

Chaos in a generalized van der Pol system and in its fractional order system

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Accepted 7 March 2006

Abstract

In this paper, chaos of a generalized van der Pol system with fractional orders is studied. Both nonautonomous and autonomous systems are considered in detail. Chaos in the nonautonomous generalized van der Pol system excited by a sinusoidal time function with fractional orders is studied. Next, chaos in the autonomous generalized van der Pol system with fractional orders is considered. By numerical analyses, such as phase portraits, Poincaré maps and bifurcation diagrams, periodic, and chaotic motions are observed. Finally, it is found that chaos exists in the fractional order system with the order both less than and more than the number of the states of the integer order generalized van der Pol system.

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1. Introduction

A van der Pol system, which is a typical nonlinear chaotic system has many interesting features, and numerous applications. It has been used for the design of various systems including biological ones, such as the heartbeats [1] or the generation of action potentials by neurons, acoustic models, the radiation of mobile phones, and as a model of electrical systems. Chaos, chaos control and chaos synchronization of nonlinear systems, including the van der Pol system have been subjected to extensive studies [2–18].

Fractional calculus is an old mathematical topic from 17th century. Although it has a long history, applications are only recent focus of interest. Many systems are known to display fractional order dynamics, such as viscoelastic systems [19], dielectric polarization, electrode–electrolyte polarization, and electromagnetic waves. Furthermore, researchers have found some systems for which the chaotic motions exist in the fractional orders [20–24].

This paper is organized as follows. In Section 2, a review and the approximation of the fractional order operator is presented. In Section 3, chaos of the generalized van der Pol system and the nonautonomous and autonomous fractional order generalized van der Pol systems is given. In Section 4, numerical simulations, such as phase portraits, Poincaré maps and bifurcation diagrams, of various nonautonomous and autonomous fractional order generalized van der Pol systems are presented. In Section 5, some conclusions are drawn.

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2. The review and the approximation of fractional order operators

There are many ways to define a fractional differential operator [25–27]. The commonly used definition for general fractional derivative is the Riemann–Liouville definition. The Riemann–Liouville definition of the fractional order derivative is

$$D^\alpha y(t) = \frac{d^n}{dt^n} D^{n-\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{y(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (1)$$

where $\Gamma(\cdot)$ is a gamma function and n is an integer such that $n-1 \leq \alpha < n$. This definition is different from the usual intuitive definition of derivative.

Thus, it is necessary to develop approximations to the fractional operators using the standard integer order operators. Fortunately, the Laplace transform which is basic engineering tool for analysing linear systems is still applicable and works:

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{n-1-k} f(t)}{dt^{n-1-k}}\right]_{t=0} \quad \text{for all } \alpha, \quad (2)$$

where n is an integer such that $n-1 \leq \alpha < n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected form

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\}. \quad (3)$$

Using the algorithm in [21,28], linear transfer function of approximations of the fractional integrator is adopted. Basically the idea is to approximate the system behavior based on frequency domain arguments. From [29], we get the table of approximating transfer functions for $1/s^\alpha$ with different fractional orders, $\alpha = 0.1 - 0.9$, in steps of 0.1, which give the maximum error 2 dB in calculations. These approximations will be used in the following study.

3. The chaos of the generalized van der Pol system and its fractional order form

The van der Pol system [30] was first given to study the oscillations in vacuum tube circuits. The equivalent state space formulation has the form of an autonomous system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= -x_1 - \varepsilon(x_1^2 - 1)x_2, \end{aligned} \quad (4)$$

where ε is a parameter. The generalized van der Pol system [31–35] has the form of a nonautonomous system which is written as

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= -x_1 - \varepsilon(1 - x_1^2)(c - ax_1^2)x_2 + b \sin \omega t, \end{aligned} \quad (5)$$

where ε, a, b, c are parameters, and ω is the circular frequency of the external excitation $b \sin \omega t$. Figs. 1 and 2 show the bifurcation diagram and Lyapunov exponent diagram for the nonautonomous generalized van der Pol system (5), and Figs. 3–6 are the phase portraits and Poincaré maps, while Poincaré maps are taken by period $\frac{2\pi}{\omega}$. The corresponding nonautonomous fractional order system is

$$\begin{aligned} \frac{d^\alpha x_1}{dt^\alpha} &= x_2, \\ \frac{d^\beta x_2}{dt^\beta} &= -x_1 - \varepsilon(1 - x_1^2)(c - ax_1^2)x_2 + b \sin \omega t, \end{aligned} \quad (6)$$

where α, β are fractional numbers.

A modified version of Eq. (6) is now proposed. The nonautonomous generalized fractional order van der Pol system (6) with two states is transformed into an autonomous generalized fractional order van der Pol system with three states

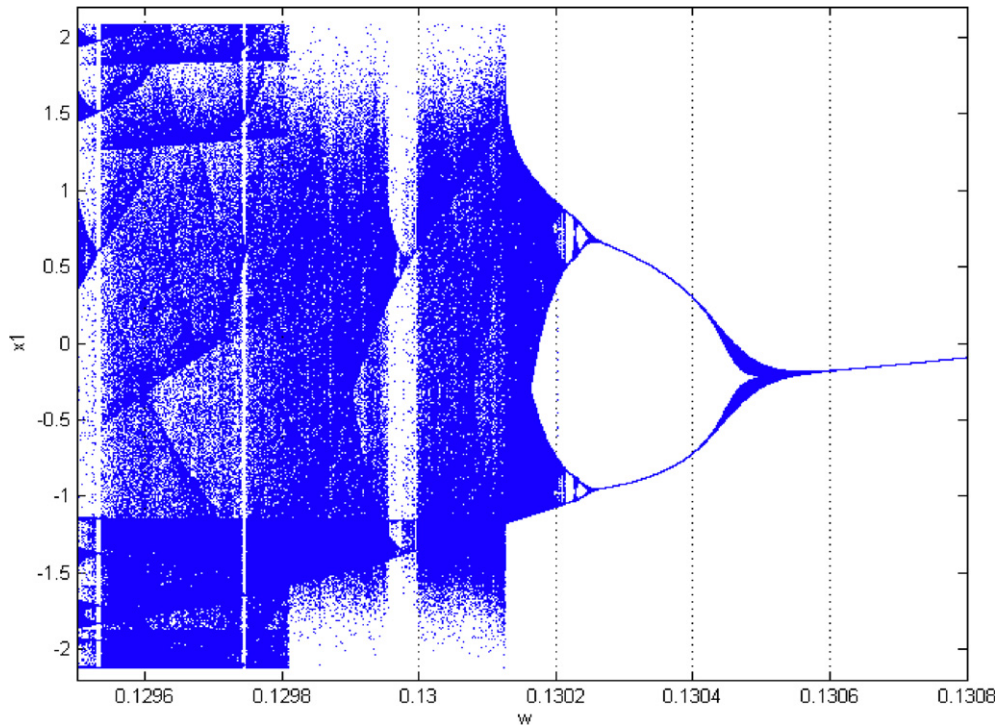


Fig. 1. The bifurcation diagram of the nonautonomous generalized van der Pol system.

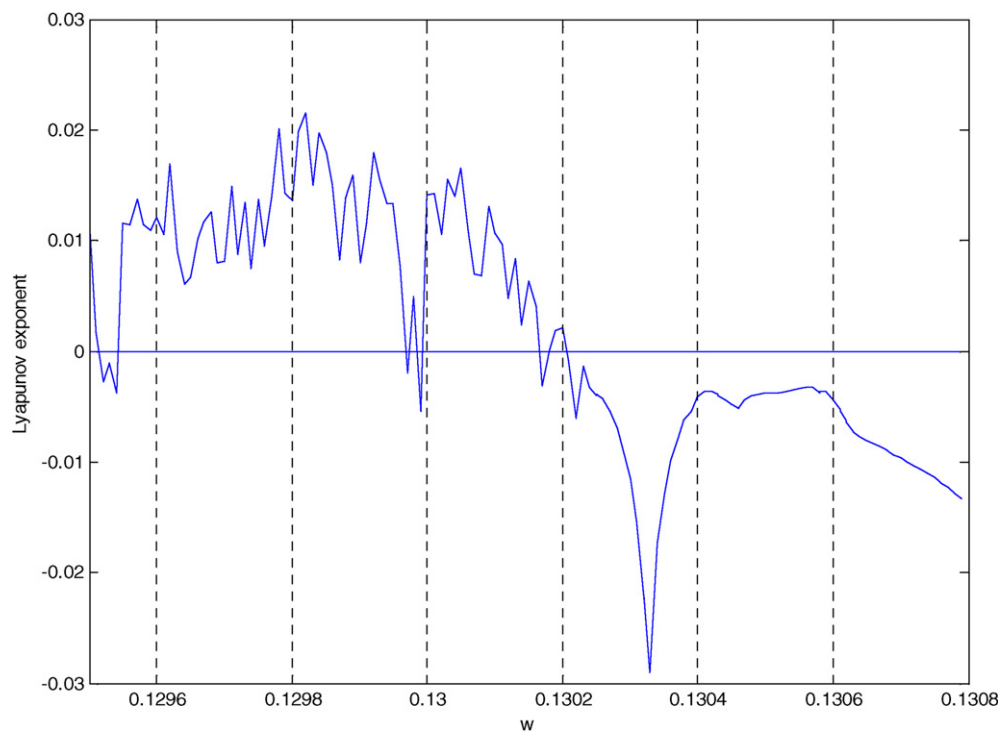


Fig. 2. Lyapunov exponent diagram of the nonautonomous generalized van der Pol system for ω between 0.1295 and 0.1308.

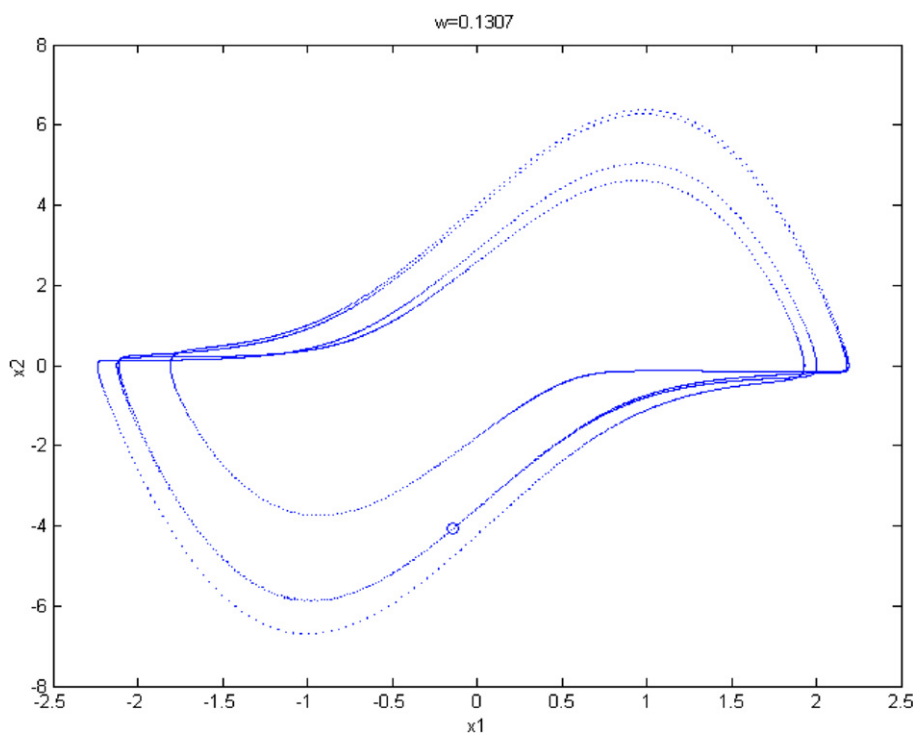


Fig. 3. The phase portrait and the Poincaré map of the nonautonomous generalized van der Pol system.

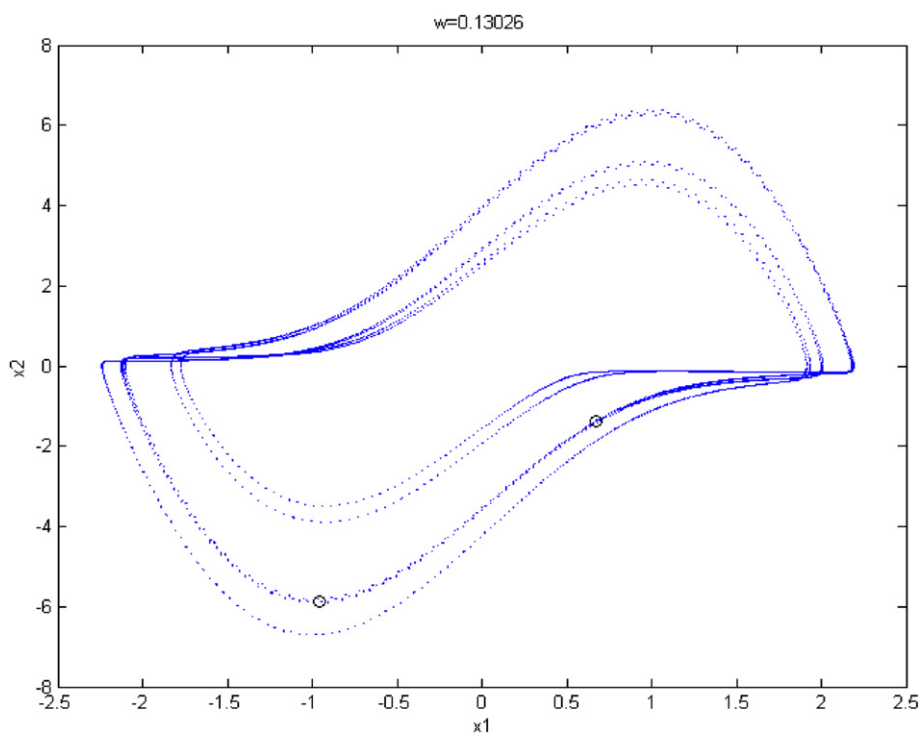


Fig. 4. The phase portrait and the Poincaré map of the nonautonomous generalized van der Pol system.

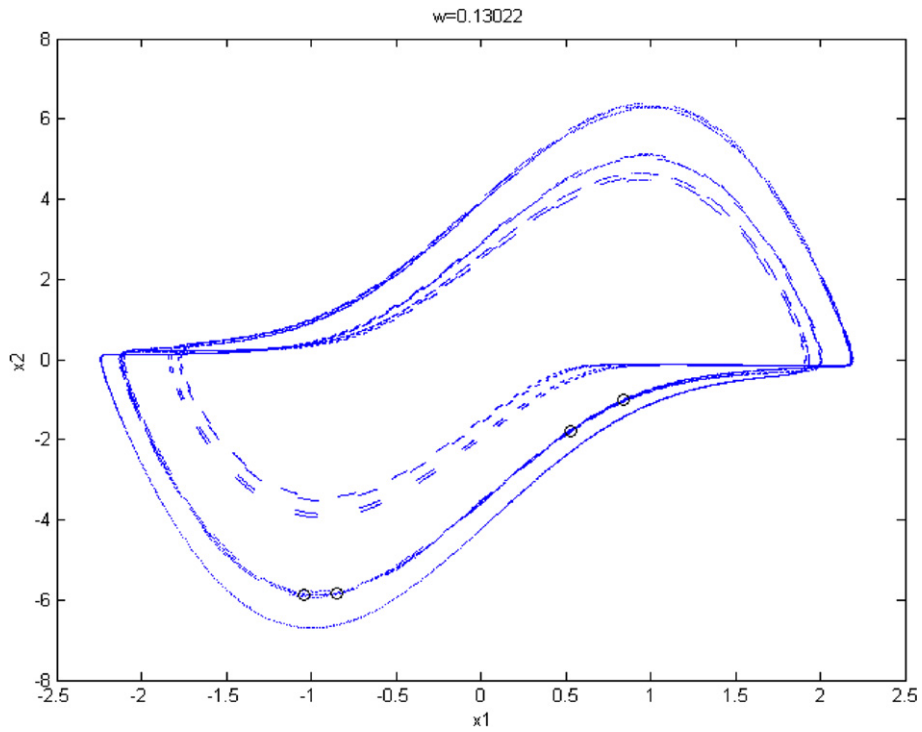


Fig. 5. The phase portrait and the Poincaré map of the nonautonomous generalized van der Pol system.

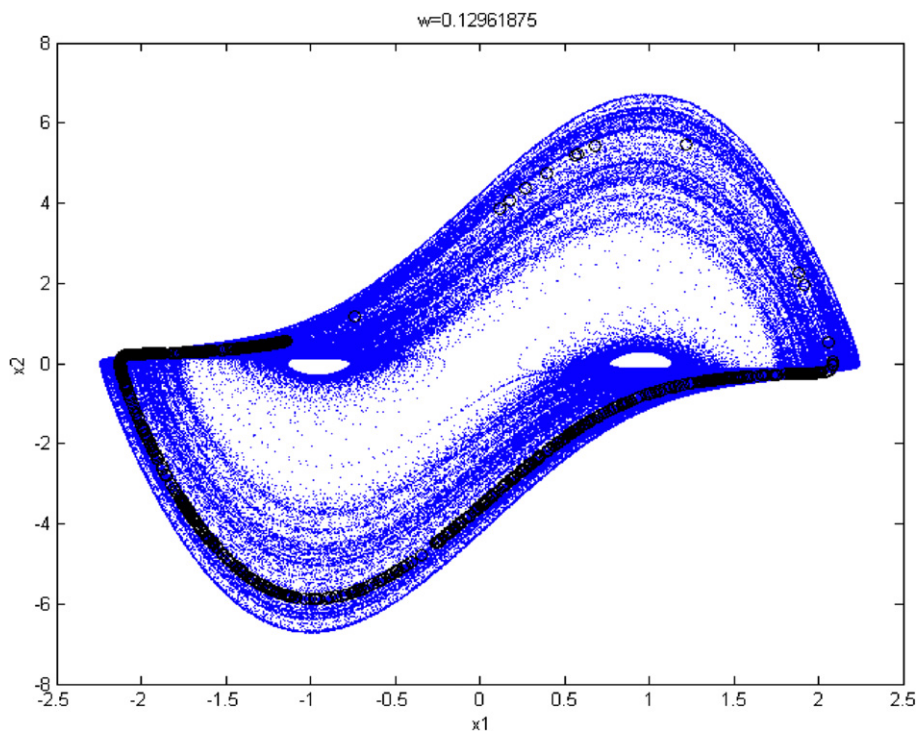


Fig. 6. The phase portrait and the Poincaré map of the nonautonomous generalized van der Pol system.

$$\begin{aligned}
\frac{d^\alpha x_1}{dt^\alpha} &= x_2, \\
\frac{d^\beta x_2}{dt^\beta} &= -x_1 - \varepsilon(1 - x_1^2)(c - ax_1^2)x_2 + b \sin \omega x_3, \\
\frac{d^\gamma x_3}{dt^\gamma} &= 1,
\end{aligned} \tag{7}$$

where α, β, γ are fractional numbers, in which the original time t in Eq. (6) is changed to a new state x_3 . When $\gamma = 1$, $x_3 = t$, Eq. (7) reduces to Eq. (6).

In this paper, we analyse and present simulation results of the chaotic dynamics produced from a new generalized fractional van der Pol system as the state of fractional order derivatives in Eq. (6) are varied from 0.1 to 1.3. It is observed that the different orders have large influences upon the overall system dynamics.

4. Numerical simulations for the fractional order generalized van der Pol systems

Our study of system (7) consists of three parts:

Part 1: γ equals one, and α, β take the same fractional numbers. The system is equivalent to a nonautonomous fractional order system with two states, Eq. (6).

Part 2: α, β, γ take the same fractional numbers.

Part 3: γ equals 1.1 or 0.9, and α, β take the same fractional numbers.

In Part 1, $\gamma = 1$, Eq. (7) reduces to Eq. (6) with two states x_1 and x_2 . α, β take the same fractional numbers and vary from 1.1 to 0.7 in steps of 0.1. Fig. 7 shows the bifurcation diagram and Figs. 8–11 are the phase portraits and Poincaré maps with $\alpha = \beta = 1.1$, while Poincaré maps are taken by period $\frac{2\pi}{\omega}$. Fig. 12 shows the bifurcation diagram and Figs. 13–16 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.9$. Fig. 17 shows the bifurcation diagram and Figs. 18–21 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.8$. Fig. 22 shows the bifurcation diagram and Figs. 23–26 are

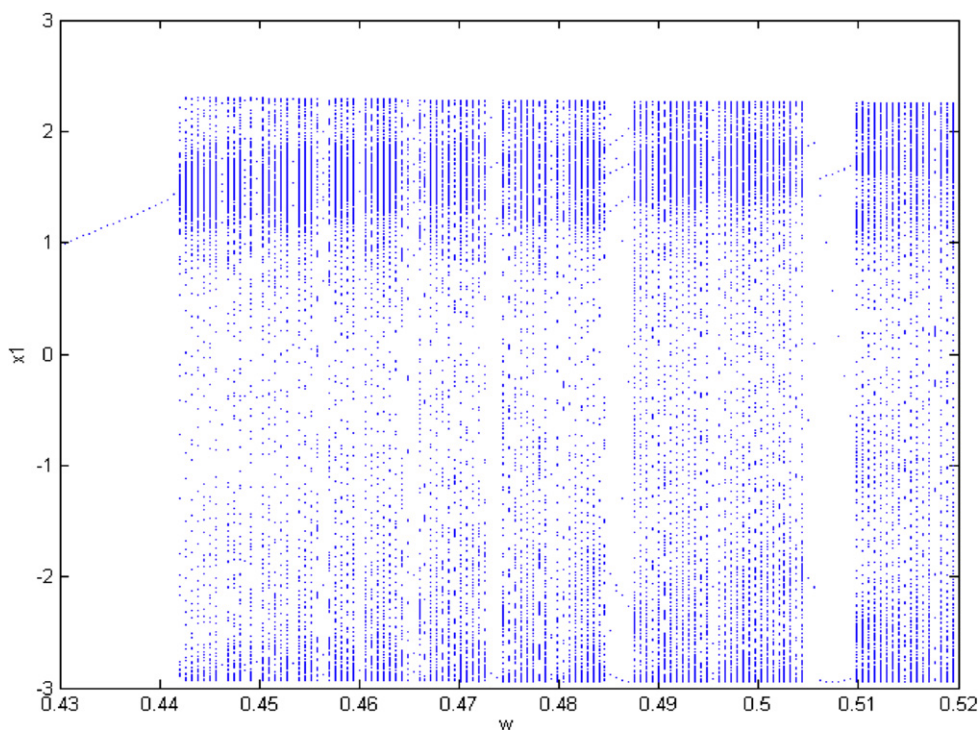


Fig. 7. The bifurcation diagram of the nonautonomous fractional order system with order $\alpha = \beta = 1.1$, $\gamma = 1$.

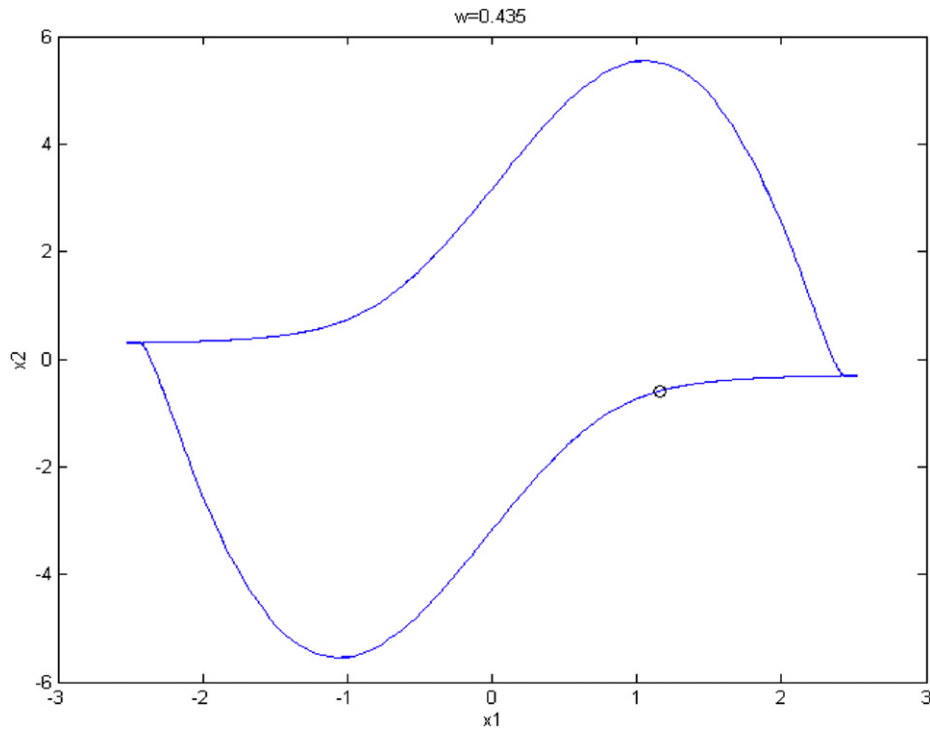


Fig. 8. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 1.1$, $\gamma = 1$, $\omega = 0.435$.

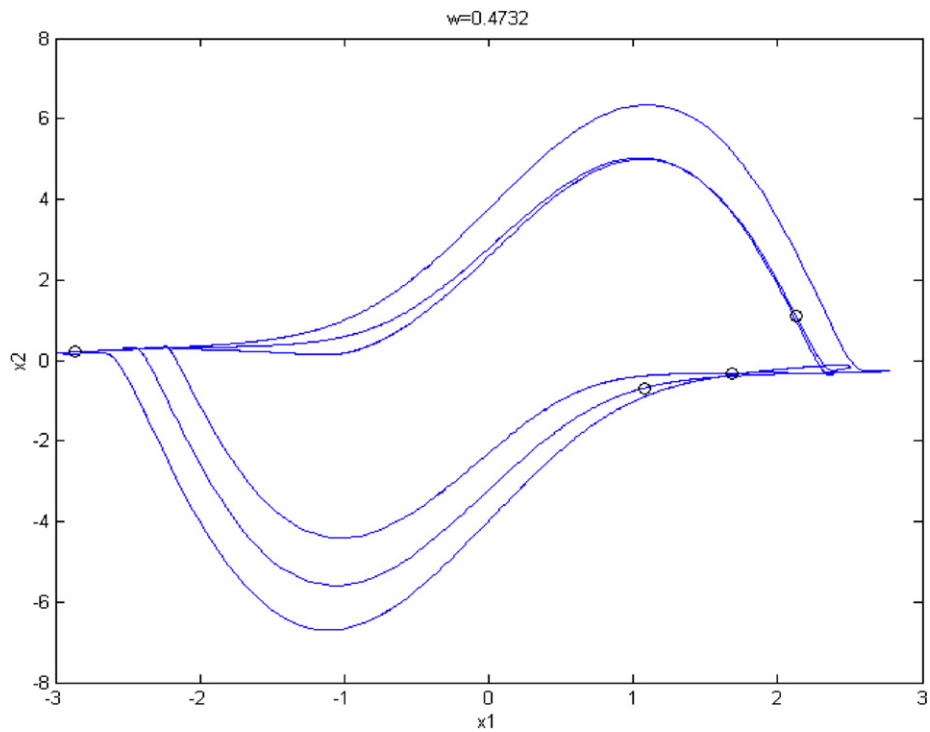


Fig. 9. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 1.1$, $\gamma = 1$, $\omega = 0.4732$.

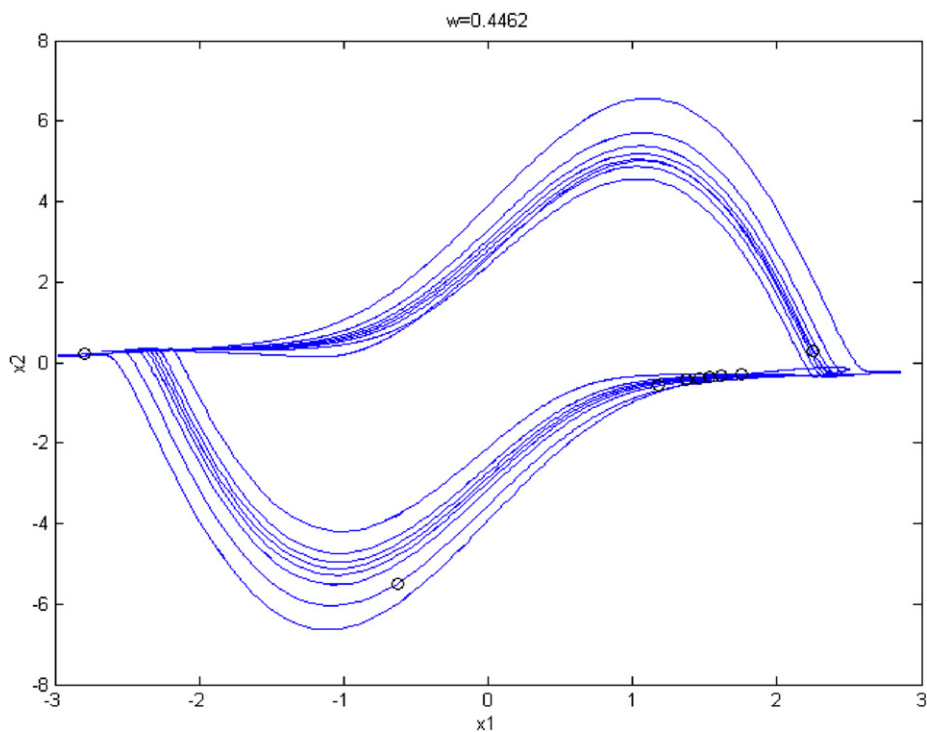


Fig. 10. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 1.1$, $\gamma = 1$, $\omega = 0.4462$.

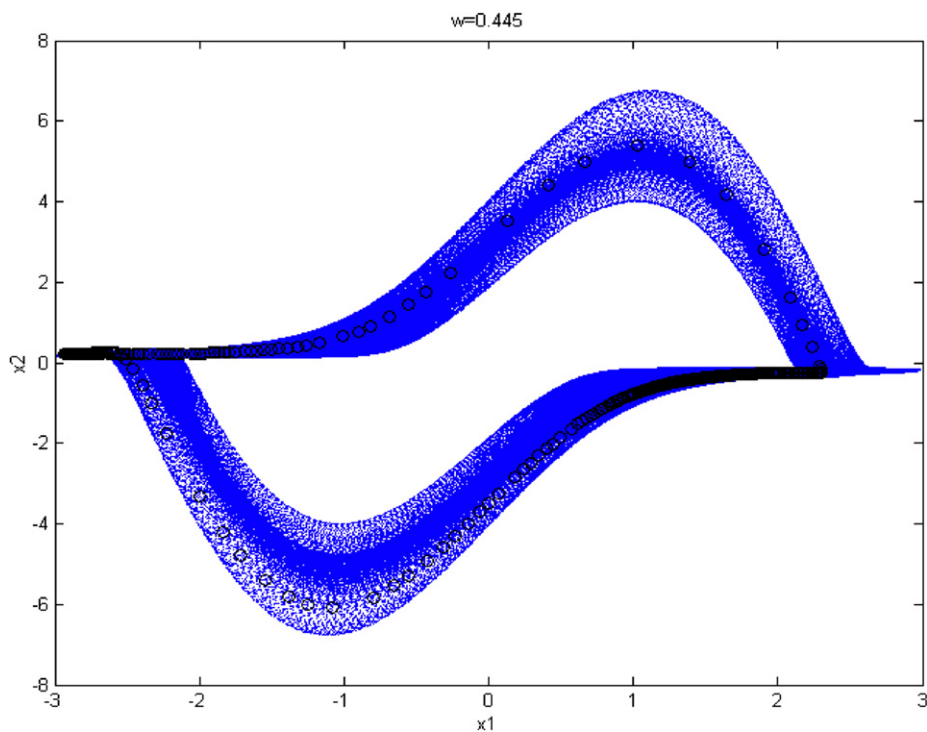


Fig. 11. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 1.1$, $\gamma = 1$, $\omega = 0.445$.

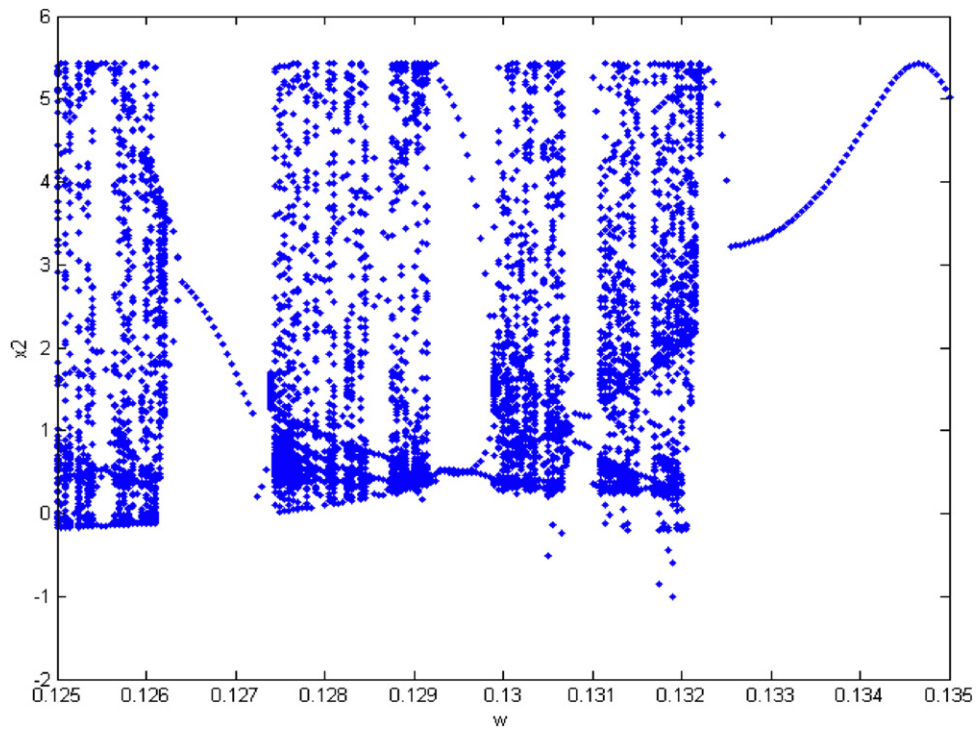


Fig. 12. The bifurcation diagram of the nonautonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1$.

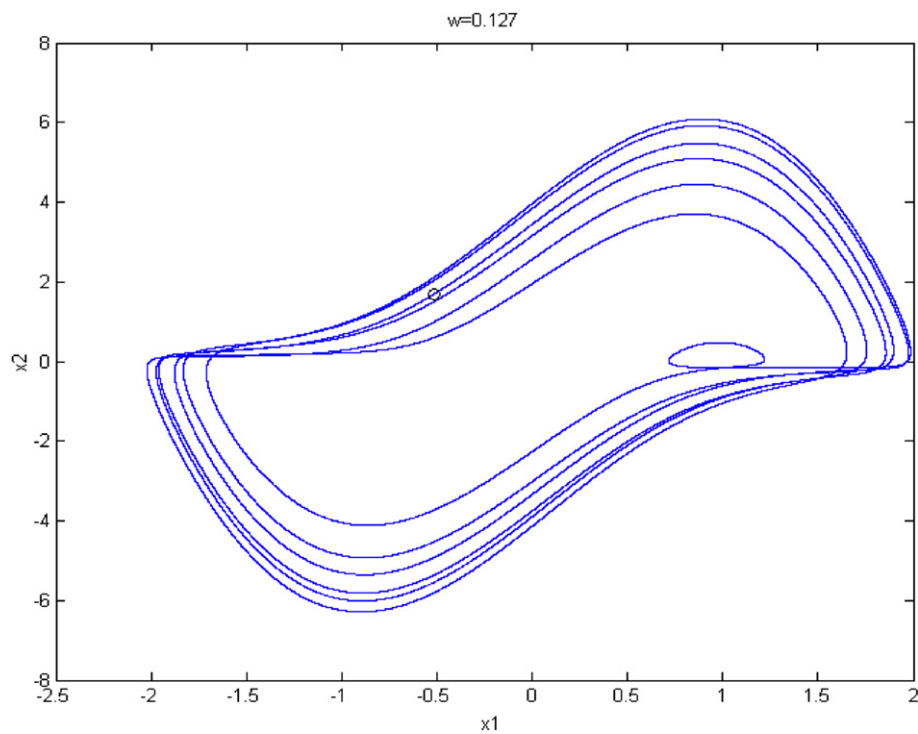


Fig. 13. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1$, $\omega = 0.127$.

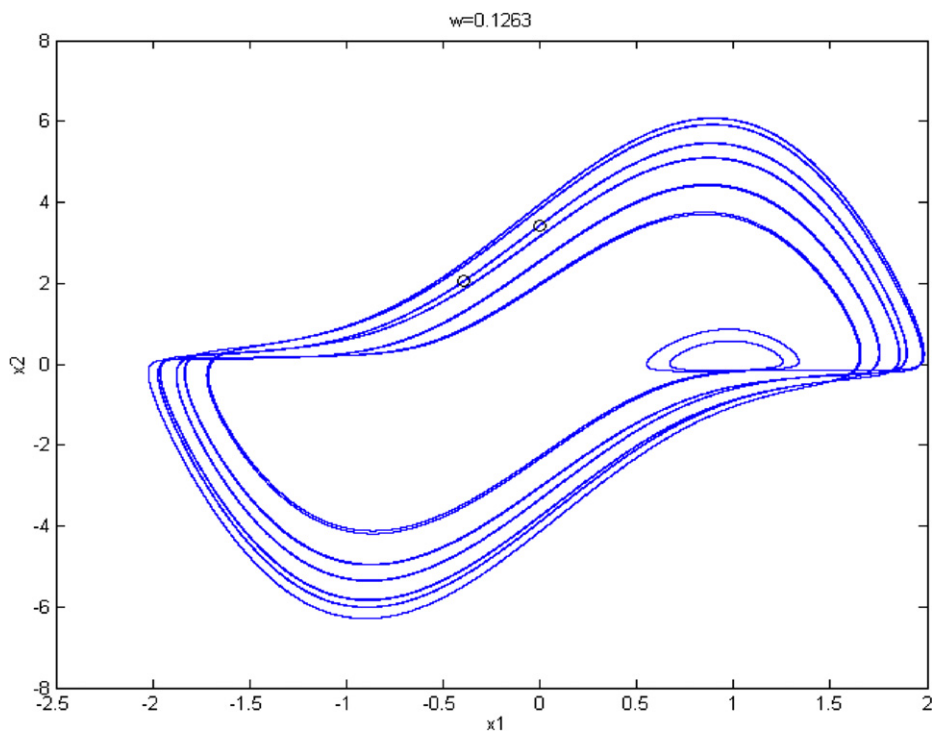


Fig. 14. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1$, $\omega = 0.1263$.

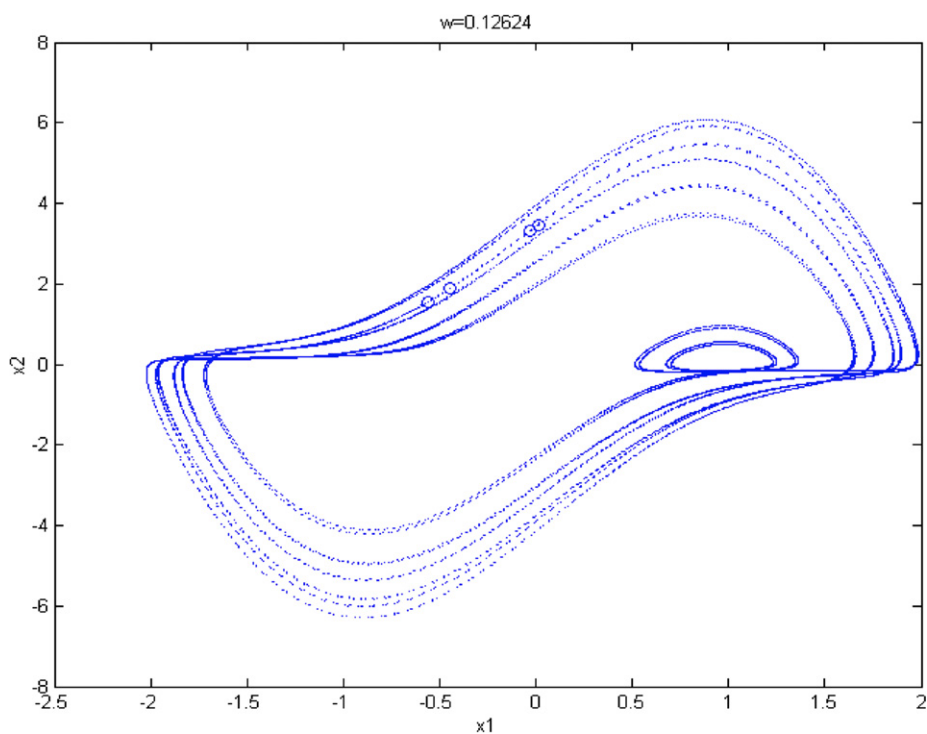


Fig. 15. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1$, $\omega = 0.12624$.

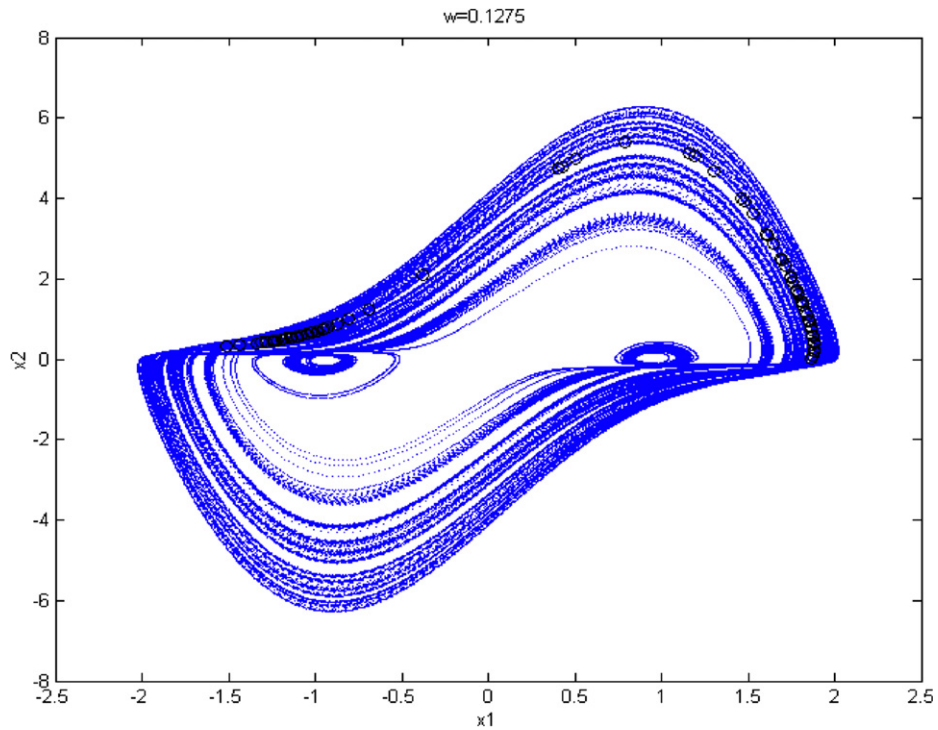


Fig. 16. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1$, $\omega = 0.1275$.

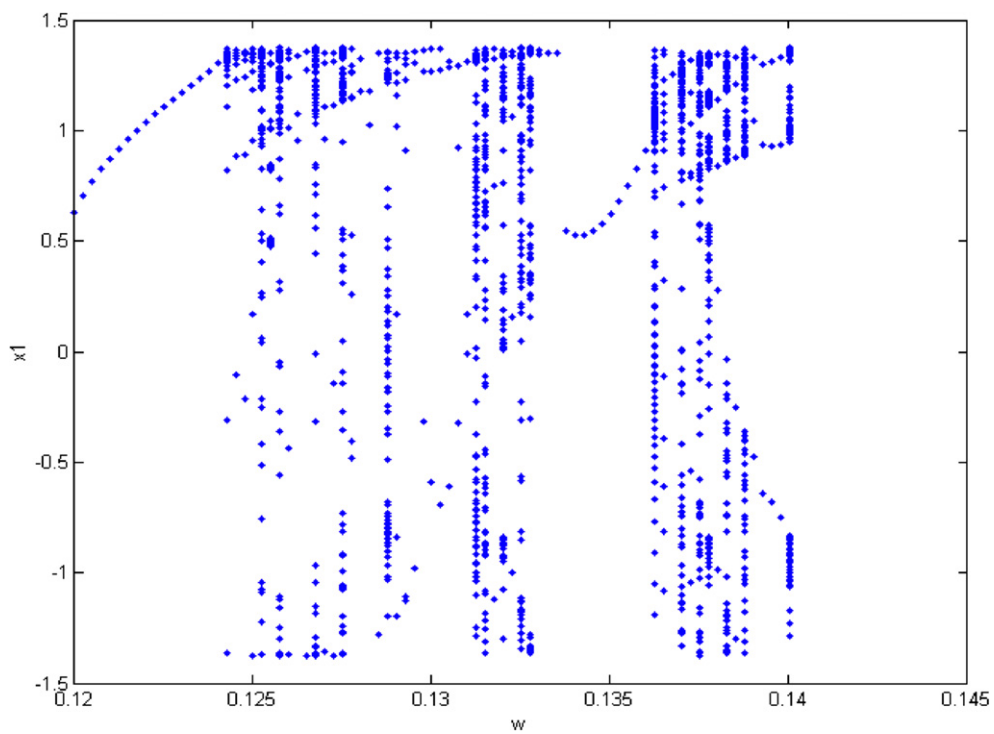


Fig. 17. The bifurcation diagram of the nonautonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1$.

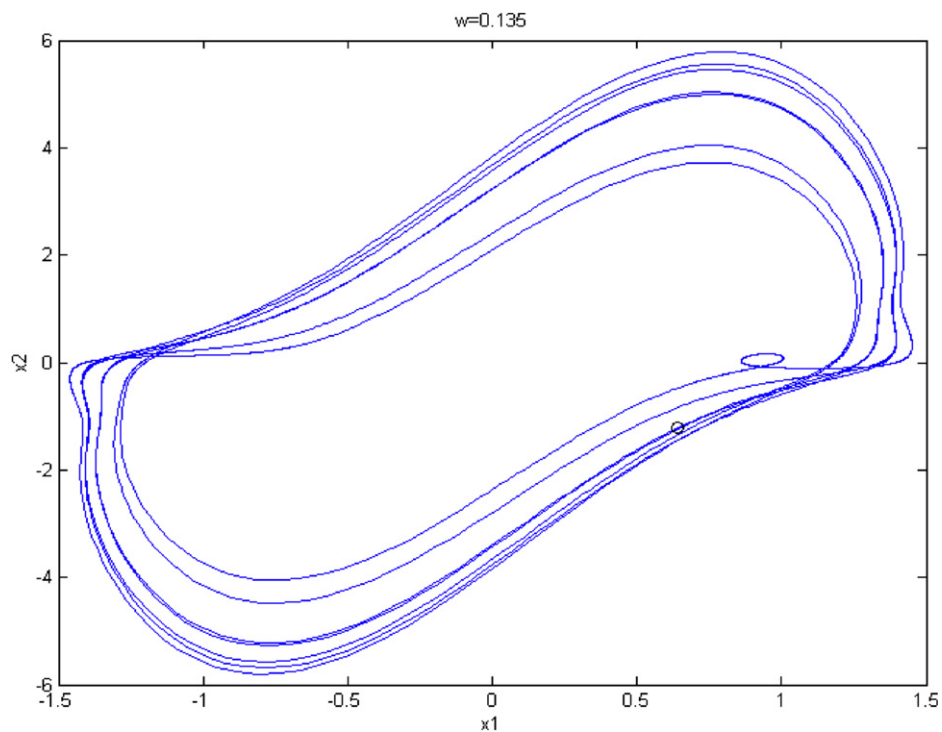


Fig. 18. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1$, $\omega = 0.135$.

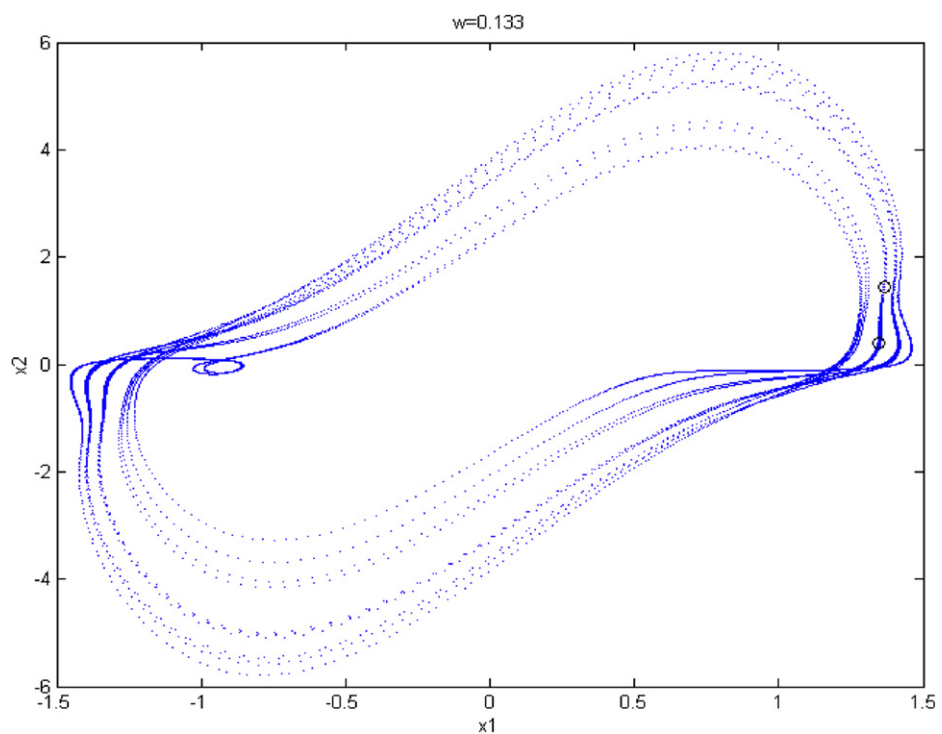


Fig. 19. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1$, $\omega = 0.133$.

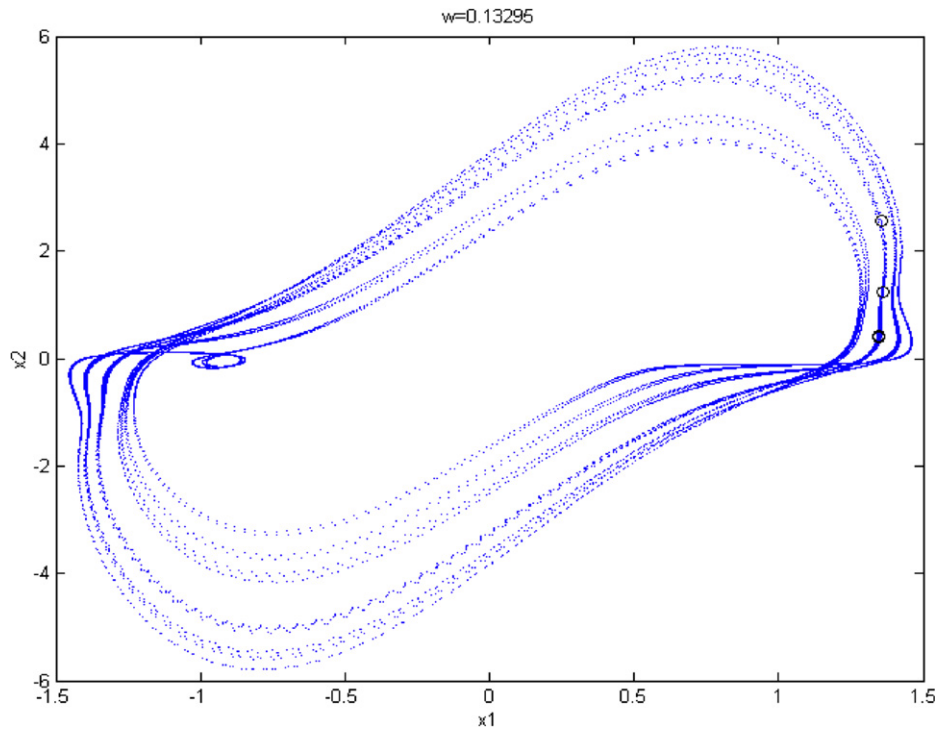


Fig. 20. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1$, $\omega = 0.13295$.

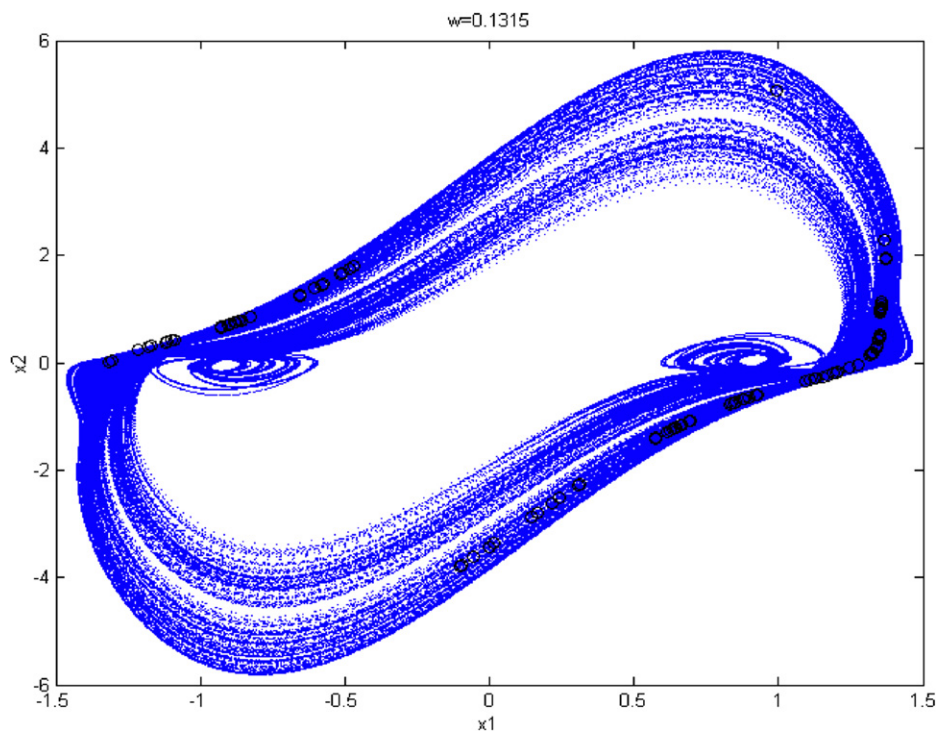


Fig. 21. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1$, $\omega = 0.1315$.

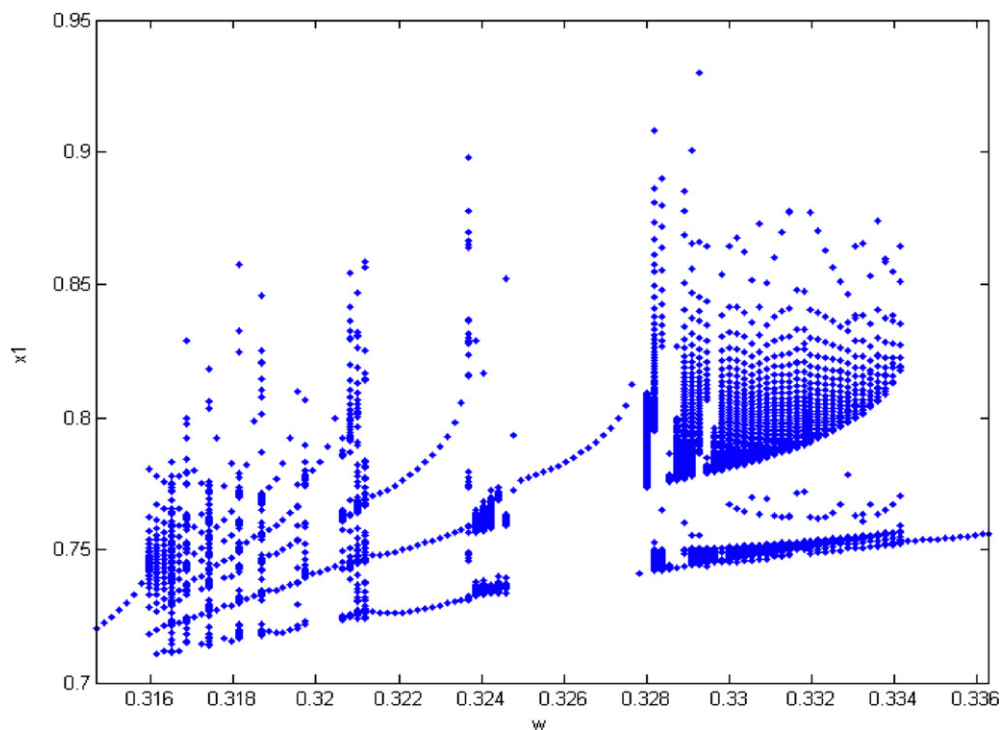


Fig. 22. The bifurcation diagram of the nonautonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1$.

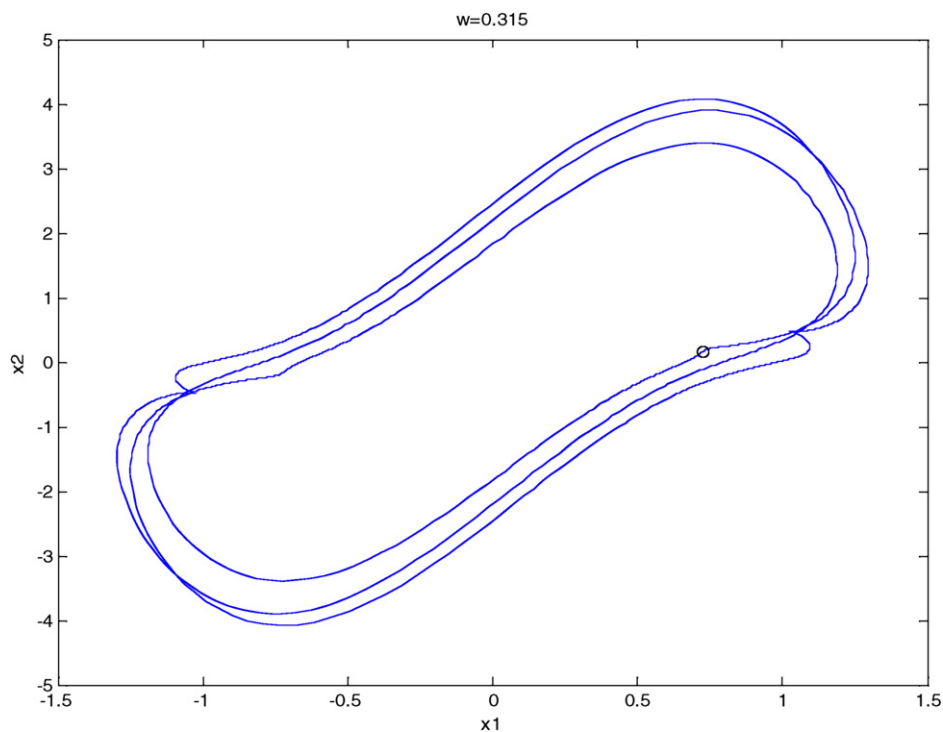


Fig. 23. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1$, $\omega = 0.315$.

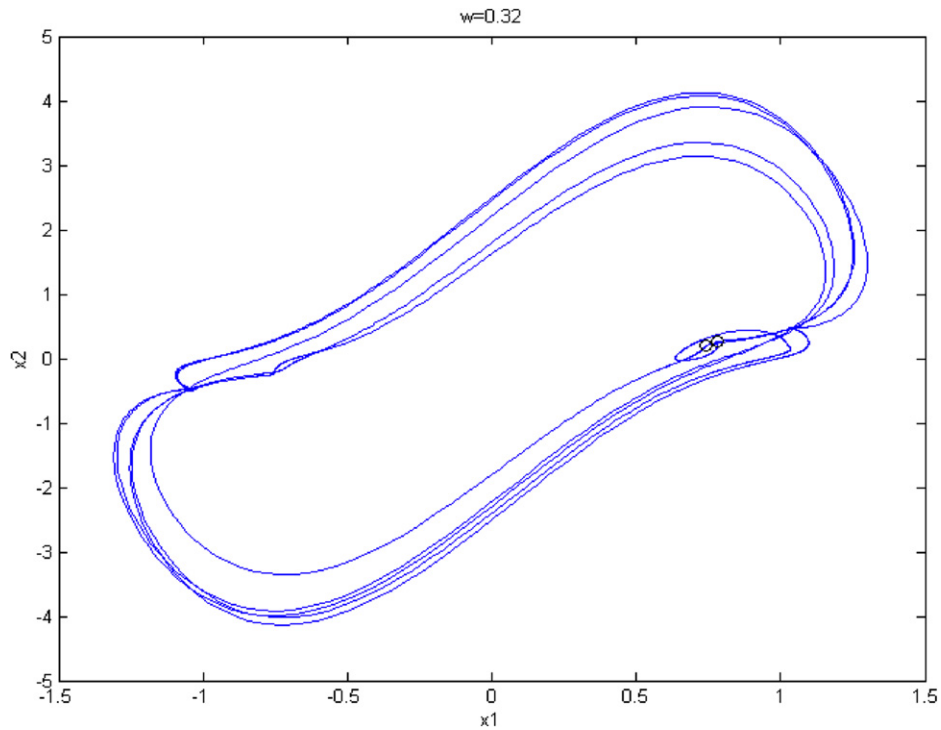


Fig. 24. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1$, $\omega = 0.32$.

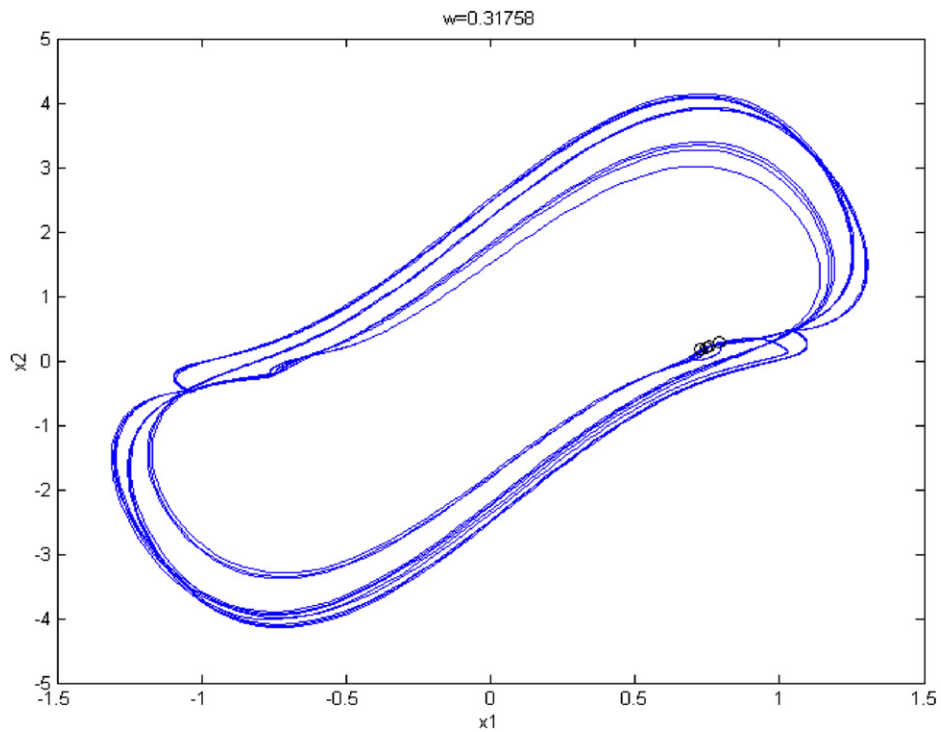


Fig. 25. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1$, $\omega = 0.31758$.

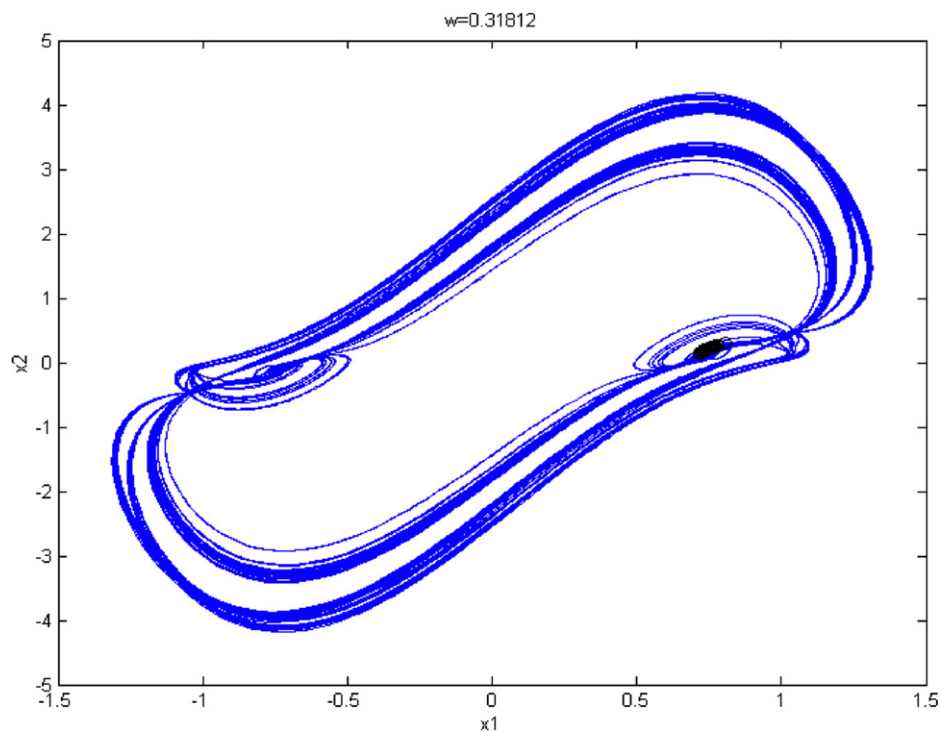


Fig. 26. The phase portrait and Poincaré maps of the nonautonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1$, $\omega = 0.31812$.

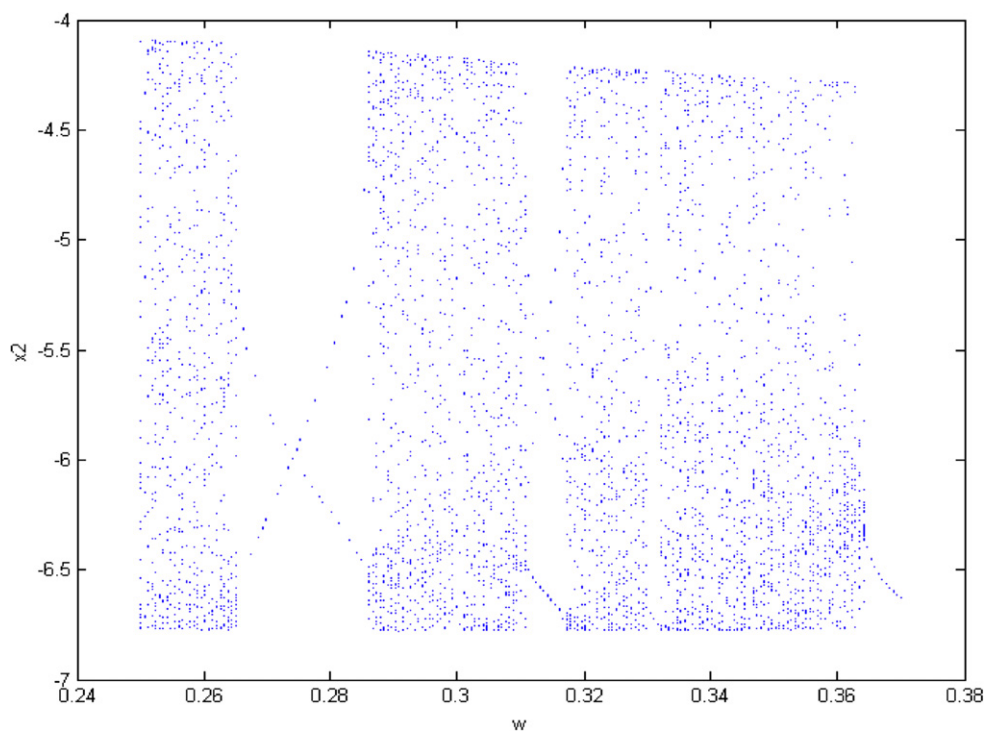


Fig. 27. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = \gamma = 1.1$.

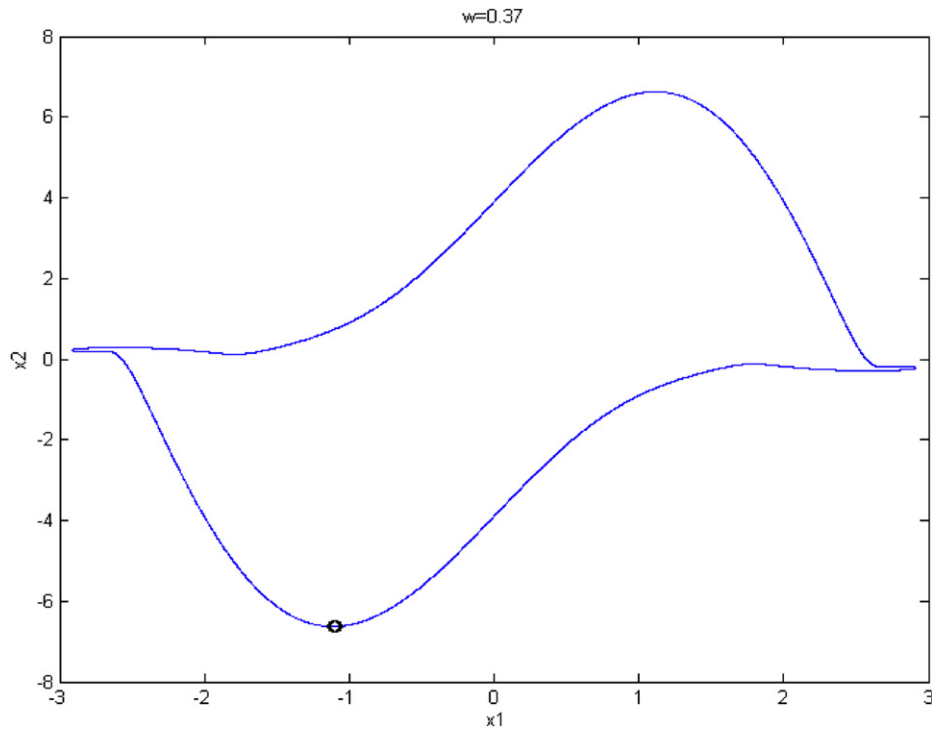


Fig. 28. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = \gamma = 1.1$, $\omega = 0.37$.

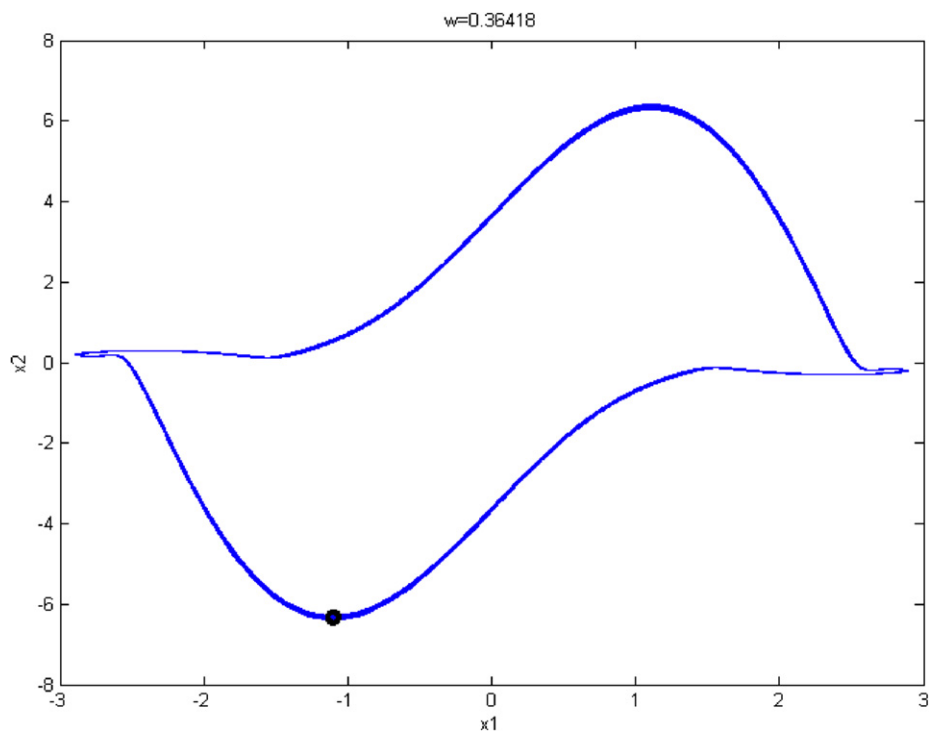


Fig. 29. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = \gamma = 1.1$, $\omega = 0.36418$.

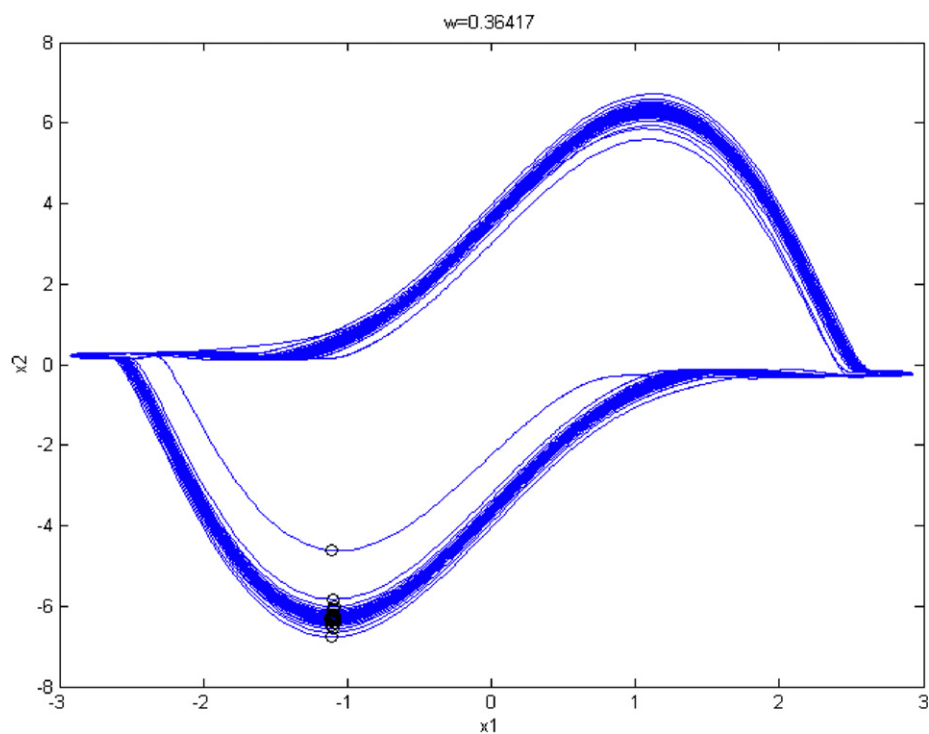


Fig. 30. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = \gamma = 1.1$, $\omega = 0.36417$.

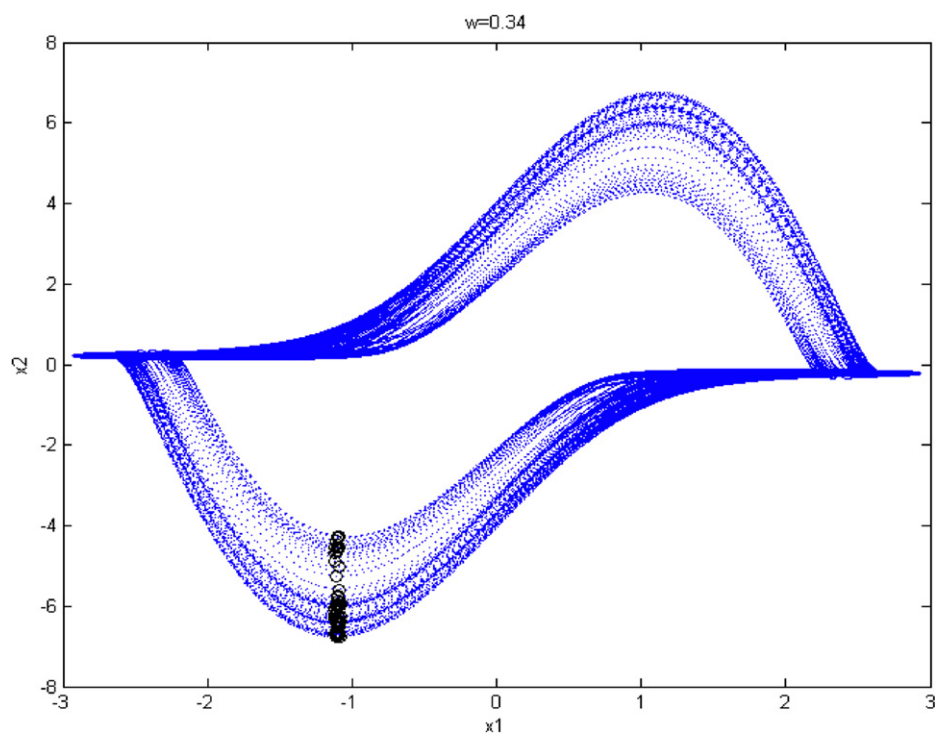


Fig. 31. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = \gamma = 1.1$, $\omega = 0.34$.

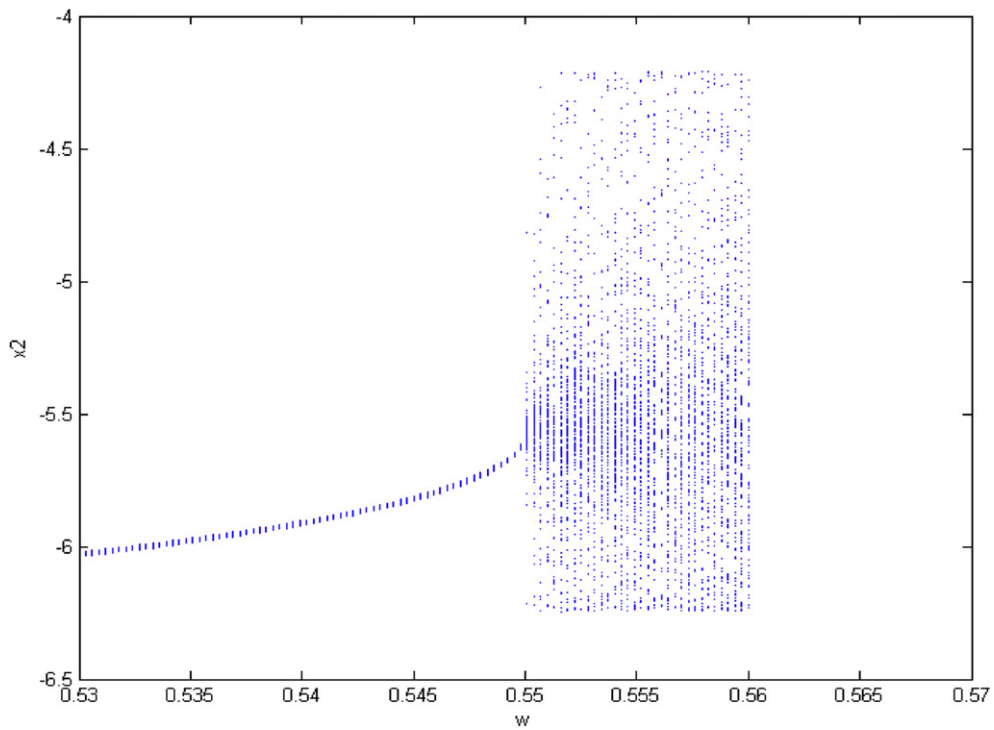


Fig. 32. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1.1$

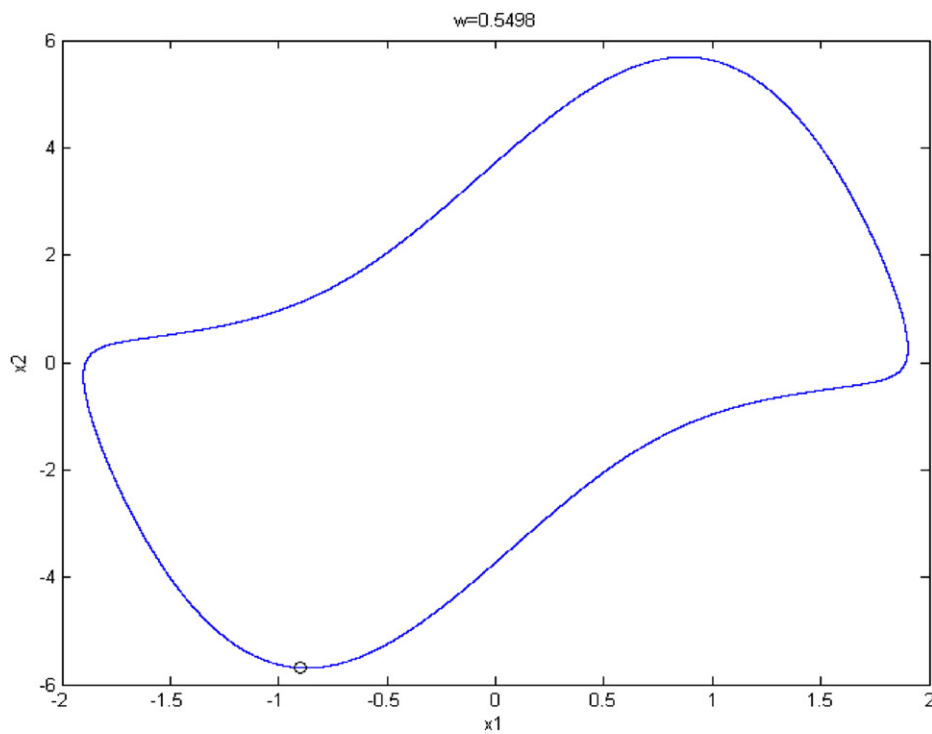


Fig. 33. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.5498$.

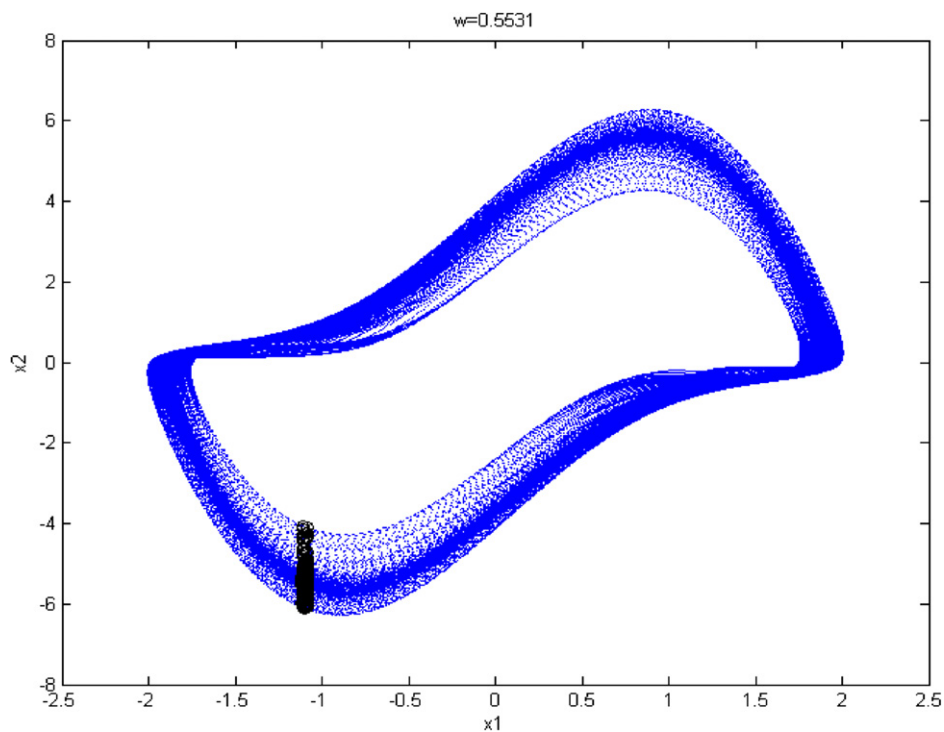


Fig. 34. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.5531$.

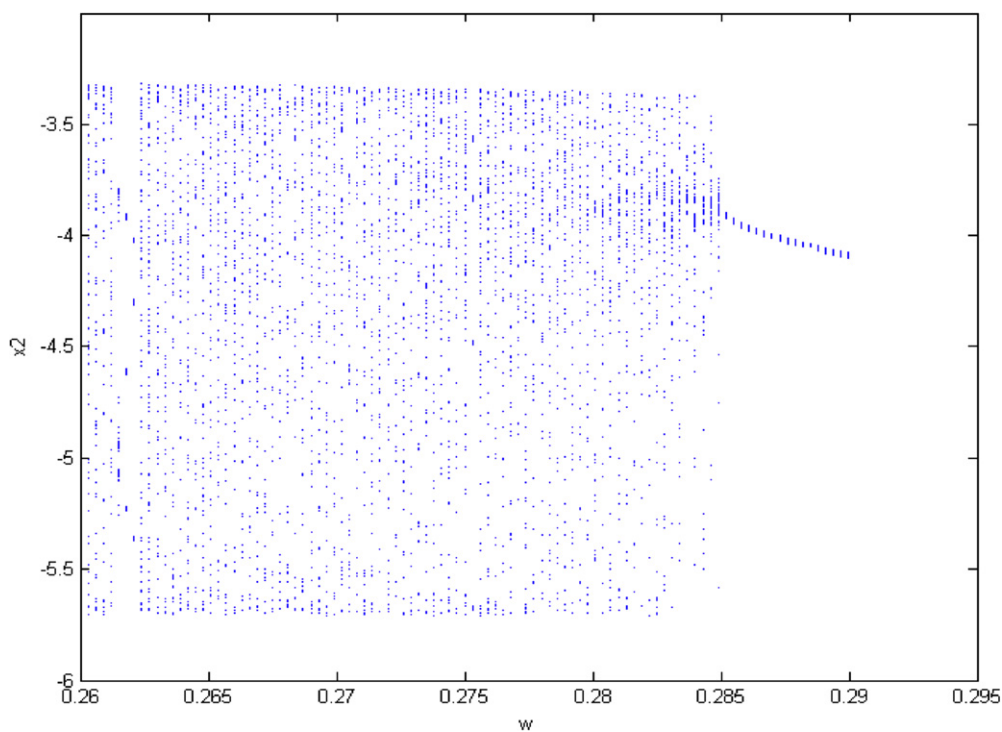


Fig. 35. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1.1$.

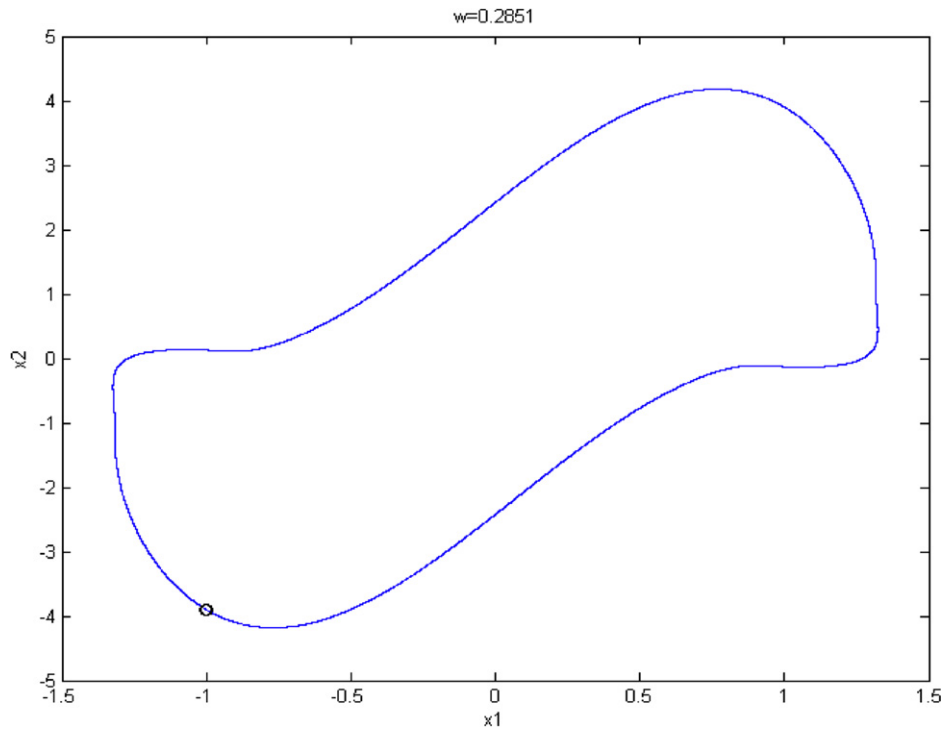


Fig. 36. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2851$.

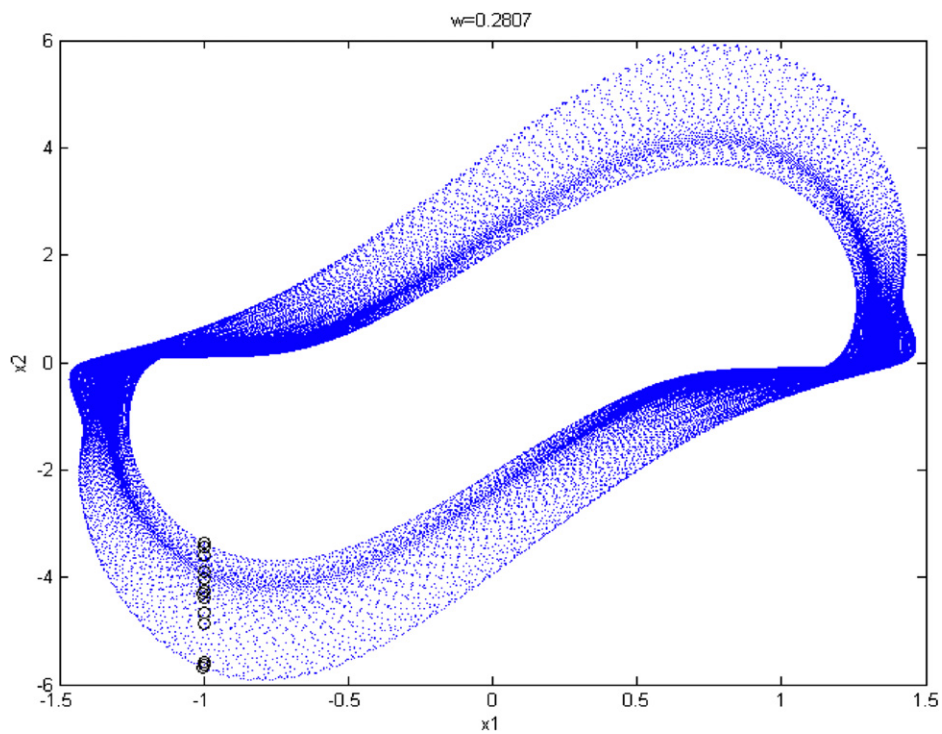


Fig. 37. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2807$.

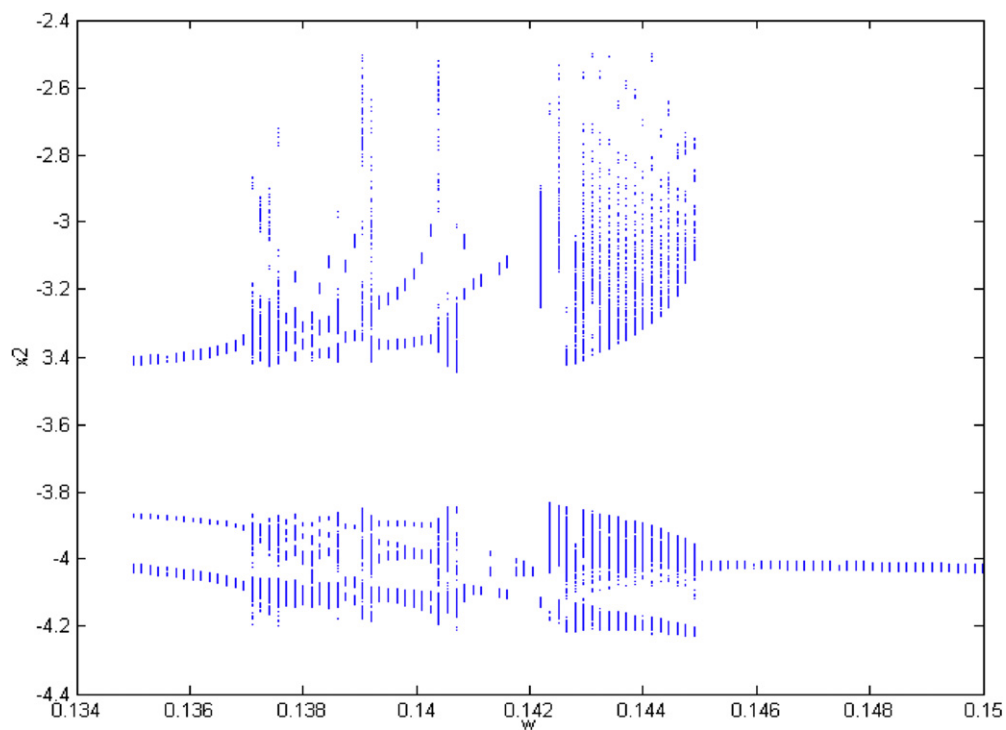


Fig. 38. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1.1$.

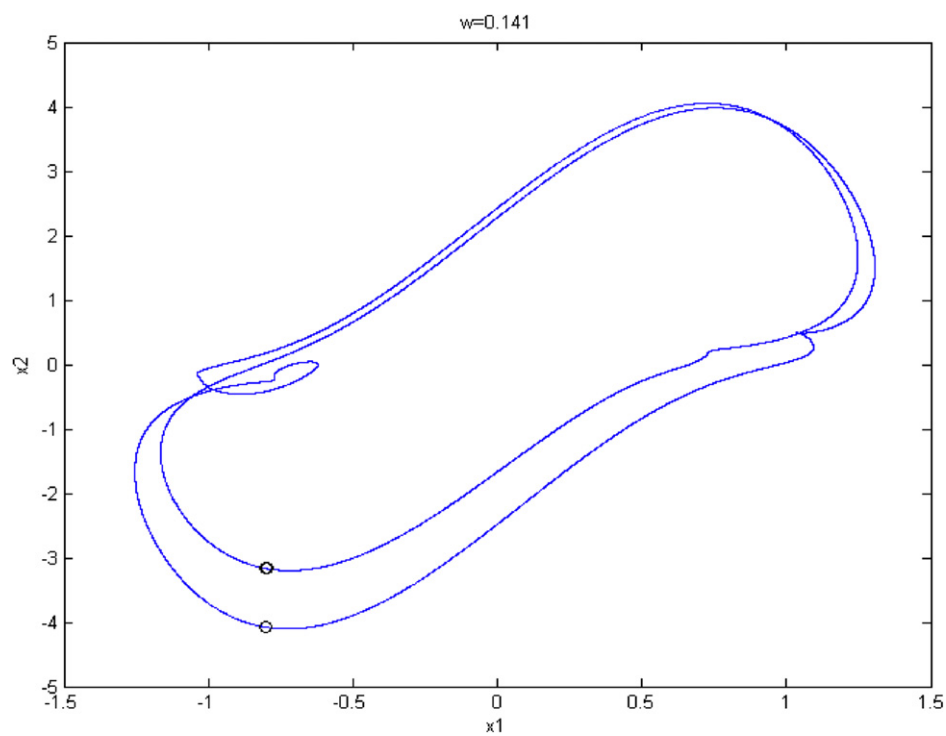


Fig. 39. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.141$.

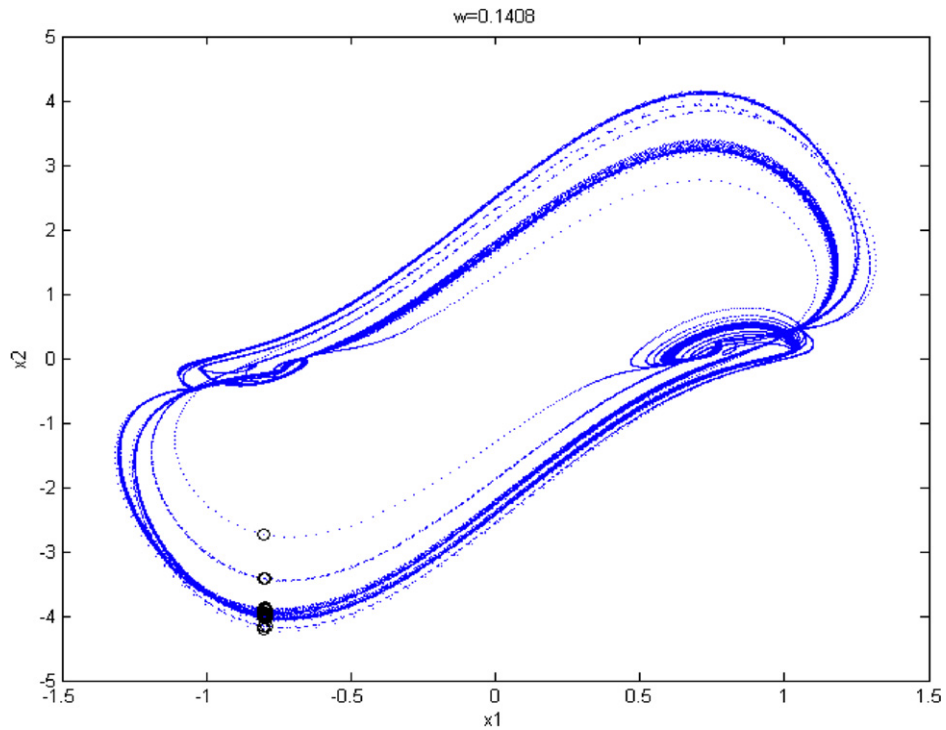


Fig. 40. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.1408$.

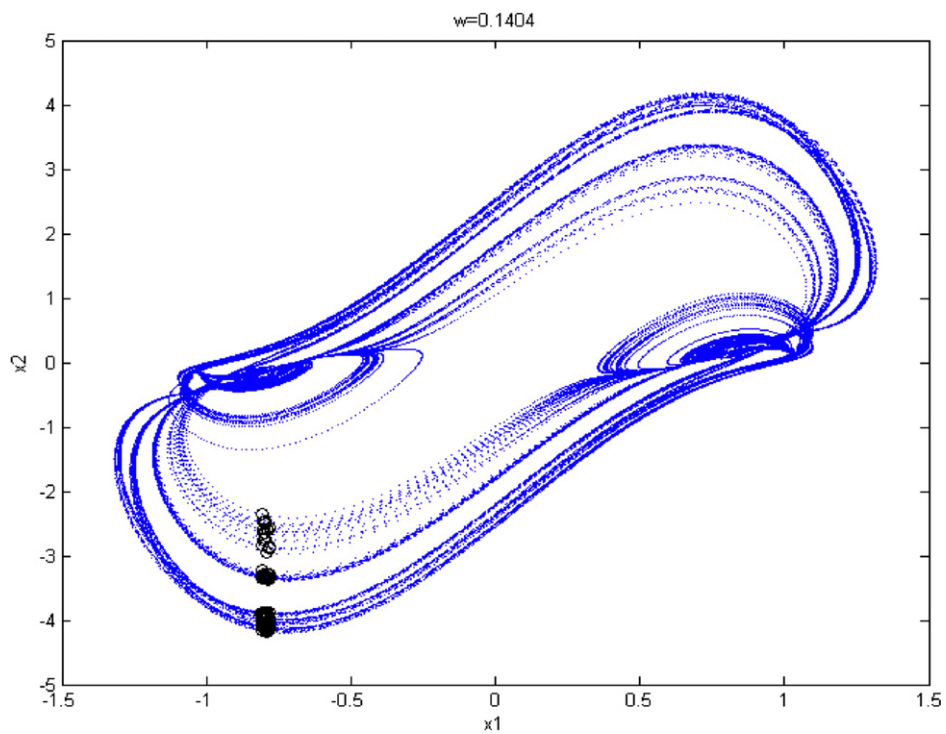


Fig. 41. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.1404$.

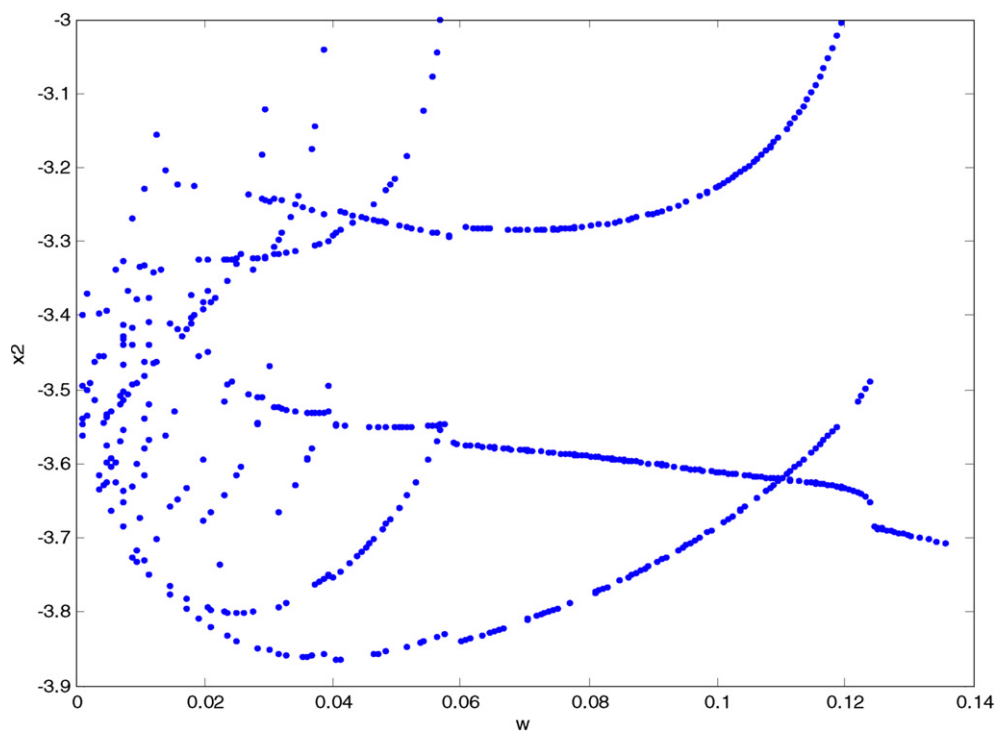


Fig. 42. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.6$, $\gamma = 1.1$.

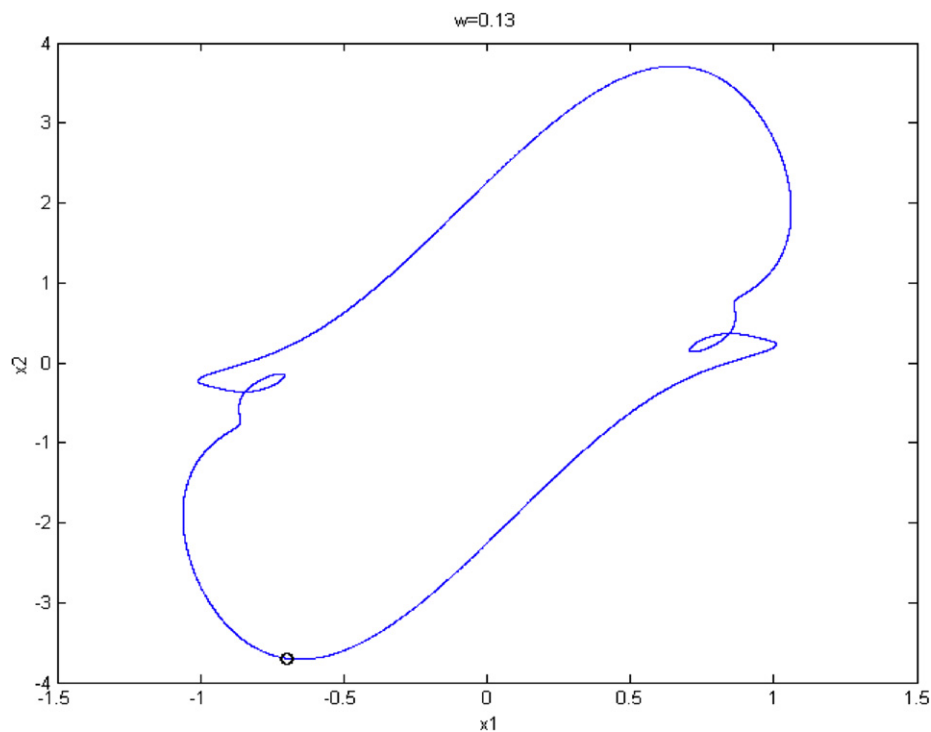


Fig. 43. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.13$.

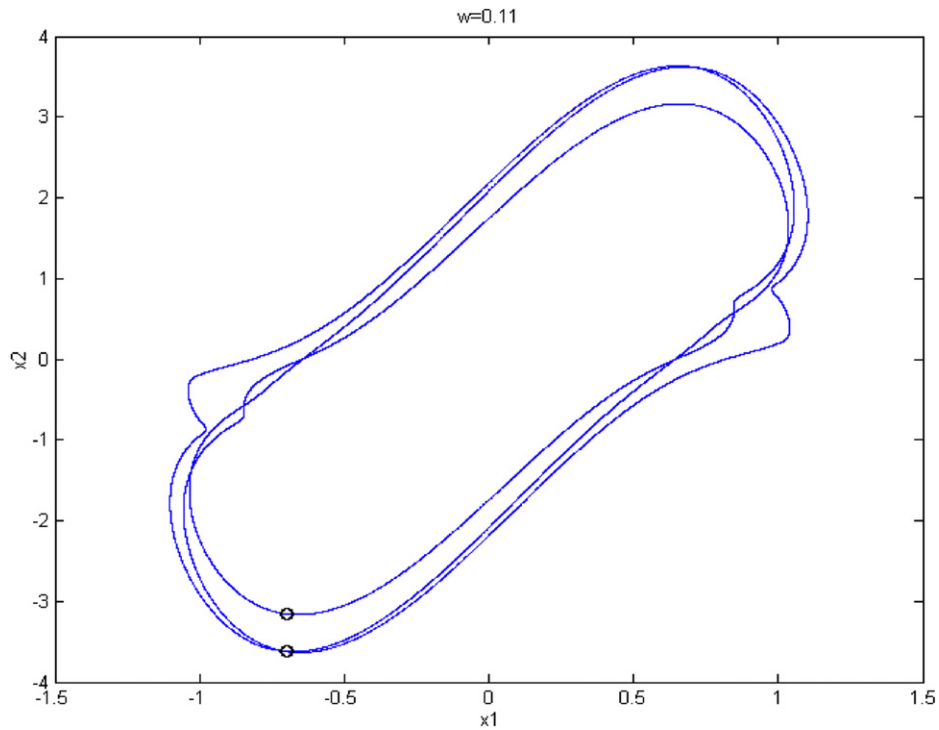


Fig. 44. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.11$.

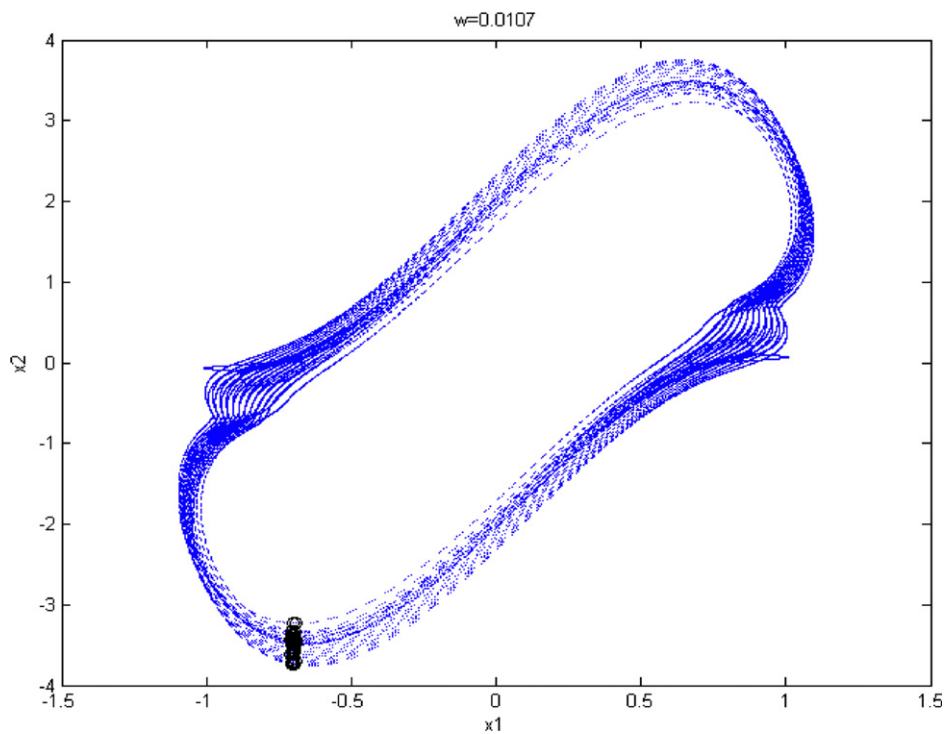


Fig. 45. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.0107$.

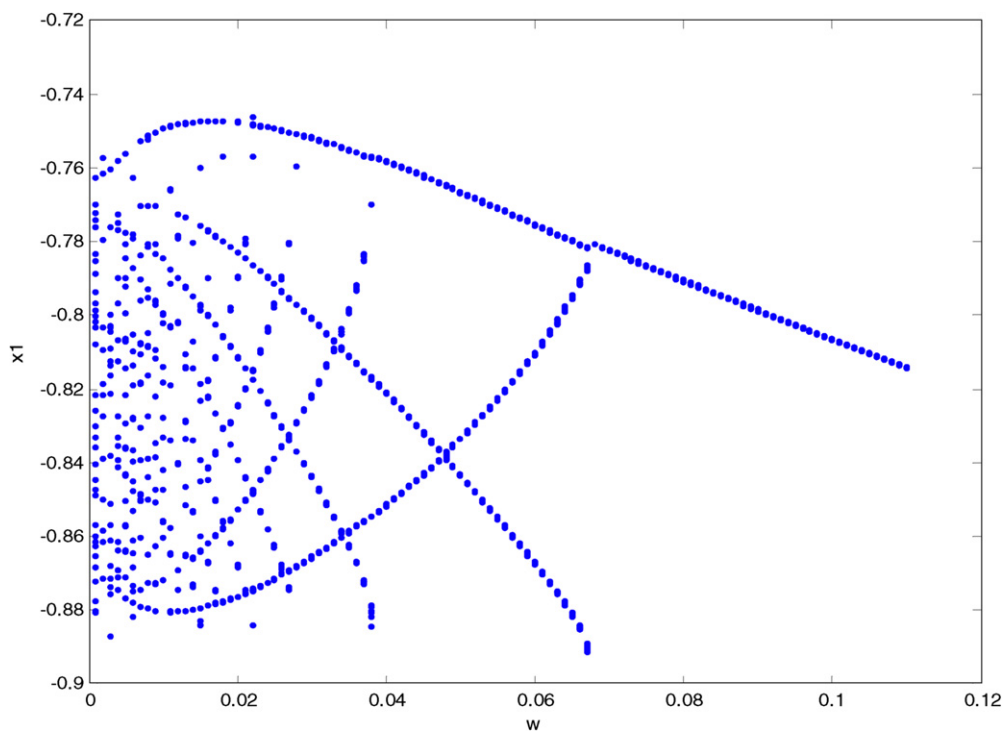


Fig. 46. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.5$, $\gamma = 1.1$.

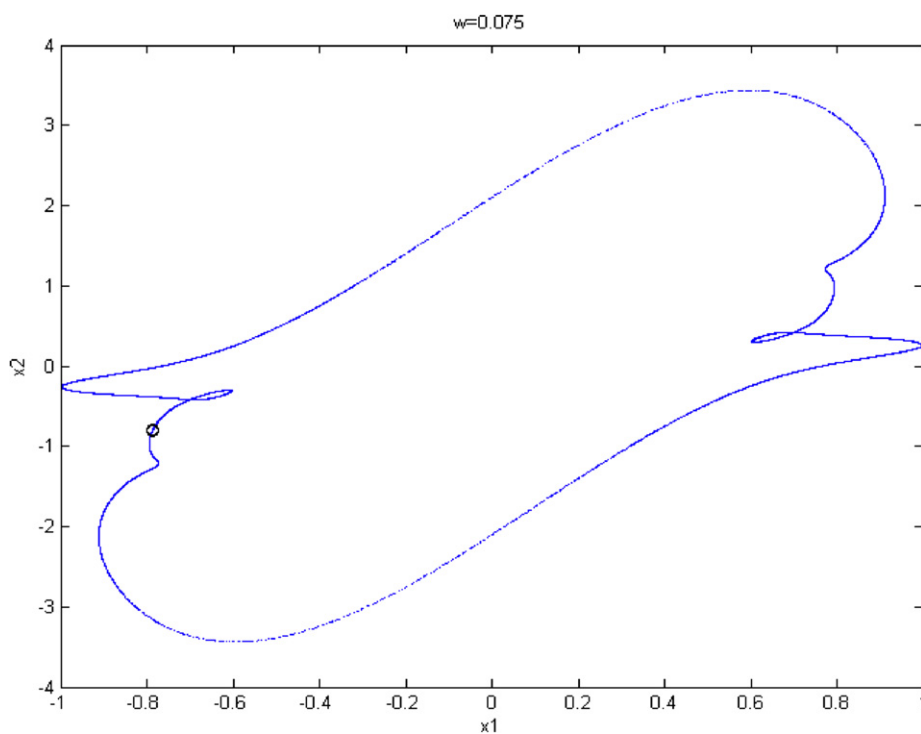


Fig. 47. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.075$.

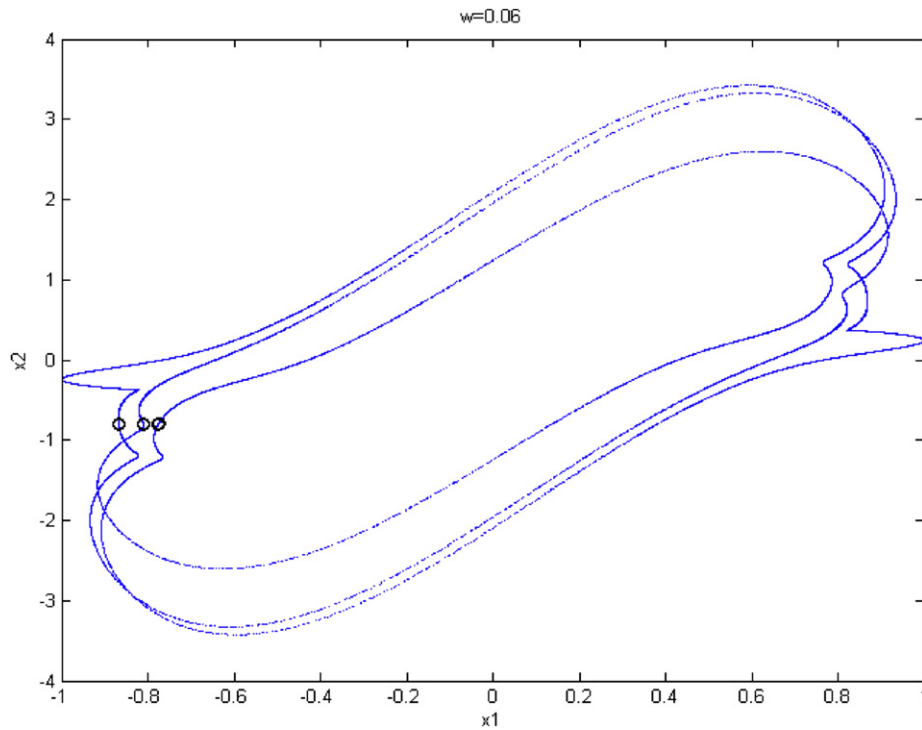


Fig. 48. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.06$.

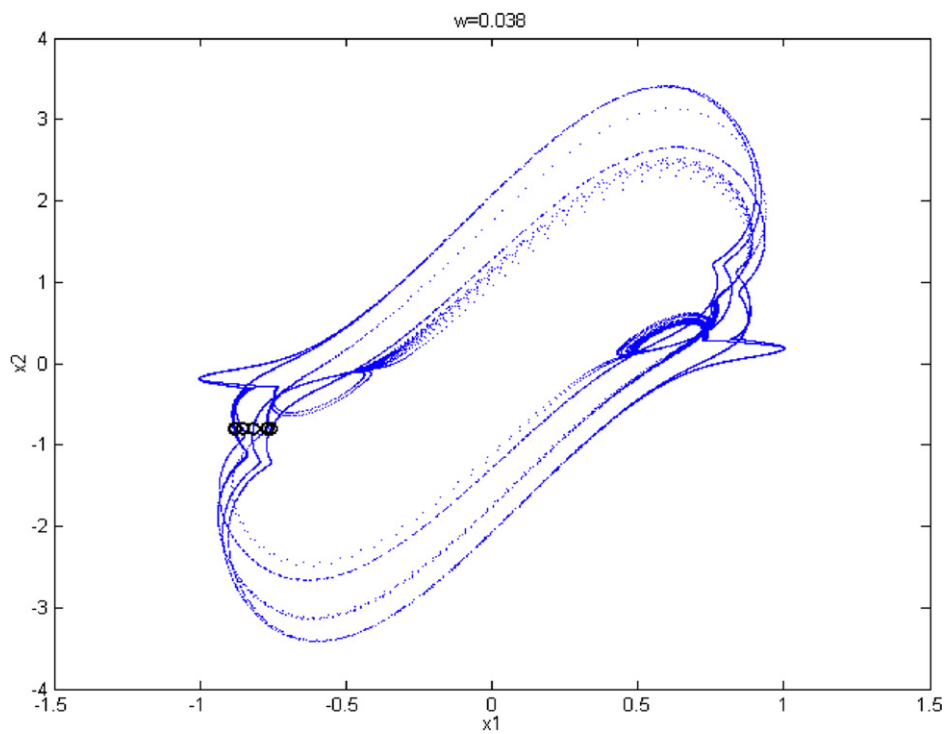


Fig. 49. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.038$.

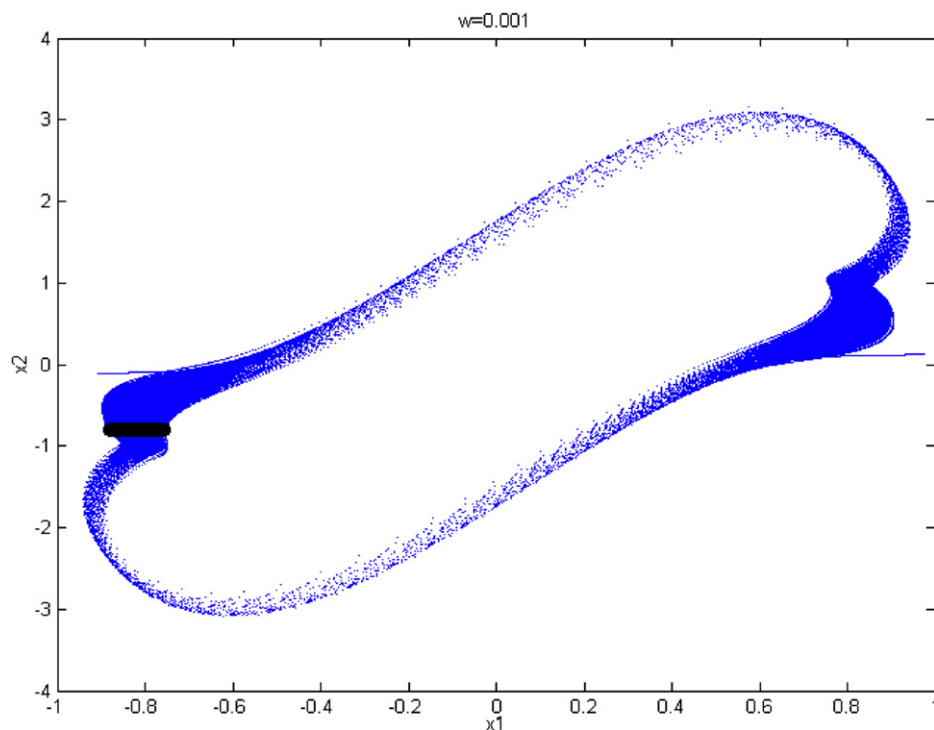


Fig. 50. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.001$.

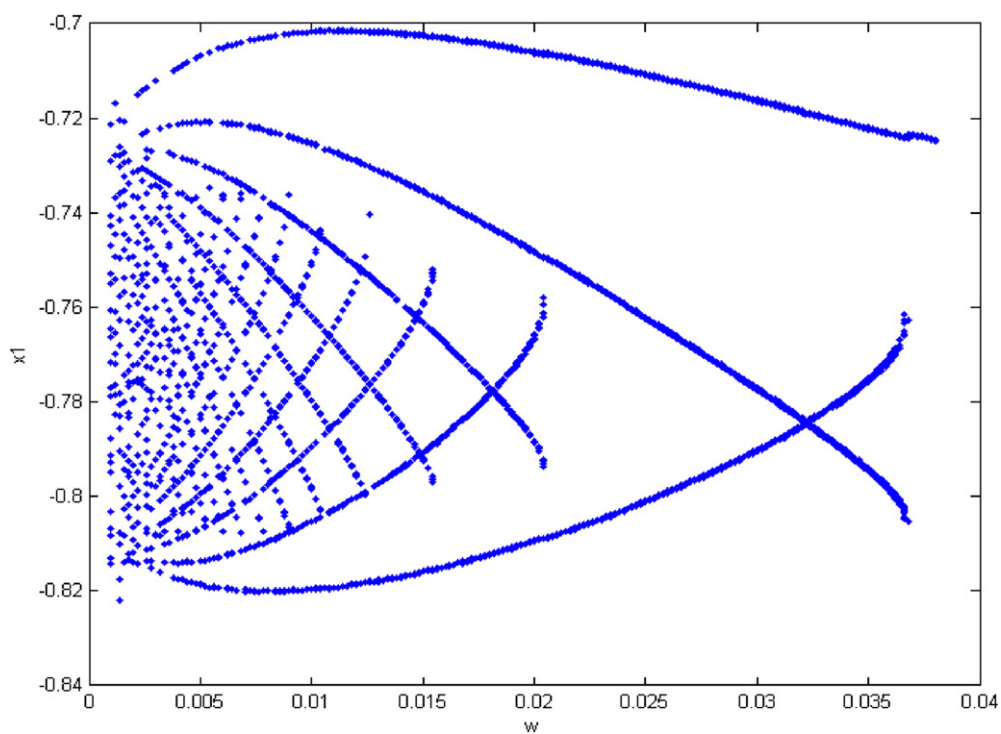


Fig. 51. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.4$, $\gamma = 1.1$.

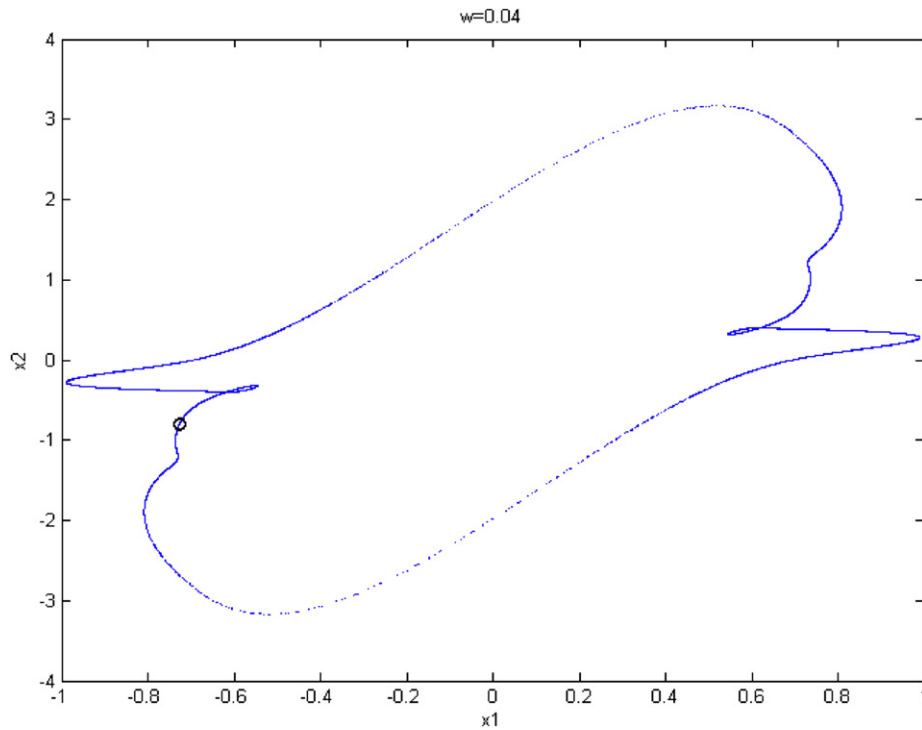


Fig. 52. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.04$.

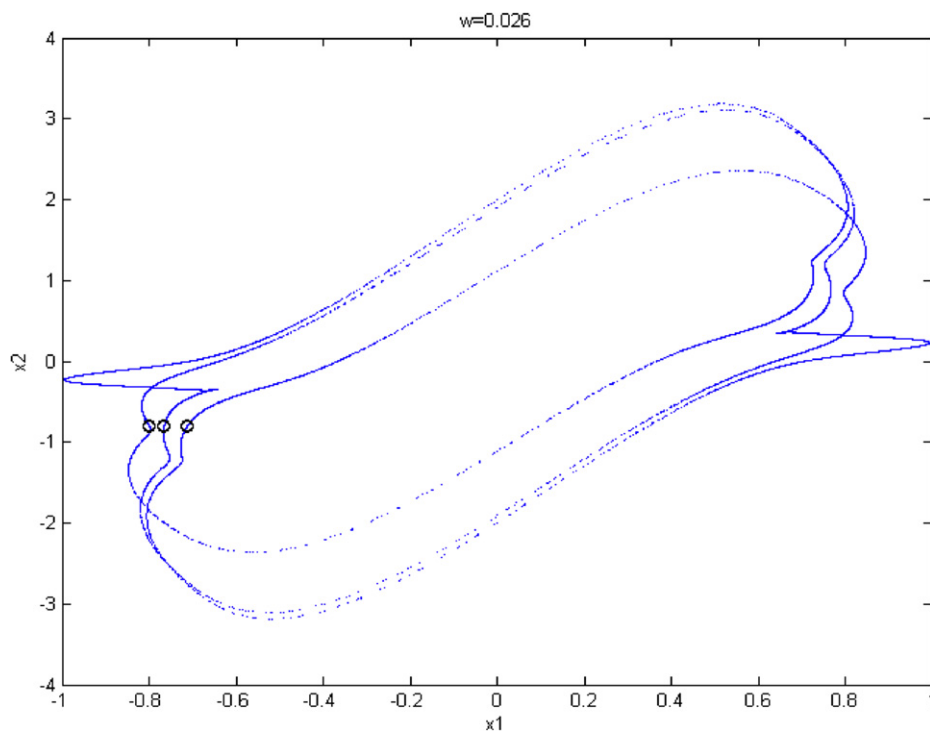


Fig. 53. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.026$.

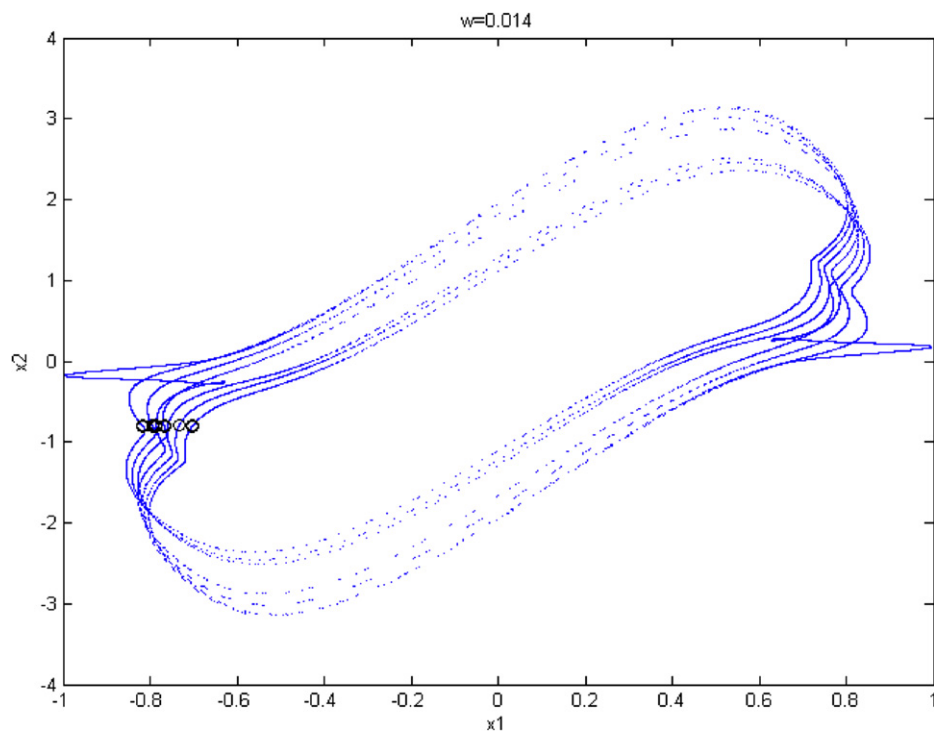


Fig. 54. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.014$.

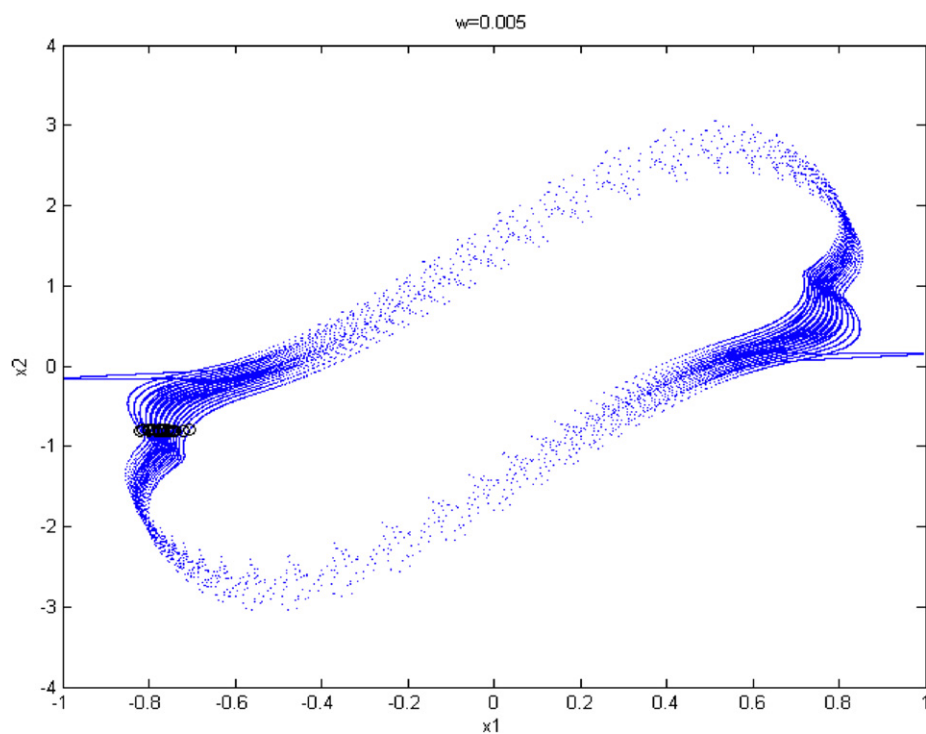


Fig. 55. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.005$.

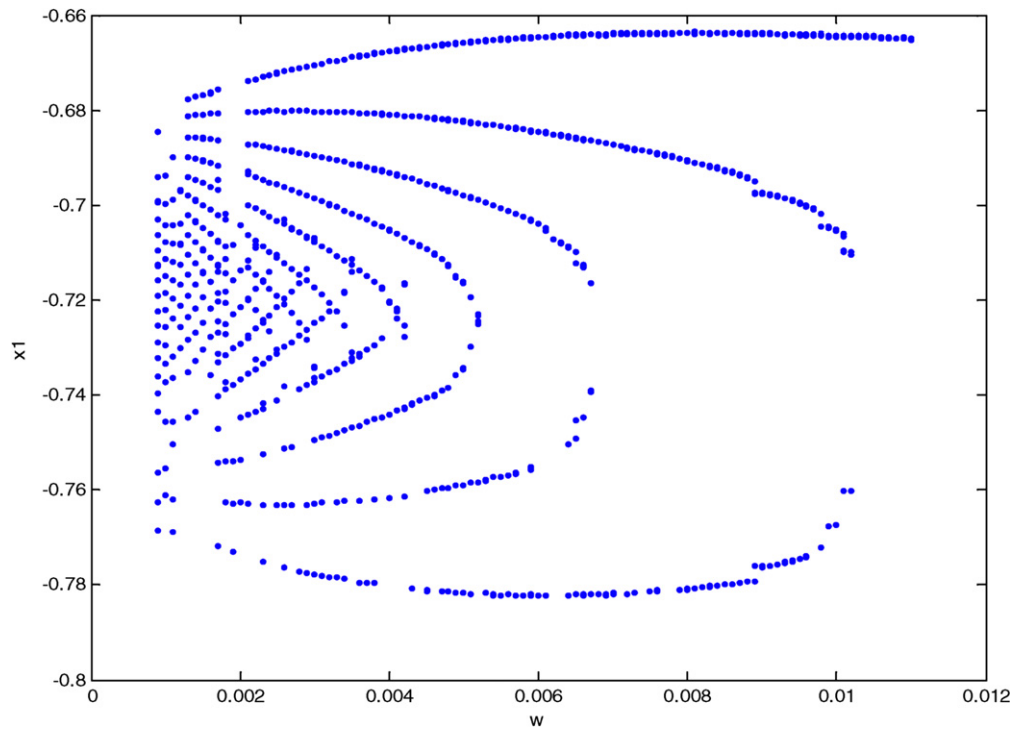


Fig. 56. The bifurcation diagram of the autonomous fractional order system with order $\alpha = \beta = 0.3$, $\gamma = 1.1$.

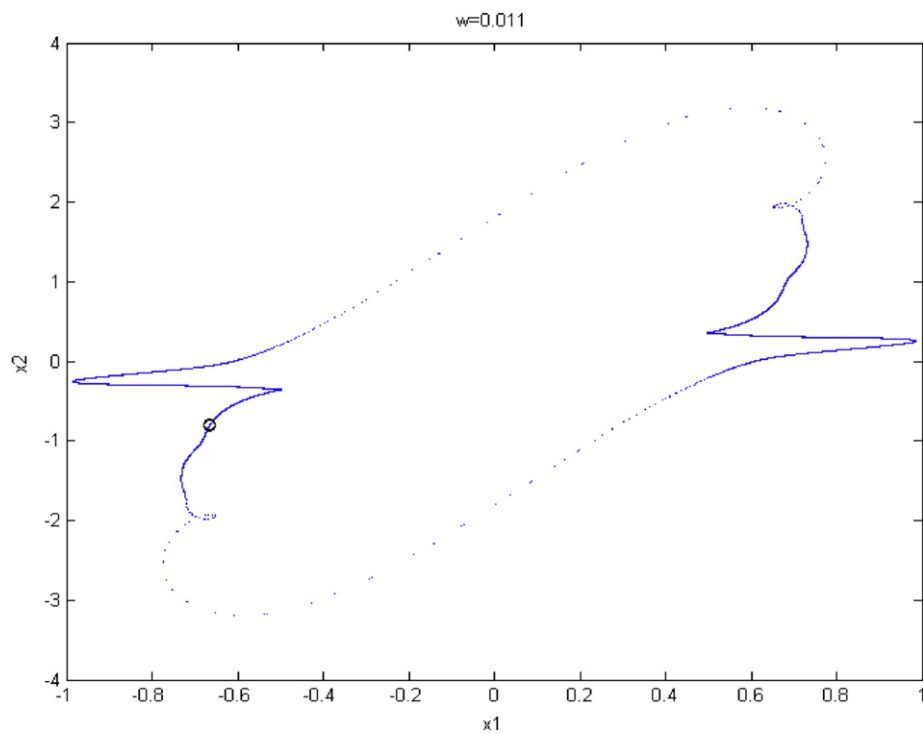


Fig. 57. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.011$.

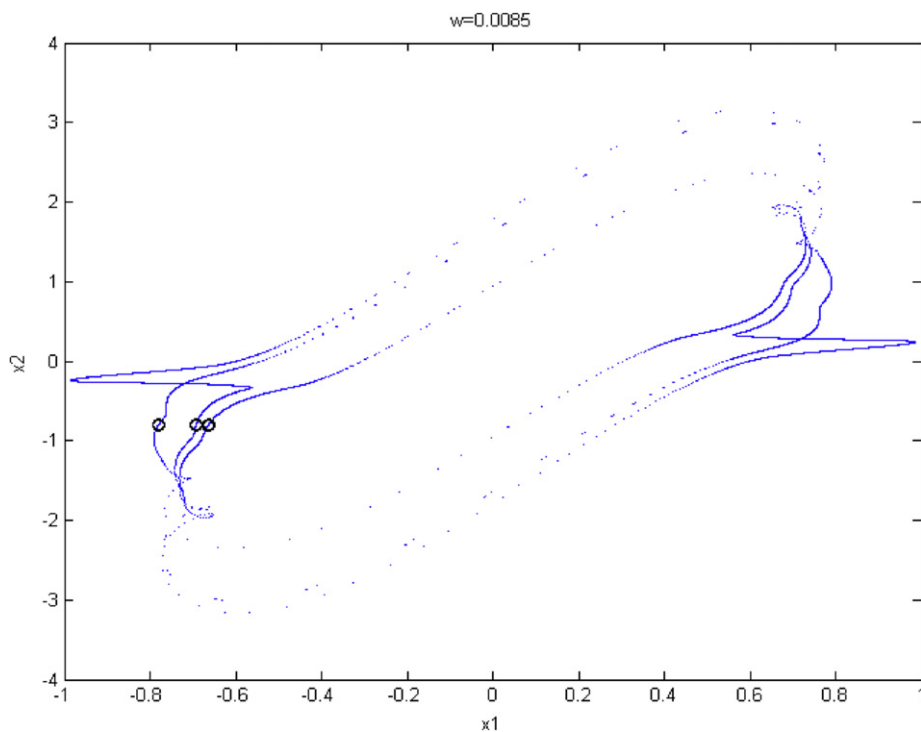


Fig. 58. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0085$.

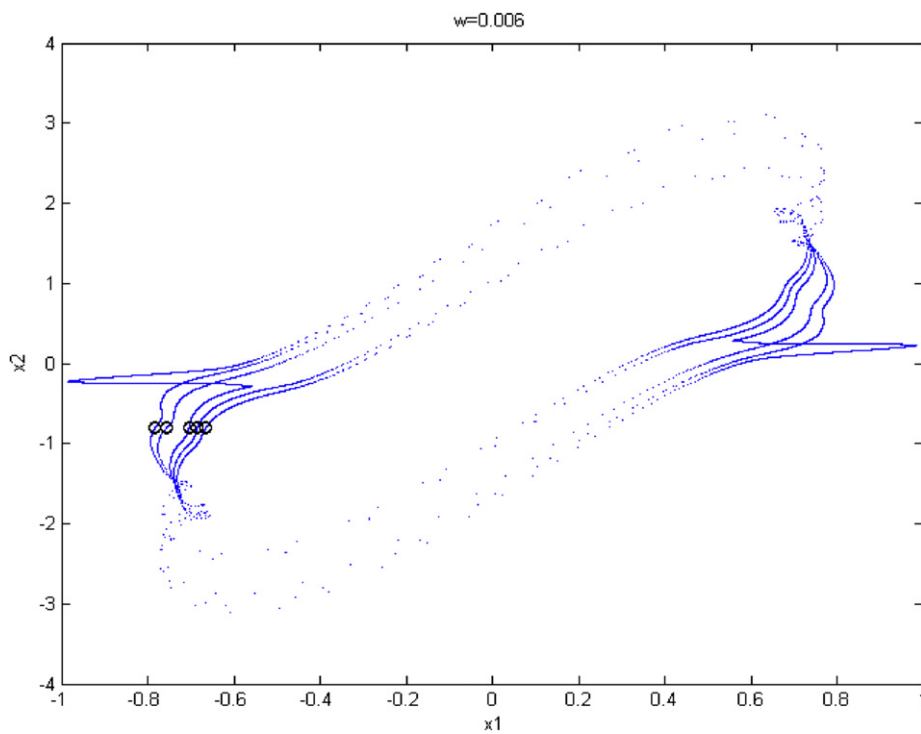


Fig. 59. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.006$.

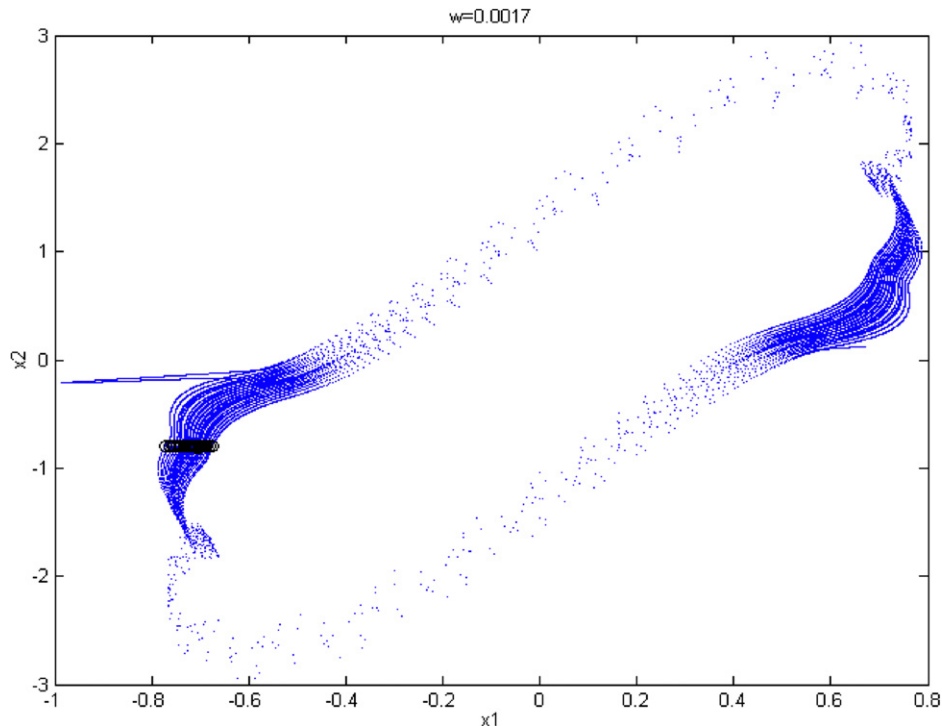


Fig. 60. The phase portrait and Poincaré maps of the autonomous fractional order system with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$.

the phase portraits and Poincaré maps with $\alpha = \beta = 0.7$. According to the results of simulation in Part 1, it is found that the chaotic motions exist in the nonautonomous system with fractional orders. The lowest total fractional order for chaos existence in this system is 1.4 (2×0.7). When the total fractional order is 1.2 (2×0.6), no chaos exists.

In Part 2, α , β , γ take the same fractional numbers, vary from 1.1 to 0.6 in steps of 0.1. Fig. 27 shows the bifurcation diagram and Figs. 28–31 are the phase portraits and Poincaré maps with $\alpha = \beta = \gamma = 1.1$. When $\alpha = \beta = \gamma$ and vary from 0.9 to 0.6 in steps of 0.1, no chaos exists.

In Part 3, $\gamma = 1.1$, α , β take same fractional numbers and vary from 1.1 to 0.3 in steps of 0.1. Fig. 32 shows the bifurcation diagram and Figs. 33 and 34 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.9$, $\gamma = 1.1$. Fig. 35 shows the bifurcation diagram and Figs. 36 and 37 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.8$, $\gamma = 1.1$. Fig. 38 shows the bifurcation diagram and Figs. 39–41 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.7$, $\gamma = 1.1$. Fig. 42 shows the bifurcation diagram and Figs. 43–45 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.6$, $\gamma = 1.1$. Fig. 46 shows the bifurcation diagram and Figs. 47–50 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.5$, $\gamma = 1.1$. Fig. 51 shows the bifurcation diagram and Figs. 52–55 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.4$, $\gamma = 1.1$. Fig. 56 shows the bifurcation diagram and Figs. 57–60 are the phase portraits and Poincaré maps with $\alpha = \beta = 0.3$, $\gamma = 1.1$. According to the results of simulation in Part 3, it is found that the chaotic motions exist when γ takes 1.1 and α , β vary from 0.9 to 0.3 in steps of 0.1. When α , β take the fractional number less than 0.3, no chaos is found.

$\gamma = 0.9$, α , β take the same fractional numbers 1.1, 1.2, 1.3 and 1.4, only periodic motions are found, no chaos exists.

5. Concluding remarks

In this paper, chaos of a generalized van der Pol system both with integer order and with fractional orders is studied. Both autonomous and nonautonomous systems are studied in some detail. It is found that chaotic motions exist in the nonautonomous generalized van der Pol system excited by a sinusoidal time function. For fractional order systems, when $\gamma = 1$, α , β vary from 1.1 to 0.7 in the steps of 0.1, chaos exists. Next, chaotic motions exist when the fraction orders $\alpha = \beta = \gamma = 1.1$. When $\gamma = 1.1$ with $\alpha = \beta$ varying from 1.1 to 0.3, chaotic motions also exist.

Acknowledgement

This research was supported by the National Science Council, Republic of China, under Grant No. NSC94-2212-E-009-013.

References

- [1] Glass L. *Theory of heart*. New York, Heidelberg, Berlin: Springer; 1990.
- [2] Mahmoud Gamal M, Farghaly Ahmed AM. Chaos control of chaotic limit cycles of real and complex van der Pol oscillators. *Chaos, Solitons & Fractals* 2004;21:915–24.
- [3] Ge Zheng-Ming, Chen Yen-Sheng. Synchronization of unidirectional coupled chaotic systems via partial stability. *Chaos, Solitons & Fractals* 2004;21:101–11.
- [4] Ge Zheng-Ming, Leu Wei-Ying. Anti-control of chaos of two-degrees-of-freedom loudspeaker system and chaos synchronization of different order systems. *Chaos, Solitons & Fractals* 2004;20:503–21.
- [5] Ge Zheng-Ming, Leu Wei-Ying. Chaos synchronization and parameter identification for loudspeaker system. *Chaos, Solitons & Fractals* 2004;21:1231–47.
- [6] Ge Zheng-Ming, Chen Chien-Cheng. Phase synchronization of coupled chaotic multiple time scale systems. *Chaos, Solitons & Fractals* 2004;20:639–47.
- [7] Ge Z-M, Chang C-M. Chaos synchronization and parameter identification of single time scale brushless dc motors. *Chaos, Solitons & Fractals* 2004;20:883–903.
- [8] Ge Zheng-Ming, Cheng Jui-Wen, Chen Yen-Sheng. Chaos anticontrol and synchronization of three time scales brushless dc motor system. *Chaos, Solitons & Fractals* 2004;22:1165–82.
- [9] Ge Z-M, Cheng J-W. Chaos synchronization and parameter identification of three time scales brushless dc motor system. *Chaos, Solitons & Fractals* 2005;24:597–616.
- [10] Ge Zheng-Ming, Lee Ching-I. Control, anticontrol and synchronization of chaos for an autonomous rotational machine system with time-delay. *Chaos, Solitons & Fractals* 2005;23:1855–64.
- [11] Ge Zheng-Ming, Chen Yen-Sheng. Adaptive synchronization of unidirectional and mutual coupled chaotic systems. *Chaos, Solitons & Fractals* 2005;26:881–8.
- [12] Chen Hsien-Keng, Ge Zheng-Ming. Bifurcation and chaos of a two-degree-of freedom dissipative gyroscope. *Chaos, Solitons & Fractals* 2005;24:125–36.
- [13] Jiang G, Zheng W, Chen G. Global chaos synchronization with channel time-delay. *Chaos, Solitons & Fractals* 2004;20:267–75.
- [14] Chen G, Liu S. On generalized synchronization of spatial chaos. *Chaos, Solitons & Fractals* 2003;15:311–8.
- [15] Li Z, Xu D. A secure communication scheme using projective chaos synchronization. *Chaos, Solitons & Fractals* 2004;22:477–81.
- [16] Jiang Guo-Ping, Tang Wallace Kit-Sang, Chen Guanrong. A simple global synchronization criterion for coupled chaotic systems. *Chaos, Solitons & Fractals* 2003;15:925–35.
- [17] Ge Z-M, Yu C-Y, Chen Y-S. Chaos synchronization and anti-control of a rotational supported simple pendulum. *JSME Int J, Ser C* 2004;47:233–41.
- [18] Ge Z-M, Hsiao C-L, Chen Y-S. Nonlinear dynamics and chaos control for a time delay Duffing system. *Int J Nonlinear Sci Numer Simul* 2005;6:187–99.
- [19] Bagley RL, Calico RA. Fractional order state equations for the control of viscoelastically damped structures. *J Guid Control Dynam* 1991;14:304–11.
- [20] Li C, Chen G. Chaos in the fractional order Chen system and its control. *Chaos, Solitons & Fractals* 2004;22:549–54.
- [21] Ahmad Wajdi M, Harb Ahmad M. On nonlinear control design for autonomous chaotic systems of integer and fractional orders. *Chaos, Solitons & Fractals* 2003;18:693–701.
- [22] Ahmad Wajdi M, Reyad El-Khazali, Yousef Al-Assaf. Stabilization of generalized fractional order chaotic systems using state feedback control. *Chaos, Solitons & Fractals* 2004;22:141–50.
- [23] Ahmad WM. Hyperchaos in fractional order nonlinear systems. *Chaos, Solitons & Fractals* 2005;26:1459–65.
- [24] Nimmo S, Evans AK. The effects of continuously varying the fractional differential order of chaotic nonlinear systems. *Chaos, Solitons & Fractals* 1999;10:1111–8.
- [25] Ahmad Wajdi M, Sprott JC. Chaos in a fractional order autonomous nonlinear systems. *Chaos, Solitons & Fractals* 2003;16:339–51.
- [26] Barbosa Ramiro S, Tenreiro Machado JA, Ferreira Isabel M, Tar József K. Dynamics of the fractional order van der Pol oscillator. In: *Proceedings of the second IEEE International Conference on Computational Cybernetics (ICCC'04)*, ISBN 3-902463-01-5, 2004. p. 373–8.
- [27] Li Changpin, Peng Guojun. Chaos in Chen's system with a fractional order. *Chaos, Solitons & Fractals* 2004;22:443–50.
- [28] Charef A, Sun HH, Tsao YY, Onaral B. Fractal system as represented by singularity function. *IEEE Trans Automat Contr* 1992;37:1465–70.
- [29] Hartley TT, Lorenzo CF, Qammer HK. Chaos in a fractional order Chua's system. *IEEE Trans CAS—I* 1995;42:485–90.
- [30] van der Pol B. Forced oscillations in a circuit with nonlinear resistance (receptance with reactive triode). *London, Edinburgh, and Dublin Philosophical Magazine* 1927;3:65–80.

- [31] Bi Qinsheng. Dynamical analysis of two coupled parametrically excited van der Pol oscillators. *Int J Nonlinear Mech* 2004;39:33–54.
- [32] dos Santos AM, Lopes SR, Viana RL. Rhythm synchronization and chaotic modulation of coupled van der Pol oscillators in a model for the heartbeat. *Physica A* 2004;338:335–55.
- [33] Addo-Asah W, Akpati HC, Mickens RE. Investigation of a generalized van der Pol oscillator differential equation. *J Sound Vibr* 1995;179:733–5.
- [34] Waluya SB, van Horssen WT. On the periodic solutions of a generalized nonlinear van der Pol oscillator. *J Sound Vibr* 2003;268:209–15.
- [35] Mickens RE. Analysis of nonlinear oscillators having non-polynomial elastic terms. *J Sound Vibr* 2002;255:789–92.