon\phi_0^2 H_0^2. \tag{39}$$

Note that $\phi V'$ counts the effective scale-dimension of the effective potential $V(\phi)$. For example, $\phi V' = nV$ if $V = \phi^n$ with the effective dimension n. Therefore, this condition implies that the inflationary phase exists only when the effective dimension of the effective potential is 4 at the inflationary phase $\phi = \phi_0$ and $H = H_0$. Since $V = \lambda\phi^4$ represents the scale-invariant potential in the 4-dimensional space, i.e. dim $(\lambda) = 0$ in 4-space, this condition will be denoted as the scaling condition [12]. As an example, the spontaneously symmetry-breaking potential of the form

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 + 6\epsilon H_0^2\left(\phi^2 - \phi_0^2\right) + 3\epsilon H_0^2\phi_0^2, \tag{40}$$

with an arbitrary coupling constant λ, can be shown to be a good candidate satisfying the scaling condition (39).

In addition, the first-order perturbation equation of H-equation (35) and ϕ-equation (36) can be shown to be

$$\epsilon\phi_0(\delta\dot{\phi} - H_0\delta\phi) = 4(c_1 + c_2 + 3c_3)(\delta\ddot{H} + 3H_0\delta\dot{H}) - \epsilon\phi_0^2\delta H, \tag{41}$$

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + \left(V''(\phi_0) - 12\epsilon H_0^2\right)\delta\phi = 6\epsilon\phi_0(\delta\dot{H} + 4H_0\delta H). \tag{42}$$

Writing $\delta H = \exp[hH_0t]\delta H_0$ and $\delta\phi = \exp[pH_0t]\delta\phi_0$, the above perturbation equations can be written as

$$\epsilon\phi_0\,(p-1)\,\delta\phi = 4(c_1 + c_2 + 3c_3)H_0\left(h^2 + 3h - \frac{\epsilon\phi_0^2}{4(c_1 + c_2 + 3c_3)H_0^2}\right)\delta H, \tag{43}$$

$$H_0\left(p^2 + 3p + \frac{V''(\phi_0) - 12\epsilon H_0^2}{H_0^2}\right)\delta\phi = 6\epsilon\phi_0\,(h+4)\,\delta H. \tag{44}$$

In addition to the trivial solution $\delta H = 0$ and $\delta\phi = 0$, consistent solutions also exist when $h = -4$ and $p = 1$. If $h = -4$, we will have the following constraint from the right-hand side of equation (43):

$$H_0^2 = \frac{\epsilon\phi_0^2}{16(c_1 + c_2 + 3c_3)}. \tag{45}$$

In addition, the existence of nontrivial solution $p = 1$ implies that

$$V''(\phi_0) = 12\epsilon H_0^2 - 4H_0^2 \tag{46}$$

from the left-hand side of equation (44). As a result,

$$\delta H = \exp[-4H_0 t]\delta H_0. \tag{47}$$

Equivalently, the linear perturbation gives

$$H = H_0 + \delta H_0 \exp[-4H_0 t] \tag{48}$$

as the solution to the Hubble parameter with a small deviation from the de Sitter background.

We will show in addition that the I-equation (33) and the constraint equation $\bar{L} = 0$ are both consistent with the above constraints in the presence of static internal space and de Sitter external space. Indeed, writing $D_t \equiv \partial_t + 3H$, $\delta\bar{L}/\delta I \equiv \bar{L}_I$ and $\delta\bar{L}/\delta\dot{I} \equiv \bar{L}_{\dot{I}}$, the I-equation

$$\left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)\frac{\delta\bar{L}}{\delta I} = \left(3H + \frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \frac{\delta\bar{L}}{\delta\dot{I}},$$

under the condition of flat and static internal space can be integrated to give

$$Y(H) \equiv \bar{L}_I - D_t\bar{L}_{\dot{I}} = 6D(c_2 + 4c_3)[\ddot{H} + 4H\dot{H}] + \frac{K_1}{a^3} = 0 \tag{49}$$

with a constant of integration K_1. During the inflationary phase, we can ignore the effect of the K_1 term. Therefore, the I-equation does not have a leading-order contribution. In addition, the first-order perturbation of this equation gives $\delta Y = 6D(c_2 + 4c_3)[\delta\ddot{H} + 4H_0\delta\dot{H}]$ which vanishes identically compatible with the perturbation δH given in equation (47). In fact, we can also compute the complete I-equation and find that

$$D_t Y = 6D(c_2 + 4c_3)[\dddot{H} + 4\dot{H}^2 + 7H\ddot{H} + 12H^2\dot{H}] = 0. \tag{50}$$

There is no leading contribution either. In addition, the perturbative equation takes the form

$$\begin{aligned} \delta D_t Y = D_t\delta Y &= 6D(c_2 + 4c_3)D_t[\delta\ddot{H} + 4H_0\delta\dot{H}] \\ &= 6D(c_2 + 4c_3)[\delta\dddot{H} + 7H\delta\ddot{H} + 12H^2\delta\dot{H}] = 0 \end{aligned} \tag{51}$$

that vanishes identically with δH given in equation (47).

As mentioned above, we still have to compute all possible constraints from the internal space scalar field ψ-equation $\bar{L} = 0$. It is interesting to find that the leading-order perturbation equation for $\bar{L} = 0$ gives

$$6\epsilon\phi_0^2 H_0^2 - V_0 = 3\epsilon\phi_0^2 H_0^4 = 12(2c_1 + 3c_2 + 12c_3)H_0^4 \tag{52}$$

incorporating the scaling constraint $V_0 = 3\epsilon\phi_0^2 H_0^2$. Therefore, the leading-order equation of $\bar{L} = 0$ gives another constraint

$$H_0^2 = \frac{\epsilon\phi_0^2}{4(2c_1 + 3c_2 + 12c_3)}. \tag{53}$$

In addition, the first-order perturbation equation of this ψ constraint can be shown to be

$$\left[3\epsilon\phi_0^2 - 12(2c_1 + 3c_2 + 12c_3)H_0^2\right][\delta\dot{H} + 4H_0\delta H] + \left(12\epsilon\phi_0 H_0^2 - V_0'\right)\delta\phi = 0. \tag{54}$$

Therefore, this equation is completely consistent with the whole system by observing that all coefficients in the above equation vanishes identically. Indeed, $3\epsilon\phi_0^2 - 12(2c_1 + 3c_2 + 12c_3)H_0^2 = 0$, following equation (53). In addition, $\delta\dot{H} + 4H_0\delta H = 0$ and $12\epsilon\phi_0 H_0^2 - V_0'$ are automatically satisfied.

In summary, the constraints (45) and (53) indicate that the coupling constants should obey the following constraint in order to admit an inflationary phase in the presence of a static internal space:

$$2c_1 + c_2 = 0. \tag{55}$$

As a result, the Hubble constant and the field parameters are related by

$$H_0^2 = \frac{\epsilon\phi_0^2}{8(c_2 + 6c_3)}. \tag{56}$$

Note that the above results indicate that (1) the static internal flat space solution is completely compatible with the conditions of the inflationary solution, (2) the perturbation equation of $\bar{L} = 0$ is a perfect identity consistent with all other constraints derived elsewhere. This indicates that the constant ψ solution is a very reasonable choice for the stationary state of the system. In addition, (3) the static internal space assumption is also a consistent choice for the existence and stability of the inflationary phase.

In addition, we can also consider the perturbation of ψ by setting $\psi = \psi_0 + \delta\psi$ and perturb the field equation (9). The result is $D_A\partial^A\delta\psi = 0$. Here we have also used the identity $\delta L = 0$ under the linear perturbation shown above. Assuming $\delta\psi(Z) = \delta\psi(z)$ such that the internal space z is completely decoupled from the 4D space time. As a result, the perturbation equation $D_A\partial^A\delta\psi = 0$ is consistent if $\delta\psi(z)$ is a harmonic function obeying $\partial^m\partial_m\delta\psi = 0$. Hence, $\delta\psi(z) =$ constant for a consistent perturbation. Therefore, the constant ψ is also a consistent choice of background solution.

Note also that the effect of the $\psi =$ constant implies that $L = 0$. Together with the constraint $2c_1 + c_2 = 0$ for the existence of an inflationary phase, one effectively has a scalar equation of the form

$$\frac{1}{2}\partial_A\phi\partial^A\phi + V(\phi) + \frac{\epsilon}{2}\phi^2\mathbf{R} = \frac{c_2}{2}\left(\mathbf{R}^{AB}{}_{CE}\right)^2 - c_2\left(\mathbf{R}^A{}_B\right)^2 - c_3\mathbf{R}^2 \tag{57}$$

with a purely geometric source. Although the equation $L = 0$ is not exactly a Klein–Gordon equation of the form $(\partial^2 + m^2 + R/5)\phi = \alpha R^2$ studied in [19], both theories appear to have a similar physical origin. It was shown that an effective re-normalized Lagrangian of the form $\alpha\phi R^2$ is a result of dimensional consideration. Indeed, the coupling constant α can only be made dimensionless, rendering a system free from introducing any additional arbitrary length scale, if the spacetime dimension $N = 6$ [19].

Note that the scalar fields ψ and ϕ, both with dimension 1, considered in this paper are designed to replace all dimensionful coupling constants with appropriate scalar fields. As a result, all coupling constants are assumed to be dimensionless in this approach. The only exceptions are some parameters associated with the SSB potential $V(\phi)$ designed to pick up a symmetry-breaking scale. The constraint equation $L = 0$ indicates that the scalar field ψ introduced here may have a close relation with the re-normalizability of the energy–momentum tensor for ϕ. In addition, a similar model has been studied in [20] with the Gauss–Bonnet (GB) Lagrangian coupled to a perfect fluid. The constraint $2c_1 + c_2 = 0$ in this paper follow first from the stability of the Friedmann equation (35) with a coefficient of the combination $(c_1 + c_2 + 3c_3)$ coupled to the quadratic interactions $\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}$. This coefficient vanishes for the GB term with $c_1 : c_2 : c_3 = 1 : -4 : 1$. This is the main difference between the model considered in [20] and the current models. In addition, we have also shown in the appendix that the Friedmann equation agrees with [21] up to a difference in the definition of sign of the coupling constants c_i. Our inflationary phase solution also agrees with [21] up to a scale due to the effect of the scalar field in this induced-gravity model. Indeed, the Friedmann equation and the I-equation implies that $H_0^2 = \Lambda/6$ and $(c_2 + 4c_3)\Lambda = 1$, respectively, after setting $V = \Lambda$ for a system without a dynamical scalar field ϕ. As a result, $H_0^2 = 1/[6((c_2 + 4c_3)]$.

5. Conclusion

It is shown that replacing the internal space dimensionful coupling constant with a dimension 1 scalar field ψ = constant works harmonically with the Kaluza–Klein inflationary universe under the constraint $2c_1 + c_2 = 0$. In addition, from the effective Lagrangian shown in equation (29), it is easy to find that any quadratic Lagrangian must present itself as combinations of the form $\bar{L}_2 = l_1\dot{H}^2 + l_2(\dot{H}H^2 + H^4) \equiv -12(c_1 + c_2 + 3c_3)[(\dot{H} + H^2)^2 + H^4] - 12(c_2 + 6c_3)[(\dot{H} + H^2)H^2]$ with a dimensionless l_i corresponding to the linear combinations of c_i defined accordingly. Here, $\bar{L}_2$ denotes the quadratic part of the effective Lagrangian $\bar{L}$. As a result, it can be shown that any quadratic Lagrangian of the combinations $\bar{L}_2 \sim l_1\dot{H}^2 + l_2(\dot{H}H^2 + H^4)$ will not contribute to the Friedmann equation.

Indeed, the quadratic terms contribute to the Friedmann equation (32) according to

$$E_2 = \bar{L}_2 + H\left(\frac{\mathrm{d}}{\mathrm{d}t} + 3H\right)\bar{L}_{\dot{H}} - H\bar{L}_H - \dot{H}\bar{L}_{\dot{H}} \to \bar{L}_2 + 3H^2\bar{L}_{\dot{H}} - H\bar{L}_H \tag{58}$$

in the de Sitter background with $L_H \equiv \delta\bar{L}_2/\delta H$ and $L_{\dot{H}} \equiv \delta\bar{L}_2/\delta\dot{H}$ shown in the above equation. It is clear that the l_1 term does not contribute to the above equation E_2 in the de Sitter space with $\dot{H}_0 = 0$. Therefore, we effectively have the quadratic Lagrangian $\bar{L}_2 = l_2(\dot{H}H^2 + H^4)$ needed to be considered for its effect on the leading-order Friedmann equation. As a result, we can show that $\bar{L}_2 \to l_2H_0^4$, $H^2\bar{L}_{\dot{H}} \to l_2H_0^4$ and $H\bar{L}_H \to 4l_2H_0^4$. Hence, the total contribution of the quadratic Lagrangian to E_2 cancels each other. Therefore, this proves that the quadratic Lagrangian does not contribute to the de Sitter solution in 4-dimension.

The perturbation equation for $\delta\phi$ indicates a constraint (46) $V''(\phi_0) = 12\epsilon H_0^2 - 4H_0^2$ which turns out to be inconsistent with the SSB scalar potential (40). Indeed, $V_0'' = 2\lambda\phi_0^2 + 12\epsilon H_0^2$ for this potential. Hence, the constraint (46) implies that $\lambda\phi_0^2 = -2H_0^2$. A negative λ indicates that the SSB potential is an unstable potential without a global minimum. In fact, we can show that the local minimum is at $\phi_m^2 = 0$ and the local maximum is at $\phi_M^2 = (1 + 6\epsilon)\phi_0^2$. Hence, the consistent initial state ϕ_0 of the scalar field will be expected to locate at the left-hand side of the maximum point ϕ_M. The scalar field will hence roll down to the local minimum which is located at $\phi = 0$. This will lead to an un-physical state with an infinite Newtonian constant G. In addition, this local minimum is also not a stable vacuum state. ϕ will eventually tunnel to its global minimum at $\phi \to \infty$. Therefore, the constraint (46) is not a physical constraint for the SSB potential and this is also true for the Coleman–Weinberg effective potential [17]. Hence, the only consistent perturbation of ϕ is $\delta\phi = 0$. This indicates that the de Sitter background is highly stable and compatible with the stable mode $\delta H \to 0$. Therefore, the system will remain stable as long as the scalar field does change very slowly. Note that the negative coupling constant appears to be a universal feature of any coupled effective potential including the Eric–Weinberg dynamical symmetry-breaking potential [18].

Indeed, the ϕ equation under the slow rollover assumptions, $|\dot{\phi}/\phi| \ll H_0$ and $|\ddot{\phi}/\phi| \ll H_0^2$, states that

$$\ddot{\phi} + 3H_0\dot{\phi} \sim 0$$

during the period where $H \sim H_0$. This gives

$$\phi \sim \phi_0 + \frac{\dot{\phi}_0}{3H_0}[1 - \exp(-3H_0t)]. \tag{59}$$

Therefore, the slow rollover approximation is indeed consistent with the dynamics of the scalar field equation that has been a focus of research interests in the literature. Therefore, ϕ does change very slowly during this inflationary phase.

In summary, we have derived abundant constraints from the assumptions: (i) $\psi(z) =$ constant and (ii) $d(t) =$ constant. These assumptions are adopted partly from the fact that they are both not appreciable in the 4-dimensional physical universe observed today. Hence, it is reasonable to freeze their dynamics at certain stage of the evolutionary process. The abundant consistency shown in this paper compatible with the abundant constraints from these assumptions implies that these assumptions are in fact rather reliable. Although we are unable to provide a dynamical reason for these assumptions from the first principle, the compatibility of these assumptions with the inflationary 4-space deserves more attention for its possible physical implications.

Our result indicates, however, that both assumptions, static ψ and static d, appear to be a consistent set of choices for the higher-derivative Kaluza–Klein models. As a result, the Kaluza–Klein higher-derivative induced-gravity theory behaves similarly to the conventional 4-dimensional induced higher-derivative gravity in the lower-energy limit, namely, only R^2 and R^2_{ab} couplings remain effective during the inflationary de Sitter phase. This is inconsistent with the 4D theories that R^2_{abcd} terms can be replaced by R^2 and R^2_{ab} couplings following the GB theorem. Therefore, the related research deserves more attention.

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Appendix

A.1. Field equation

The field equation of the Lagrangian $L = -\epsilon\phi^2\mathbf{R}/2 - c_1\left(\mathbf{R}^{AB}{}_{CE}\right)^2 - c_2\left(\mathbf{R}^A{}_B\right)^2 - c_3\mathbf{R}^2 - \partial_A\phi\partial^A\phi/2 - V$ can be derived by the variation of $\mathbf{g}_{AB}$. The result is

$$\begin{aligned}\frac{\epsilon\phi^2}{2}\left(\frac{1}{2}\mathbf{R}\mathbf{g}_{AB} - \mathbf{R}_{AB}\right) &+ \frac{1}{2}\mathbf{g}_{AB}\left(c_1\left(\mathbf{R}^{AB}{}_{CE}\right)^2 + c_2\left(\mathbf{R}^A{}_B\right)^2 + c_3\mathbf{R}^2 - L_\phi\right)\\ &= 2\left(c_3\mathbf{R}\mathbf{R}_{AB} + c_2\mathbf{R}_{AC}\mathbf{R}^C_B + c_1\mathbf{R}_{ACDE}\mathbf{R}^{CDE}_B\right) - 2c_3(\mathbf{g}_{AB}D^2 - D_AD_B)\mathbf{R}\\ &\quad - \frac{c_2}{2}\mathbf{g}_{AB}D^2\mathbf{R} - (4c_1 + c_2)D^2\mathbf{R}_{AB} + 2(2c_1 + c_2)D_AD_C\mathbf{R}_B{}^C\\ &\quad + \frac{\epsilon}{2}(D_A\partial_B - \mathbf{g}_{AB}D^2)\phi^2 + \frac{1}{2}\partial_A\phi\partial_B\phi. \end{aligned} \tag{A.1}$$

In addition, the scalar equation can be shown to be

$$D^2\phi = V' + \epsilon\phi\mathbf{R}. \tag{A.2}$$

In order to derive the field equation in a covariant way, we may write the variation of the Riemann curvature tensor as $\delta\mathbf{R}^D{}_{CBA} = -D_A\delta\Gamma^D_{BC} + D_B\delta\Gamma^D_{AC}$ as if $\delta\Gamma^A_{BC}$ is a type T(1,2) tensor. The derivation has nothing to do with whether $\delta\Gamma^A_{BC}$ is a tensor or not. Rather, by imagining $\delta\Gamma^A_{BC}$ as a tensor and using all the related properties of the tensor, it helps in reducing the effort in deriving these equations, especially when integration-by-part is required. In addition, we have also used the Bianchi identity $D_CD_D\mathbf{R}^{ACDB} = D^2\mathbf{R}^{AB} - D_CD^A\mathbf{R}^{BC}$ in converting the differentiation of the Riemann tensor into the differentiation of the Ricci tensor. Note also that our result agrees with [21] up to a difference in the definition of sign in c_i.

In particular, we can show explicitly that the Friedmann equation for static internal space agrees with equation (15) in [21].

In addition, the static I-equation (28) can be written as

$$\begin{aligned} D\bar{L} &= D_t Y = 6D(c_2+4c_3)D_t[\ddot{H}+4H\dot{H}] \\ &= 6D(c_2+4c_3)D_t\partial_t[\dot{H}+2H^2] = -D(c_2+4c_3)D^2R. \end{aligned} \tag{A.3}$$

Here, $D_t \equiv \partial_t + 3H$. As a result, the static internal space I-equation can be written as

$$\bar{L} + (c_2+4c_3)D^2R = 0, \tag{A.4}$$

which is identical with the mn-component of the Einstein equation (A.1). Note that $R_{mn} = R_{an} = 0$ in the static internal space condition. Therefore, the only thing left over from the mn-component Einstein equation is identical to equation (A.4). Note that this result agrees with equation (17) in [21].

As shown earlier, $\mathbf{J}_{AB} = \mathbf{G}_{AB} - \mathbf{T}_{AB} = \frac{1}{2}\mathbf{R}\mathbf{g}_{AB} - \mathbf{R}_{AB} - \mathbf{T}_{AB} = 0$ can be decomposed into three different equations: $\mathbf{J}_{tt} = 0$, $\mathbf{g}^{ij}\mathbf{J}_{ij} = 0$ and $\mathbf{g}^{mn}\mathbf{J}_{mn} = 0$ in the $(4+D)$-dimensional spacetime described by the GFRW metric (6). These equations correspond to the Friedmann equation (24), the a-variational equation (25) and the d-variational equation (26). Therefore, this proves that equations (24)–(26) and (36) are the complete set of field equations in the GFRW spacetime. We choose to ignore one of the redundant equations (25) in this paper without losing any physical information as implied by the Bianchi identity (11).

In summary, the reduced formulae shown in this paper can be helpful in extracting some useful information without going into the details of the field equations. For example, the existence of the inflationary solution $H = H_0$ has to do with the leading-order equations. It can be done by ignoring any term like $f(H)\dot{H}$, with $f(H)$ an arbitrary function of H. On the other hand, the stability of the inflationary solution has to do with those leading-order terms linear in time differentiation of δH. We can freely ignore the terms like $\dot{H}^2$. In particular, $(\mathrm{d}/\mathrm{d}t)(f(H)\delta H) = f(H)\delta\dot{H}$ can be used to skip unrelated terms, with $f(H)$ an arbitrary function of H, with the close formula shown in this paper.

A.2. Curvature tensor

For completeness of the calculation, we will list all non-vanishing components of the Ricci tensors, scalar curvatures and terms present in the Lagrangian

$$\mathbf{R}^{ti}{}_{tj} = [\dot{H} + H^2]\delta^i_j, \tag{A.5}$$

$$\mathbf{R}^{ij}{}_{kl} = \left(H^2 + \frac{k_1}{a^2}\right)\left(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k\right), \tag{A.6}$$

$$\mathbf{R}^{tm}{}_{tn} = (\dot{I} + I^2)\delta^m_n, \tag{A.7}$$

$$\mathbf{R}^{im}{}_{jn} = HI\delta^m_n\delta^i_j, \tag{A.8}$$

$$\mathbf{R}^{mn}{}_{pq} = \left(I^2 + \frac{k_2}{\mathrm{d}^2}\right)\left(\delta^m_p\delta^n_q - \delta^m_q\delta^n_p\right). \tag{A.9}$$

In addition, one has

$$\mathbf{R}^t{}_t = -[3(\dot{H} + H^2) + D(\dot{I} + I^2)], \tag{A.10}$$

$$\mathbf{R}^i{}_j = -\left[\dot{H} + 3H^2 + 2\frac{k_1}{a^2} + DHI\right]\delta^i_j, \tag{A.11}$$

$$\mathbf{R}^m{}_n = -\left[\dot{I} + DI^2 + (D-1)\frac{k_2}{\mathrm{d}^2} + 3HI\right]\delta^m_n, \tag{A.12}$$

$$\mathbf{R} = -\left[6\left(\dot{H} + 2H^2 + \frac{k_1}{a^2} + DHI\right) + D(D-1)\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right) + 2D(\dot{I} + I^2)\right], \tag{A.13}$$

and

$$\begin{aligned}\left(\mathbf{R}^{AB}{}_{CD}\right)^2 &= 12(\dot{H} + H^2)^2 + 4D(\dot{I} + I^2)^2 \\ &\quad + 12DH^2I^2 + 12\left(H^2 + \frac{k_1}{a^2}\right)^2 + 2D(D-1)\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right)^2,\end{aligned} \tag{A.14}$$

$$\begin{aligned}\left(\mathbf{R}^A{}_B\right)^2 &= 12(\dot{H} + H^2)^2 + D(D+1)(\dot{I} + I^2)^2 + 12\left(H^2 + \frac{k_1}{a^2}\right)^2 \\ &\quad + D(D-1)^2\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right)^2 + 3D(D+3)H^2I^2 \\ &\quad + 12(\dot{H} + H^2)\left(H^2 + \frac{k_1}{a^2}\right) + 6D(\dot{H} + H^2 + HI)(\dot{I} + I^2) \\ &\quad + 6DHI\left(\dot{H} + 3H^2 + 2\frac{k_1}{a^2}\right) + 2D(D-1)(\dot{I} + I^2)\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right) \\ &\quad + 6D(D-1)HI\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right),\end{aligned} \tag{A.15}$$

$$\begin{aligned}(\mathbf{R})^2 &= 36(\dot{H} + H^2)^2 + 4D^2(\dot{I} + I^2)^2 + 36D^2H^2I^2 + 36\left(H^2 + \frac{k_1}{a^2}\right)^2 \\ &\quad + D^2(D-1)^2\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right)^2 + 72(\dot{H} + H^2)\left(H^2 + \frac{k_1}{a^2} + DHI\right) \\ &\quad + 12D(D-1)\left(\dot{H} + 2H^2 + \frac{k_1}{a^2}\right)\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right) \\ &\quad + 24D\left(\dot{H} + 2H^2 + \frac{k_1}{a^2}\right)(\dot{I} + I^2) + 72DHI\left(H^2 + \frac{k_1}{a^2}\right) \\ &\quad + 4D^2(D-1)(\dot{I} + I^2)\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right) \\ &\quad + 24D^2(\dot{I} + I^2)HI + 12D^2(D-1)HI\left(I^2 + \frac{k_2}{\mathrm{d}^2}\right).\end{aligned} \tag{A.16}$$

A.3. Compactification

Another way to derive the decoupled field equation is to assume that the N-space decouples according to $M^N \to M^4 \times M^D$ via the following metric decomposition:

$$\mathrm{d}s^2 = g_{ab}(x)\,\mathrm{d}x^a\,\mathrm{d}x^b + \mathrm{d}^2(x)h_{mn}(z)\,\mathrm{d}z^m\,\mathrm{d}z^n \tag{A.17}$$

with the internal space metric $g_{mn}(x, z) = \mathrm{d}^2(x)h_{mn}(z)$.

Non-vanishing spin connections are

$$\Gamma^a_{bc} = \Gamma^a_{bc}; \qquad \Gamma^c_{mn} = -E^c g_{mn}; \qquad \Gamma^m_{cn} = E_c\delta^m_n; \qquad \Gamma^m_{np} = \Gamma^m_{np}, \tag{A.18}$$

with $E_c \equiv \partial_c d(x)/d(x)$ a vector-like function. As a result, we can also show that all non-vanishing curvature tensors are

$$\mathbf{R}^{ab}{}_{cd} = R^{ab}{}_{cd}, \tag{A.19}$$

$$\mathbf{R}^{am}{}_{bn} = -\nabla^a E_b \delta^m_n, \tag{A.20}$$

$$\mathbf{R}^{mn}{}_{pq} = -E^a E_a \left(\delta^m_p \delta^n_q - \delta^m_q \delta^n_p\right), \tag{A.21}$$

$$\mathbf{R}^a{}_b = R^a{}_b + D\nabla^a E_b, \tag{A.22}$$

$$\mathbf{R}^m{}_n = (D_a + DE_a)E^b \delta^m_n, \tag{A.23}$$

$$\mathbf{R} = R + 2D\nabla'_a E^a, \tag{A.24}$$

with $\nabla_a E_b \equiv (D_a + E_a)E_b$ and $\nabla'_a E_b \equiv [D_a + (D+1)/2E_a]E_b$.

The Lagrangian $L = -\frac{\epsilon}{2}\phi^2\mathbf{R} - c_1\left(\mathbf{R}^{AB}{}_{CE}\right)^2 - c_2\left(\mathbf{R}^A{}_B\right)^2 - c_3\mathbf{R}^2 - \partial_A\phi\partial^A\phi/2 - V$ can be shown to be

$$L(E_a) = \frac{\epsilon\phi^2}{2}[R + 2D\nabla'_a E^a] - c_3[R^2 + 4DR\nabla'_a E^a + 4D^2(\nabla'_a E^a)^2] - \partial_A\phi\partial^A\phi/2 - V \tag{A.25}$$

$$-c_2\left(R^a{}_b{}^2 + 2DR^{ab}\nabla_a E_b + D^2[(D_a + DE_a)E_b][(D^a + DE^a)E^b]\right) \tag{A.26}$$

$$-c_1\left[R^{ab}{}_{cd}{}^2 + 4D(\nabla_a E^a)^2 + 2D(D-1)(E_a E^a)^2\right]. \tag{A.27}$$

Note that $E_a = \partial_a d/d$, therefore the I-equation can be derived by varying the above effective Lagrangian with respect to d. We can first derive the field equation with respect to δE_a and perform another integration-by-part to find the field equation of δd. Note also that only terms linear in E_a will contribute to the I-equation once $I = 0$ is imposed for the static internal space solution. Therefore, we only need to consider

$$L = D\phi^2 D_a E^a - 4c_3 DRD_a E^a - 2c_2 DR^{ab} D_a E_b \sim D[(c_2 + 4c_3)D_a R - D_a\phi^2]E^a. \tag{A.28}$$

In addition, there is also a term derived from the volume measure $\sqrt{\mathbf{g}} \propto d^D$. Therefore, the I-equation can be shown to be

$$D[\bar{L} - (c_2 + 4c_3)D^2 R] = 0. \tag{A.29}$$

This is identical to the I-equation (A.4) $\bar{L} + (c_2 + 4c_3)D_t D_t R = 0$ shown above. And also note that equation (A.29) holds for the case with $d = d(x)$ in the presence of a constant background internal space $d =$ constant. Therefore, the conditions of the inflationary phase remain the same for inhomogeneous $d(x)$ in the presence of a static internal background.

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