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非線性零壹整數規劃在人員指派最佳化之應用

A Note on Optimization of a Nonlinear 0-1 Integer Program for Assigning Persons to Certain Projects

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摘要：本文旨在建立一個使人員配置最佳化以得到最大獲利之數學模型，同時敘述找出有效率的解法之定理，並舉範例一則以展示解法。接著提出一個廣義的人員指派問題之規劃，將其轉化爲一個整數規劃模型，導出一個吾人認爲最有效指派方案。實證結果顯示提出的模型能有效解決限量人員配置問題，在管理意涵上，可以作爲業界處理人力資源之參考。

關鍵詞：整數規劃、演算法、非線性、人力派遣

Abstract: A model is developed for assigning persons to certain projects so that the assignment maximized the profit. Also, a theorem is stated and, consequently, an efficient solution algorithm is proposed and an example is presented. A mathematical programming model for the extended assignment problem is proposed, which is then expressed as a classical integer linear programming model to determine the assignments with the maximum efficiency. Experimental results showed that the proposed model has solved the capacity allocation problem efficiently.

***Key words**:* Integer Program, Algorithm, Nonlinear, Assignment

1. Introduction

When a new item is innovated by a company, usually, the company can own the patent of the new item for a certain period of time. Thus, the company is able to monopoly the trade for the said period of time. However, a new item does not mean a profitable one. In fact, there usually is a gap between the two and it needs to take some efforts. This work develops a model and the solution algorithm to find an optimal assignment from the available manpower so that the profit is maximized. Genetic algorithm can avoid many improper solutions, and speed up the searching efficiency (Wang, 2002). Toroslu and Arslanoglu (2007) proposed that genetic algorithms are proven to be very successful for multi-objective optimization problems. This paper also presented a genetic algorithm as an aid for project assignment. Chu and Beasley（1997）indicated that for solving the assignment problem of allocating personnel, managers emphasized the effective manpower is practically important in certain professions. The resulting assignments can achieve the maximum efficiency in resources utilization (Chen and Lu, 2007).

First, to formulate the model/mathematical programming of the problem above, and then find an optimal decision/solution for .the model/ programming

Let $\{J_j : j = 1,2,\cdots, N\}$ be a collection of N projects (e.g. the above mentioned new items) for each j, it requires a_j unit of time to finish the project J_j when one person working on the project alone. The available time for J_j is T_j, that is to say J_j must be accomplished in a period of T_j. In order to make an optimal decision, the total number of persons available for the assignments is K. The 0-1 integer programming (Flouda, 1995, Hiller and Liberman, 2005) is formulated below to determine an assignment so that the overall profit is maximized.

To formulate the model, first, the variables are defined, for each j=1,2,…,N, let x_j be the number of persons assigned to the project J_j. Let P_j the unit profit of the item over the time period is T_j. Then, when x_j persons are assigned to project J_j, clearly, the project J_j produces the profit $P_j(T_j - \frac{a_i}{x_j})$.

Thus, the objective of this program is to maximize the total profit $\sum_{j=1}^{N} P_j(T_j - \frac{a_i}{x_j})$.

The problem to be solved is to find this maximum under the constraint that the total number of persons (manpower) to be assigned is K.

Thus, the solution of the 0-1 integer program below provides the optimal assignment.

$$\max \sum_j P_j \left(T_j - \frac{a_j}{x_j} \right)$$

$$s.t.$$

$$\sum_j x_j \le K$$

$$x_j \ge \frac{a_j}{T_j}$$

$$x_j \in Z \quad \text{and } x_j \ge 1, \forall j = 1, 2, \cdots, N, .$$

where P_j, T_j and a_j are positive real numbers
and Z is the set of all integers

This is a nonlinear 0-1 integer program (Pedregal, 2004, Lazimy, 1982), in order to solve it efficiently, an algorithm for solving this type of the problems is presented in the following section.

2. Algorithm

To solve this problem, the model can be reduced to the following form

$$\text{(P)} \qquad \begin{aligned} &\min \sum_j \frac{P_j a_j}{x_j} \\ &s.t. \sum_j x_j \le K \\ &\quad x_j \ge \frac{a_j}{T_j} \\ &\quad x_j \in Z \end{aligned}$$

Let $A_j = P_j a_j$ and $Q_j = \dfrac{a_j}{T_j}$, then (P) can be rewritten as

$$\begin{aligned} &\min \sum_j \frac{A_j}{x_j} \\ &s.t. \sum_j x_j \le K \\ &\quad x_j \ge Q_j \\ &\quad x_j \in Z \end{aligned} \qquad (1)$$

This is a non linear integer program. Usually, solving a non linear program can be difficult. This paper will develop a solution algorithm for (P). It is easy to see that if $x_1, x_2, \cdots, x_N$ is optimal then $\sum_j x_j = K$. To solve (P), a sequence $x(k) = (x_1(k), x_2(k), \cdots, x_N(k))$, $k = 0,1,2,\cdots,l$ is constructed iteratively. Let the initial value be $x(0) = (x_1(0), x_2(0), \cdots, x_N(0))$

The following three statements are clear.

If $\sum_j x_j(0) > K$ then the problem is not feasible.

If $\sum_j x_j(0) = K$ then the initial solution is the unique solution.

If $\sum_j x_j(0) < K$ then the problem can be solved as follows:

For $m = 1, 2, \cdots, l$

Let $$\max_j \left\{ \frac{A_j}{x_j(m)} - \frac{A_j}{x_j(m)+1} \right\} = \frac{A_{j_0}}{x_{j_0}(m)\left(x_{j_0}(m)+1\right)} \quad (2)$$

$$x_j(m+1) = \begin{cases} x_j(m) & \text{if } j \neq j_0 \\ x_j(m)+1 & \text{if } j = j_0 \end{cases} \quad \text{for} \quad j = 1,2,\cdots,N$$

If $\sum_j x_j(m+1) = K$ then $x(m+1) = \left(x_1(m+1), x_2(m+1), \cdots, x_N(m+1)\right)$, is an optimal solution. Otherwise next m.

The above algorithm is a direct consequence of the Theorem below (Liu *et al.*, 2006; Liu and Konvalina, 1991):

THEOREM

Let $x(m) = (x_1(m), x_2(m)), \cdots, x_N(m))$ be a vector of N variables for each $m \in \{0,1,2,\cdots,l\}$.

Also let $j_0 \in \{1, 2, \cdots, N\}$ such that

$$\frac{A_{j_0}}{x_{j_0}(m)(x_{j_0}(m)+1)} = \max_j \left(\frac{A_j}{x_j(m)} - \frac{A_j}{x_j(m)+1} \right) \quad (3)$$

Given $x_j(0) = 1$ for all $j = 1,2,\cdots,N$.

Let

$$x_j(m+1)=\begin{cases} x_j(m) & \text{if } j \neq j_0 \\ x_j(m)+1 & \text{if } j = j_0 \end{cases} \quad \text{for} \quad j=1,2,\cdots,N \tag{4}$$

Then $\sum_j x_j(m+1)=K$ implies $x(m+1)=\left(x_1(m+1),x_2(m+1),\cdots,x_N(m+1)\right)$ is an optimal solution of (P).

Proof. The theorem can be proven by mathematical induction. The proof is omitted.

Intuitively, the sequence constructed in theorem is a monotonic sequence of vectors in R^N .The consecutive terms in the sequence are differed by one component with value 1. In each iteration, the term that maximizes the difference of the objective function value is chosen. Thus, to choose *j* such that $\frac{A_j}{x_j(m)}-\frac{A_j}{x_j(m)+1}$ is greater than $\frac{A_t}{x_t(m)}-\frac{A_t}{x_t(m)+1}$ for all $t=1,2,\cdots,N$.

This algorithm is very efficient since for all *j*, $\frac{A_j}{x_j(m)}-\frac{A_j}{x_j(m)+1}$ can be calculated simultaneously. This is a simple but powerful algorithm. Next, an example is presented to illustrate the algorithm.

3. Example

$$T_1=18,\quad T_2=22,\quad T_3=20 \text{ and } \mathrm{T}_4=34$$

$$P_1=108,\quad P_2=122,\quad P_3=120 \text{ and } P_4=93$$

$$a_1=24,\quad a_2=28,\quad a_3=18 \text{ and } a_4=32$$

Consider 2 cases: Case 1 $K=5$

Case 2 $K=10$

Solution:

$$Q_1=\frac{a_1}{T_1}=\frac{24}{18},\quad Q_2=\frac{a_2}{T_2}=\frac{28}{22}\quad Q_3=\frac{a_3}{T_3}=\frac{18}{20}\quad Q_4=\frac{a_4}{T_4}=\frac{32}{34}$$

The initial solution

$$x_1(0)=2,\quad x_2(0)=2,\quad x_3(0)=1,\quad x_4(0)=1$$

Case 1: The problem is infeasible since $\sum_j x_j(0)=6>K=5$

Case 2: $K = 10$ this is a standard problem, with $x(0)$ above as an initial solution. $A_1 = 108 \times 24 = 2596$, $A_2 = 3416$, $A_3 = 2160$, $A_4 = 2974$

The integer program as follows:

$$\min \frac{2596}{x_1} + \frac{3416}{x_2} + \frac{2160}{x_3} + \frac{2974}{x_4}$$
$$s.t. \quad x_1 + x_2 + x_3 + x_4 \le 10$$
$$x_1 \ge 24/18$$
$$x_2 \ge 28/22$$
$$x_3 \ge 18/20$$
$$x_4 \ge 32/34$$
$$x_1, x_2, x_3, x_4 \in Z$$

This problem is solved by our proposed algorithm as follows:

$$D_j(m) = \frac{A_j}{x_j(m)} - \frac{A_j}{x_j(m)+1} = \frac{A_j}{x_j(m)(x_j(m)+1)}$$

Solution Table

j	1	2	3	4	$\sum x_j(m)$
$x(0)$	2	2	1	1	6
$D_j(0)$	$\frac{2596}{6} = 432.6$	$\frac{3416}{6} = 569.6$	$\frac{2160}{2} = 1080$	$\frac{2976}{2} = 1488$	
$x(1)$	2	2	1	1+1=2	7
$D_j(1)$	$\frac{2596}{6} = 432.6$	$\frac{3416}{6} = 569.6$	$\frac{2160}{2} = 1080$	$\frac{2976}{6} = 496$	
$x(2)$	2	2	1+1=2	2	8
$D_j(2)$	$\frac{2596}{6} = 432.6$	$\frac{3416}{6} = 569.6$	$\frac{2160}{6} = 360$	$\frac{2976}{6} = 496$	
$x(2)$	2	2+1=3	2	2	9
$D_j(3)$	$\frac{2596}{6} = 432.6$	$\frac{3416}{12} = 284.7$	$\frac{2160}{6} = 360$	$\frac{2976}{6} = 496$	
$x(3)$	2	3	2	2+1=3	10

The solution $x = (x_1, x_2, x_{3,} x_4) = (2, 3, 2, 3)$ is optimal. The optimal objective value is $1298 + 1138.6 + 1080 + 992 = 4557$

The example above demonstrates a systematic way of using the algorithm. Next example is an application of the algorithm for dealing with a problem on assigning the jobs for a group of workers.

4. Conclusions and Remarks

In general, a nonlinear integer program is not easy to solve efficiently. Especially, when the number of variables is large the problem becomes complicated. (Fletcher, 1987) The algorithm presented in this work is very simple and efficient. Also, it is easy to apply for finding the solutions. The example above demonstrates the solution method using the presented algorithm. Because of using this algorithm, the problem (P) is solved efficiently.

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