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Notes on high-energy limit of bosonic closed string scattering amplitudes

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Abstract

We study bosonic closed string scattering amplitudes in the high-energy limit. We find that the methods of decoupling of high-energy zero-norm states and the high-energy Virasoro constraints, which were adopted in the previous works to calculate the ratios among high-energy open string scattering amplitudes of different string states, persist for the case of closed string. However, we clarify the previous saddle-point calculation for high-energy open string scattering amplitudes and claim that only (t, u) channel of the amplitudes is suitable for saddle-point calculation. We then discuss three evidences to show that saddle-point calculation for high-energy closed string scattering amplitudes is not reliable. By using the relation of tree-level closed and open string scattering amplitudes of Kawai, Lewellen and Tye (KLT), we calculate the high-energy closed string scattering amplitudes for *arbitrary* mass levels. For the case of high-energy closed string four-tachyon amplitude, our result differs from the previous one of Gross and Mende, which is NOT consistent with KLT formula, by an oscillating factor.

1. Introduction

Recently high-energy, fixed-angle behavior of string scattering amplitudes [1–3] was intensively reinvestigated [4–10]. The motivation was to uncover the long-sought hidden stringy space–time symmetry. An important new ingredient of this approach is the zero-norm states (ZNS) [11–13] in the old covariant first quantized (OCFQ) string spectrum. One utilizes the de-

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coupling of zero-norm states to obtain relations among scattering amplitudes. An infinite number of linear relations among high-energy scattering amplitudes of different string states were derived. Moreover, these linear relations can be used to fix the proportionality constants among high-energy scattering amplitudes of different string states at each fixed mass level algebraically. Thus there is only one independent component of high-energy scattering amplitude at each fixed mass level. On the other hand, a saddle-point method was also developed to calculate the general formula of tree-level high-energy scattering amplitudes of four arbitrary string states to verify the ratios among the high-energy scattering amplitudes of different string states calculated by the above algebraic methods. Moreover, these high-energy scattering amplitudes can be expressed in terms of high-energy four tachyon scattering amplitude as conjectured by Gross in 1988 [2]. However, all the above calculations were focused only on the case of open string theory.

In this paper, we generalize the calculations to high-energy closed string scattering amplitudes. We find that the methods of decoupling of high-energy zero-norm states and the high-energy Virasoro constraints, which were adopted in the previous works to calculate the ratios among high-energy open string scattering amplitudes of different string states, persist for the case of closed string. The result is simply the tensor product of two pieces of open string ratios of high-energy scattering amplitudes. However, we clarify the previous saddle-point calculation for high-energy open string scattering amplitudes and claim that only (t,u) channel of the amplitudes is suitable for saddle-point calculation. We then discuss three evidences to show that saddle-point calculation for high-energy closed string scattering amplitudes is not reliable. By using the relation of tree-level closed and open string scattering amplitudes of Kawai, Lewellen and Tye (KLT) [14], we calculate the tree-level high-energy closed string scattering amplitudes for *arbitrary* mass levels. For the case of high-energy closed string four-tachyon amplitude, our result differs from the previous one of Gross and Mende [1], which is NOT consistent with KLT formula, by an oscillating factor. This means that the high-energy closed string amplitudes *do not factorize* into product of two high-energy open string amplitudes in contrast to the conventional wisdom [1,15].

2. Decoupling of zero norm states

In this section, we calculate the ratios among high-energy closed string scattering amplitudes of different string states by the decoupling of high-energy closed string ZNS. Since the calculation is similar to that of open string, we will, for simplicity, work on the first massive level $M^2 = 8(n-1) = 8$ $(n=2)$ only. At this mass level, the corresponding open string Ward identities are ($M^2 = 2$ for open string, $\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$) [16]

$$k_\mu \theta_\nu \mathcal{T}^{\mu\nu} + \theta_\mu \mathcal{T}^\mu = 0, \tag{1}$$

$$\left(\frac{3}{2}k_\mu k_\nu + \frac{1}{2}\eta_{\mu\nu}\right)\mathcal{T}^{\mu\nu} + \frac{5}{2}k_\mu \mathcal{T}^\mu = 0, \tag{2}$$

where θ_ν is a transverse vector. In Eqs. (1) and (2), we have chosen, say, the second vertex $V_2(k_2)$ to be the vertex operators constructed from zero-norm states and $k_\mu \equiv k_{2\mu}$. The other three vertices can be any string states. Note that Eq. (1) is the type I Ward identity while Eq. (2) is the type II Ward identity which is valid only at $D = 26$. The high-energy limits of Eqs. (1) and (2) were calculated to be

$$M\mathcal{T}^{3\to 1}_{TP} + \mathcal{T}^1_T = 0, \tag{3}$$

$$M\mathcal{T}^{4\to 2}_{LL} + \mathcal{T}^2_L = 0, \tag{4}$$

$$3M^2\mathcal{T}^{4\to 2}_{LL}+\mathcal{T}^2_{TT}+5M\mathcal{T}^2_L=0. \tag{5}$$

In the above equations, we have defined the following orthonormal polarization vectors for the second string vertex $V_2(k_2)$

$$e_P=\frac{1}{M}(E_2,k_2,0)=\frac{k_2}{M}, \tag{6}$$

$$e_L=\frac{1}{M}(k_2,E_2,0), \tag{7}$$

$$e_T=(0,0,1) \tag{8}$$

in the center-of-mass (c.m.) frame contained in the plane of scattering. We have also denoted the naive power counting for orders in energy [4,5] in the superscript of each amplitude according to the following rules, $e_L\cdot k\sim E^2, e_T\cdot k\sim E^1$. Note that since $\mathcal{T}^1_{TP}$ is of subleading order in energy, in general $\mathcal{T}^1_{TP}\neq\mathcal{T}^1_{TL}$. A simple calculation of Eqs. (3)–(5) shows that [16]

$$\mathcal{T}^1_{TP}:\mathcal{T}^1_T=1:-\sqrt{2}=1:-M, \tag{9}$$

$$\mathcal{T}^2_{TT}:\mathcal{T}^2_{LL}:\mathcal{T}^2_L=4:1:-\sqrt{2}=2M^2:1:-M. \tag{10}$$

It is interesting to see that, in addition to the leading order amplitudes in Eq. (10), the subleading order amplitudes in Eq. (9) are also proportional to each other. This does not seem to happen at higher mass level.

We are now back to the closed string calculation. The OCFQ closed string spectrum at this mass level are $(\square\square+\square+\bullet)\otimes(\square\square+\square+\bullet)'$. In addition to the spin-four positive-norm state $\square\square\otimes\square\square'$, one has 8 ZNS, each of which gives a Ward identity. In the high-energy limit, we have $\theta^{\mu\nu}=e^\mu_L e^\nu_L-e^\mu_T e^\nu_T$ or $\theta^{\mu\nu}=e^\mu_L e^\nu_T+e^\mu_T e^\nu_L$, $\theta^\mu=e^\mu_L$ or e^μ_T and one replace $\eta_{\mu\nu}$ by $e^\mu_T e^\nu_T$. In the following, we list only high-energy Ward identities which relate amplitudes with even-energy power in the high-energy expansion:

(1) $\square\square\otimes\square'$:

$$M(\mathcal{T}_{LL,LL}-\mathcal{T}_{TT,LL})+\mathcal{T}_{LL,L}-\mathcal{T}_{TT,L}=0, \tag{11}$$

$$M\mathcal{T}_{LT,PT}+\mathcal{T}_{LT,T}=0. \tag{12}$$

(2) $\square\square\otimes\bullet'$:

$$3M^2(\mathcal{T}_{LL,LL}-\mathcal{T}_{TT,LL})+(\mathcal{T}_{LL,TT}-\mathcal{T}_{TT,TT})+5M(\mathcal{T}_{LL,L}-\mathcal{T}_{TT,L})=0. \tag{13}$$

(3) $\square\otimes\square\square'$:

$$M(\mathcal{T}_{LL,LL}-\mathcal{T}_{LL,TT})+\mathcal{T}_{L,LL}-\mathcal{T}_{L,TT}=0, \tag{14}$$

$$M\mathcal{T}_{PT,LT}+\mathcal{T}_{T,LT}=0. \tag{15}$$

(4) $\square\otimes\square'$:

$$M^2\mathcal{T}_{LL,LL}+M\mathcal{T}_{LL,L}+M\mathcal{T}_{L,LL}+\mathcal{T}_{L,L}=0, \tag{16}$$

$$M^2\mathcal{T}_{PT,PT}+M\mathcal{T}_{PT,T}+M\mathcal{T}_{T,PT}+\mathcal{T}_{T,T}=0. \tag{17}$$

(5) $\square\otimes\bullet'$:

$$3M^3\mathcal{T}_{LL,LL}+M\mathcal{T}_{LL,TT}+5M^2\mathcal{T}_{LL,L}+3M^2\mathcal{T}_{L,LL}+\mathcal{T}_{L,TT}+5M^2\mathcal{T}_{L,L}=0. \tag{18}$$

(6) $\bullet\otimes\square\square'$:

$$3M^2(\mathcal{T}_{LL,LL}-\mathcal{T}_{LL,TT})+(\mathcal{T}_{TT,LL}-\mathcal{T}_{TT,TT})+5M(\mathcal{T}_{L,LL}-\mathcal{T}_{L,TT})=0. \tag{19}$$

(7) $\bullet\otimes\square'$:

$$3M^3\mathcal{T}_{LL,LL}+M\mathcal{T}_{TT,LL}+5M^2\mathcal{T}_{L,LL}+3M^2\mathcal{T}_{LL,L}+\mathcal{T}_{TT,L}+5M^2\mathcal{T}_{L,L}=0. \tag{20}$$

(8) $\bullet\otimes\bullet'$:

$$\begin{aligned}&9M^4\mathcal{T}_{LL,LL}+3M^2\mathcal{T}_{LL,TT}+3M^2\mathcal{T}_{TT,LL}+15M^3\mathcal{T}_{LL,L}\\&\quad+15M^3\mathcal{T}_{L,LL}+5M\mathcal{T}_{TT,L}+5M\mathcal{T}_{L,TT}+25M^2\mathcal{T}_{L,L}+\mathcal{T}_{TT,TT}=0.\end{aligned} \tag{21}$$

Those Ward identities which relate amplitudes with odd-energy power in the high-energy expansion are omitted as they are subleading order in energy. The mass M in Eqs. (11) to (21) should now be interpreted as the closed string mass $M^2=8$. Eqs. (12), (15) and (17) are subleading order amplitudes, and one can then solve the other 8 equations to give the ratios

$$\begin{aligned}&\mathcal{T}_{TT,TT}:\mathcal{T}_{TT,LL}:\mathcal{T}_{LL,TT}:\mathcal{T}_{LL,LL}:\mathcal{T}_{TT,L}:\mathcal{T}_{L,TT}:\mathcal{T}_{LL,L}:\mathcal{T}_{L,LL}:\mathcal{T}_{L,L}\\&\quad=1:\frac{1}{2M^2}:\frac{1}{2M^2}:\frac{1}{4M^4}:-\frac{1}{2M}:-\frac{1}{2M}:-\frac{1}{4M^3}:-\frac{1}{4M^3}:\frac{1}{4M^2}.\end{aligned} \tag{22}$$

Eq. (22) is exactly the tensor product of two pieces of open string ratios calculated in Eq. (10).

3. Virasoro constraints

We consider the mass level $M^2=8$ ($n=2$). The most general state is

$$\begin{aligned}|2\rangle&=\left\{\frac{1}{2!}\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\frac{1}{2}\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{\frac{1}{2!}\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_1^1}\tilde\alpha_{-1}^{\tilde\mu_2^1}+\frac{1}{2}\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-2}^{\tilde\mu_1^2}\right\}|0,k\rangle\\&=\frac{1}{4}\left\{\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_1^1}\tilde\alpha_{-1}^{\tilde\mu_2^1}+\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-2}^{\tilde\mu_1^2}\right\}|0,k\rangle.\end{aligned} \tag{23}$$

The Virasoro constraints are

$$L_1|2\rangle\sim\left\{k^{\mu_1^1}\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-1}^{\mu_1^2}\right\}\otimes\left\{\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_1^1}\tilde\alpha_{-1}^{\tilde\mu_2^1}+\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-2}^{\tilde\mu_1^2}\right\}=0, \tag{24a}$$

$$\tilde L_1|2\rangle\sim\left\{\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{k^{\mu_1^1}\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_2^1}+\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-1}^{\tilde\mu_1^2}\right\}=0, \tag{24b}$$

$$L_2|2\rangle\sim\left\{\boxed{\mu_1^1}\boxed{\mu_2^1}\,\eta^{\mu_1^1\mu_2^1}+2k^{\mu_1^2}\boxed{\mu_1^2}\right\}\otimes\left\{\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_1^1}\tilde\alpha_{-1}^{\tilde\mu_2^1}+\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-2}^{\tilde\mu_1^2}\right\}=0, \tag{24c}$$

$$\tilde L_2|2\rangle\sim\left\{\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\eta^{\tilde\mu_1^1\tilde\mu_2^1}+2k^{\tilde\mu_1^2}\boxed{\tilde\mu_1^2}\right\}=0. \tag{24d}$$

Taking the high-energy limit in the above equations by letting $(\mu_i,\nu_i)\to(L,T)$, and

$$k^{\mu_i}\to Me^L,\qquad \eta^{\mu_1\mu_2}\to e^Te^T, \tag{25}$$

we obtain

$$\left\{M\boxed{L}\boxed{\mu}+\boxed{\mu}\right\}\alpha_{-1}^{\mu}\otimes\left\{\boxed{\tilde\mu_1^1}\boxed{\tilde\mu_2^1}\,\tilde\alpha_{-1}^{\tilde\mu_1^1}\tilde\alpha_{-1}^{\tilde\mu_2^1}+\boxed{\tilde\mu_1^2}\,\tilde\alpha_{-2}^{\tilde\mu_1^2}\right\}=0, \tag{26a}$$

$$\left\{\boxed{\mu_1^1}\boxed{\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{M\boxed{L}\boxed{\tilde\mu}+\boxed{\tilde\mu}\right\}\tilde\alpha_{-1}^{\tilde\mu}=0, \tag{26b}$$

$$\left\{\boxed{T|T}+2M\boxed{L}\right\}\otimes\left\{\boxed{\tilde{\mu}_1^1|\tilde{\mu}_2^1}\,\tilde{\alpha}_{-1}^{\tilde{\mu}_1^1}\tilde{\alpha}_{-1}^{\tilde{\mu}_2^1}+\boxed{\tilde{\mu}_1^2}\,\tilde{\alpha}_{-2}^{\tilde{\mu}_1^2}\right\}=0, \tag{26c}$$

$$\left\{\boxed{\mu_1^1|\mu_2^1}\,\alpha_{-1}^{\mu_1^1}\alpha_{-1}^{\mu_2^1}+\boxed{\mu_1^2}\,\alpha_{-2}^{\mu_1^2}\right\}\otimes\left\{\boxed{T|T}+2M\boxed{L}\right\}=0, \tag{26d}$$

which lead to the following equations

$$\left\{M\boxed{L|\mu}+\boxed{\mu}\right\}\otimes\boxed{\tilde{\mu}_1^1|\tilde{\mu}_2^1}=0, \tag{27a}$$

$$\left\{M\boxed{L|\mu}+\boxed{\mu}\right\}\otimes\boxed{\tilde{\mu}_1^2}=0, \tag{27b}$$

$$\boxed{\mu_1^1|\mu_2^1}\otimes\left\{M\boxed{L|\tilde{\mu}}+\boxed{\tilde{\mu}}\right\}=0, \tag{27c}$$

$$\boxed{\mu_1^2}\otimes\left\{M\boxed{L|\tilde{\mu}}+\boxed{\tilde{\mu}}\right\}=0, \tag{27d}$$

$$\left\{\boxed{T|T}+2M\boxed{L}\right\}\otimes\boxed{\tilde{\mu}_1^1|\tilde{\mu}_2^1}=0, \tag{27e}$$

$$\left\{\boxed{T|T}+2M\boxed{L}\right\}\otimes\boxed{\tilde{\mu}_1^2}=0, \tag{27f}$$

$$\boxed{\mu_1^1|\mu_2^1}\otimes\left\{\boxed{T|T}+2M\boxed{L}\right\}=0, \tag{27g}$$

$$\boxed{\mu_1^2}\otimes\left\{\boxed{T|T}+2M\boxed{L}\right\}=0. \tag{27h}$$

The remaining indices μ, $\tilde{\mu}$ in the above equations can be set to be T or L, and we obtain

$$M\boxed{L|L}\otimes\boxed{L|L}+\boxed{L}\otimes\boxed{L|L}=0, \tag{28a}$$

$$M\boxed{L|L}\otimes\boxed{T|T}+\boxed{L}\otimes\boxed{T|T}=0, \tag{28b}$$

$$M\boxed{T|L}\otimes\boxed{T|L}+\boxed{T}\otimes\boxed{T|L}=0, \tag{28c}$$

$$M\boxed{L|L}\otimes\boxed{L}+\boxed{L}\otimes\boxed{L}=0, \tag{29a}$$

$$M\boxed{T|L}\otimes\boxed{T}+\boxed{T}\otimes\boxed{T}=0, \tag{29b}$$

$$M\boxed{L|L}\otimes\boxed{L|L}+\boxed{L|L}\otimes\boxed{L}=0, \tag{30a}$$

$$M\boxed{T|T}\otimes\boxed{L|L}+\boxed{T|T}\otimes\boxed{L}=0, \tag{30b}$$

$$M\boxed{T|L}\otimes\boxed{T|L}+\boxed{T|L}\otimes\boxed{T}=0, \tag{30c}$$

$$M\boxed{L}\otimes\boxed{L|L}+\boxed{L}\otimes\boxed{L}=0, \tag{31a}$$

$$M\boxed{T}\otimes\boxed{T|L}+\boxed{T}\otimes\boxed{T}=0, \tag{31b}$$

$$\boxed{T|T}\otimes\boxed{L|L}+2M\boxed{L}\otimes\boxed{L|L}=0, \tag{32a}$$

$$\boxed{T|T}\otimes\boxed{T|T}+2M\boxed{L}\otimes\boxed{T|T}=0, \tag{32b}$$

$$\boxed{T\,|\,T} \otimes \boxed{L} + 2M \boxed{L} \otimes \boxed{L} = 0, \tag{33}$$

$$\boxed{L\,|\,L} \otimes \boxed{T\,|\,T} + 2M \boxed{L\,|\,L} \otimes \boxed{L} = 0, \tag{34a}$$

$$\boxed{T\,|\,T} \otimes \boxed{T\,|\,T} + 2M \boxed{T\,|\,T} \otimes \boxed{L} = 0, \tag{34b}$$

$$\boxed{L} \otimes \boxed{T\,|\,T} + 2M \boxed{L} \otimes \boxed{L} = 0. \tag{35}$$

Since the transverse component of the highest spin state $\alpha_{-1}^{T} \cdots \alpha_{-1}^{T} \otimes \tilde{\alpha}_{-1}^{T} \cdots \tilde{\alpha}_{-1}^{T}$ at each fixed mass level gives the leading order scattering amplitude, there should have even number of T at each fixed mass level. Thus Eqs. (28c), (29b), (30c) and (31b) are subleading order in energy and are therefore irrelevant. Set $\boxed{T\,|\,T} \otimes \boxed{T\,|\,T} = 1$, we can solve the ratios from the remaining equations. The final result is

$\epsilon_{TT,TT}$	1
$\epsilon_{TT,LL} = \epsilon_{LL,TT}$	$1/(2M^2)$
$\epsilon_{LL,LL}$	$1/(4M^4)$
$\epsilon_{TT,L} = \epsilon_{L,TT}$	$-1/(2M)$
$\epsilon_{LL,L} = \epsilon_{L,LL}$	$-1/(4M^3)$
$\epsilon_{L,L}$	$1/(4M^2)$

which is exactly the tensor product of two pieces of open string ratios. This result is consistent with Eq. (22) from the decoupling of high-energy zero-norm state in Section 2.

4. Saddle point calculation

In this section, we calculate the tree-level high-energy closed string scattering amplitudes for arbitrary mass levels. We first review the calculation of high-energy open string scattering amplitude. The (s,t) channel scattering amplitude with $V_2 = \alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}\cdots\alpha_{-1}^{\mu_n}|0,k\rangle$, the highest spin state at mass level $M^2 = 2(n-1)$, and three tachyons $V_{1,3,4}$ is [6]

$$\mathcal{T}_{n;st}^{\mu_1\mu_2\ldots\mu_n} = \sum_{l=0}^{n}(-)^l \binom{n}{l} B\left(-\frac{s}{2}-1+l, -\frac{t}{2}-1+n-l\right) k_1^{(\mu_1}\cdots k_1^{\mu_{n-l}} k_3^{\mu_{n-l+1}}\cdots k_3^{\mu_n)}, \tag{36}$$

where $B(u,v) = \int_0^1 dx\, x^{u-1}(1-x)^{v-1}$ is the Euler beta function. It is now easy to calculate the general high-energy scattering amplitude at the $M^2 = 2(n-1)$ level

$$\mathcal{T}_n^{TTT\cdots}(s,t) \simeq \left[-2E^3 \sin\phi_{\text{c.m.}}\right]^n \mathcal{T}_n(s,t) \tag{37}$$

where $\mathcal{T}_n(s,t)$ is the high-energy limit of $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$ with $s+t+u = 2n-8$, and was previously [4,6] miscalculated to be

$$\tilde{\mathcal{T}}_{n;st} \simeq \sqrt{\pi}(-1)^{n-1}2^{-n}E^{-1-2n}\left(\sin\frac{\phi_{\text{c.m.}}}{2}\right)^{-3}\left(\cos\frac{\phi_{\text{c.m.}}}{2}\right)^{5-2n} \times \exp\left[-\frac{s\ln s + t\ln t - (s+t)\ln(s+t)}{2}\right]. \tag{38}$$

One can now generalize this result to multi-tensors. The (s,t) channel of open string high-energy scattering amplitude at mass level (n_1,n_2,n_3,n_4) was calculated to be [4,6]

$$\mathcal{T}^{T^1\ldots T^2\ldots T^3\ldots T^4\ldots}_{n_1n_2n_3n_4;st}=\left[-2E^3\sin\phi_{\text{c.m.}}\right]^{\sum n_i}\mathcal{T}_{\sum n_i}(s,t). \tag{39}$$

In the above calculations, the scattering angle $\phi_{\text{c.m.}}$ in the center of mass frame is defined to be the angle between $\vec{k}_1$ and $\vec{k}_3$. $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$ and $u=-(k_1+k_3)^2$ are the Mandelstam variables. $M_i^2=2(n_i-1)$ with n_i the mass level of the ith vertex. T^i in Eq. (39) is the transverse polarization of the ith vertex defined in Eq. (8). All other 4-point functions at mass level (n_1,n_2,n_3,n_4) were shown to be proportional to Eq. (39).

The corresponding (t,u) channel scattering amplitudes of Eqs. (37) and (39) can be obtained by replacing (s,t) in Eq. (38) by (t,u)

$$\begin{aligned}\mathcal{T}_n(t,u)\simeq{}&\sqrt{\pi}(-1)^{n-1}2^{-n}E^{-1-2n}\left(\sin\frac{\phi_{\text{c.m.}}}{2}\right)^{-3}\left(\cos\frac{\phi_{\text{c.m.}}}{2}\right)^{5-2n}\\&\times\exp\left[-\frac{t\ln t+u\ln u-(t+u)\ln(t+u)}{2}\right].\end{aligned} \tag{40}$$

We now claim that only (t,u) channel of the amplitude, Eq. (40), is suitable for saddle-point calculation. The previous saddle-point calculation for the (s,t) channel amplitude, Eq. (38), in the high-energy expansion is misleading. The corrected high-energy calculation of the (s,t) channel amplitude will be given in Eq. (57). The reason is as following. When calculating Eq. (37) from Eq. (36), one calculates the high-energy limit of

$$\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)},\qquad s+t+u=2n-8, \tag{41}$$

in Eq. (36) by expanding the Γ function with the Stirling formula

$$\Gamma(x)\sim\sqrt{2\pi}x^{x-1/2}e^{-x}. \tag{42}$$

However, the above expansion is not suitable for negative real x as there are poles for $\Gamma(x)$ at $x=-n$, negative integers. Unfortunately, our high-energy limit

$$s\sim 4E^2\gg 0, \tag{43a}$$

$$t\sim -4E^2\sin^2\left(\frac{\phi_{\text{c.m.}}}{2}\right)\ll 0, \tag{43b}$$

$$u\sim -4E^2\cos^2\left(\frac{\phi_{\text{c.m.}}}{2}\right)\ll 0, \tag{43c}$$

contains this dangerous situation in the (s,t) channel calculation of Eq. (38). On the other hand, the corresponding high-energy expansion of (t,u) channel scattering amplitude in Eq. (40) is well defined. Another evidence for this point is the following. When one uses the saddle point method to calculate the high-energy open string scattering amplitudes in the (s,t) channel, the saddle-point we identified was [6–8]

$$x_0=\frac{s}{s+t}=\frac{1}{1-\sin^2(\phi/2)}>1, \tag{44}$$

which is out of the integration range $(0,1)$. Therefore, we cannot trust the saddle point calculation for the (s,t) channel scattering amplitude. On the other hand, the corresponding saddle-point calculation for the (t,u) channel scattering amplitude is safe since the saddle-point x_0 is within the integration range $(1,\infty)$. This subtle situation becomes crucial and relevant when one tries to calculate the high-energy closed string scatterings amplitude and compare them with the open string ones.

We now discuss the high-energy closed string scattering amplitudes. There exists a celebrated formula by Kawai, Lewellen and Tye (KLT), which expresses the relation between tree amplitudes of closed and open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$)

$$A^{(4)}_{\text{closed}}(s,t,u) = \sin(\pi k_2 \cdot k_3) A^{(4)}_{\text{open}}(s,t) \bar{A}^{(4)}_{\text{open}}(t,u). \tag{45}$$

To calculate the high-energy closed string scattering amplitudes, one encounters the difficulty of calculation of high-energy open string amplitude in the (s,t) channel discussed above. To avoid this difficulty, we can use the well-known formula

$$\Gamma(x) = \frac{\pi}{\sin(\pi x)\Gamma(1-x)} \tag{46}$$

to calculate the large negative x expansion of the Γ function. We first discuss the high-energy four-tachyon scattering amplitude which already existed in the literature. We can express the open string (s,t) channel amplitude in terms of the (t,u) channel amplitude,

$$\begin{aligned} A^{(4\text{-tachyon})}_{\text{open}}(s,t) &= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} \\ &= \frac{\sin(\pi u/2)}{\sin(\pi s/2)} \frac{\Gamma(-\frac{t}{2}-1)\Gamma(-\frac{u}{2}-1)}{\Gamma(\frac{s}{2}+2)} \\ &\equiv \frac{\sin(\pi u/2)}{\sin(\pi s/2)} A^{(4\text{-tachyon})}_{\text{open}}(t,u), \end{aligned} \tag{47}$$

which we know how to calculate the high-energy limit. Note that for the four-tachyon case, $\bar{A}^{(4)}_{\text{open}}(t,u) = A^{(4)}_{\text{open}}(t,u)$ in Eq. (45). The KLT formula, Eq. (45), can then be used to express the closed string four-tachyon scattering amplitude in terms of that of open string in the (t,u) channel

$$A^{(4\text{-tachyon})}_{\text{closed}}(s,t,u) = \frac{\sin(\pi t/2)\sin(\pi u/2)}{\sin(\pi s/2)} A^{(4\text{-tachyon})}_{\text{open}}(t,u) A^{(4\text{-tachyon})}_{\text{open}}(t,u). \tag{48}$$

The high-energy limit of open string four-tachyon amplitude in the (t,u) channel can be easily calculated to be

$$A^{(4\text{-tachyon})}_{\text{open}}(t,u) \simeq (stu)^{-3/2} \exp\left(-\frac{s\ln s + t\ln t + u\ln u}{2}\right), \tag{49}$$

which gives the corresponding amplitude in the (s,t) channel

$$A^{(4\text{-tachyon})}_{\text{open}}(s,t) \simeq \frac{\sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-3/2} \exp\left(-\frac{s\ln s + t\ln t + u\ln u}{2}\right). \tag{50}$$

The high-energy limit of closed string four-tachyon scattering amplitude can then be calculated, through the KLT formula, to be

$$A^{(4\text{-tachyon})}_{\text{closed}}(s,t,u) \simeq \frac{\sin(\pi t/2)\sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-3} \exp\left(-\frac{s\ln s + t\ln t + u\ln u}{4}\right). \tag{51}$$

The exponential factor in Eq. (49) was first discussed by Veneziano [17]. Our result for the high-energy closed string four-tachyon amplitude in Eq. (51) differs from the one calculated in the literature [1] by an oscillating factor $\frac{\sin(\pi t/2)\sin(\pi u/2)}{\sin(\pi s/2)}$ [18]. *We stress here that our results for Eqs.* (49), (50) *and* (51) *are consistent with the KLT formula, while the previous calculation in* [1] *is NOT.*

One might try to use the saddle-point method to calculate the high-energy closed string scattering amplitude. The closed string four-tachyon scattering amplitude is

$$\begin{aligned} A^{(4\text{-tachyon})}_{\text{closed}}(s,t,u) &= \int dx\,dy \exp\left(\frac{k_1\cdot k_2}{2}\ln|z| + \frac{k_2\cdot k_3}{2}\ln|1-z|\right) \\ &= \int dx\,dy\,\left(x^2+y^2\right)^{-2}\left[(1-x)^2+y^2\right]^{-2} \\ &\quad\times \exp\left\{-\frac{s}{8}\ln\left(x^2+y^2\right) - \frac{t}{8}\ln\left[(1-x)^2+y^2\right]\right\} \\ &\equiv \int dx\,dy\,\left(x^2+y^2\right)^{-2}\left[(1-x)^2+y^2\right]^{-2}\exp\left[-Kf(x,y)\right], \end{aligned} \tag{52}$$

where $K=\frac{s}{8}$ and $f(x,y)=\ln(x^2+y^2)-\tau\ln[(1-x)^2+y^2]$ with $\tau=-\frac{t}{s}$. One can then calculate the "saddle-point" of $f(x,y)$ to be

$$\nabla f(x,y)|_{x_0=\frac{1}{1-\tau},y_0=0}=0. \tag{53}$$

The high-energy limit of the closed string four-tachyon scattering amplitude is then calculated to be

$$\begin{aligned} A^{(4\text{-tachyon})}_{\text{closed}}(s,t,u) &\simeq \frac{2\pi}{K\sqrt{\det\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}}}\exp\left[-Kf(x_0,y_0)\right] \\ &\simeq (stu)^{-3}\exp\left(-\frac{s\ln s+t\ln t+u\ln u}{4}\right), \end{aligned} \tag{54}$$

which is consistent with the previous one calculated in the literature [1], but is different from our result in Eq. (51). However, one notes that

$$\frac{\partial^2 f(x_0,y_0)}{\partial x^2}=\frac{2(1-\tau)^3}{\tau}=-\frac{\partial^2 f(x_0,y_0)}{\partial y^2}, \qquad \frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}=0, \tag{55}$$

which means that (x_0,y_0) is NOT the local minimum of $f(x,y)$, and one should not trust this saddle-point calculation. This is the third evidence to see that there is no clear definition of saddle-point in the calculation of the high-energy open string scattering amplitude in the (s,t) channel, and thus the invalid saddle-point calculation of high-energy closed string scattering amplitude.

Finally we calculate the high-energy closed string scattering amplitudes for arbitrary mass levels. The (t,u) channel open string scattering amplitude with $V_2=\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}\cdots\alpha_{-1}^{\mu_n}|0,k\rangle$, the highest spin state at mass level $M^2=2(n-1)$, and three tachyons $V_{1,3,4}$ can be calculated to be

$$\mathcal{T}^{\mu_1\mu_2\ldots\mu_n}_{n;tu}=\sum_{l=0}^{n}\binom{n}{l}B\left(-\frac{t}{2}+n-l-1,-\frac{u}{2}-1\right)k_1^{(\mu_1}\cdots k_1^{\mu_{n-l}}k_3^{\mu_{n-l+1}}\cdots k_3^{\mu_n)}. \tag{56}$$

In calculating Eq. (56), we have used the Mobius transformation $y = \frac{x-1}{x}$ to change the integration region from $(1, \infty)$ to $(0, 1)$. One notes that Eq. (56) is NOT the same as Eq. (36) with (s, t) replaced by (t, u), as one would have expected from the four-tachyon case discussed in the paragraph after Eq. (45). In the high-energy limit, one easily sees that

$$\mathcal{T}_n(s,t) \simeq (-)^n \frac{\sin(\pi u/2)}{\sin(\pi s/2)} \mathcal{T}_n(t,u), \tag{57}$$

which is the generalization of Eq. (47) to arbitrary mass levels. Eq. (57) is the correction of Eqs. (37) and (38) as claimed in the paragraph after Eq. (40). The (s, t) channel of high-energy open string scattering amplitudes at mass level (n_1, n_2, n_3, n_4) can then be written as, apart from an overall constant,

$$\begin{aligned} A^{(4)}_{\text{open}}(s,t) &\simeq (-)^{\sum n_i} \frac{\sin(\pi u/2)}{\sin(\pi s/2)} \left[-2E^3 \sin\phi_{\text{c.m.}}\right]^{\sum n_i} \mathcal{T}_{\sum n_i}(t,u) \\ &\simeq (-)^{\sum n_i} \frac{\sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{\frac{\sum n_i - 3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right). \end{aligned} \tag{58}$$

Finally the total high-energy open string scattering amplitude is the sum of (s, t), (t, u) and (u, s) channel amplitudes, and can be calculated to be

$$\begin{aligned} A^{(4)}_{\text{open}} \simeq{}& (-)^{\sum n_i} \frac{\sin(\pi s/2) + \sin(\pi t/2) + \sin(\pi u/2)}{\sin(\pi s/2)} \\ & \times (stu)^{\frac{\sum n_i - 3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right). \end{aligned} \tag{59}$$

By using Eqs. (45) and (57), the high-energy closed string scattering amplitude at mass level (n_1, n_2, n_3, n_4) is calculated to be, apart from an overall constant,

$$\begin{aligned} A^{(4)}_{\text{closed}}(s,t,u) &\simeq (-)^{\sum n_i} \frac{\sin(\pi t/2)\sin(\pi u/2)}{\sin(\pi s/2)} \left[-2E^3 \sin\phi_{\text{c.m.}}\right]^{2\sum n_i} \mathcal{T}_{\sum n_i}(t,u)^2 \\ &\simeq (-)^{\sum n_i} \frac{\sin(\pi t/2)\sin(\pi u/2)}{\sin(\pi s/2)} \\ &\quad \times (stu)^{\sum n_i - 3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right), \end{aligned} \tag{60}$$

where $\mathcal{T}_{\sum n_i}(t, u)$ is given by Eq. (40). For the case of four-tachyon scattering amplitude at mass level $(0, 0, 0, 0)$, Eq. (60) reduces to Eq. (51). All other high-energy closed string scattering amplitudes at mass level (n_1, n_2, n_3, n_4) are proportional to Eq. (60). The proportionality constants are the tensor product of two pieces of open string ratios.

5. Conclusion

In conclusion, we have used the methods of decoupling of high-energy zero-norm states and the high-energy Virasoro constraints to calculate the ratios among high-energy closed string scattering amplitudes of different string states. The result is exactly the tensor product of two pieces of open string ratios calculated before. However, we clarify the previous saddle-point calculation for high-energy open string scattering amplitudes and show that only (t, u) channel of the amplitudes is suitable for saddle-point calculation. We also discuss three evidences, Eqs. (43), (44) and (55), to show that saddle-point calculation for high-energy closed string scattering amplitudes is not reliable. Instead of using saddle-point calculation adopted before, we then propose to use the formula of Kawai, Lewellen and Tye (KLT) to calculate the high-energy closed string scattering amplitudes for *arbitrary* mass levels. For the case of high-energy closed string four-tachyon amplitude, our result differs from the previous one of Gross and Mende, which is NOT consistent with KLT formula, by an oscillating factor. The oscillating prefactors in Eqs. (59) and (60) imply the existence of infinitely many zeros and poles in the string scattering amplitudes even in the high-energy limit. Physically, the presence of poles simply reflects the fact that there are infinite number of resonances in the string spectrum [18], and the presence of zeros reflects the coherence of string scattering.

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