Anisotropic higher derivative gravity and inflationary universe

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(Received 23 May 2006; published 16 August 2006)

Stability analysis of the Kantowski-Sachs type universe in pure higher derivative gravity theory is studied in detail. The nonredundant generalized Friedmann equation of the system is derived by introducing a reduced one-dimensional generalized Kantowski-Sachs type action. Existence and stability of inflationary solution in the presence of higher derivative terms are also studied in detail. Implications to the choice of physical theories are discussed in detail in this paper.

DOI: 10.1103/PhysRevD.74.043522 PACS numbers: 98.80.−k, 04.50.+h

I. INTRODUCTION

Inflationary theory is a nice resolution for the flatness, monopole, and horizon problems of our present universe described by the standard big bang cosmology [1]. In particular, our universe is homogeneous and isotropic to a very high degree of precision [2,3]. Such a universe can be described by the well-known Friedmann-Robertson-Walker (FRW) metric [4].

One expects that gravitational physics could be different from the standard Einstein models near the Planck scale [5,6]. For example, quantum gravity or string corrections could lead to interesting cosmological applicastions [5]. Indeed, some investigations have already addressed the possibility of deriving inflation from higher order gravitational corrections [7–10].

For example, a general analysis of the stability condition for a variety of pure higher derivative gravity theories could be useful choosing physical models. In particular, it has been shown that a stability condition should hold for any potential candidate of inflationary universe in the flat FRW space [10].

In addition, there is no particular reason for our universe to be initially isotropic to such a high degree of precision. Even if anisotropy can be smoothed out by the proposed inflationary process, it is also interesting to study the stability of the FRW space during the post-inflationary epoch. Nonetheless, it is interesting to study the cases where our universe starts out from an initially anisotropic universe. As a result, our universe is expected to evolve from certain anisotropic universe to a stable and isotropic universe. Indeed, it has been shown that there exists such kind of anisotropic solution for a NS-NS model with a metric, a dilaton, and an axion field [11]. Such inflationary solution is also shown to be stable against small field perturbations [12]. Note also that similar stability analysis has also been studied in various fields of interest [13,14].

Higher derivative terms should also be important for the Planck scale physics [10,13]. For example, higher order corrections from quantum gravity or string theory have been considered as the inflationary models [15]. In addition, higher derivative terms also arise as the quantum corrections to the matter fields [15]. The stability analysis of the pure higher derivative gravity models has hence been shown in Ref. [10]. Therefore, it is interesting to study the implication of this stability analysis in different models.

Recently, there are also growing interests in the study of Kantowski-Sachs (KS) type anisotropic universes [16–18]. Hence we will try to study the existence and stability conditions of an inflationary de Sitter final state in the presence of higher derivative theory in Kantowski-Sachs spaces. In particular, it will be applied to study a large class of pure gravity models with inflationary KS/FRW solutions in this paper. Any KS type solution that leads itself to an asymptotic FRW metric at time infinity will be referred to as the KS/FRW solution in this paper for convenience.

It will be shown that the existence of a stable de Sitter background is closely related to the choices of the coupling constants. We will try to generalize the work in Refs. [19,20] in order to obtain a model-independent formula for the nonredundant field equations in the Kantowski-Sachs (KS) type anisotropic space.

We will first derive a stability equation which turns out to be identical to the stability equation for the existence of the inflationary de Sitter solution discussed in Refs. [10,20]. Note that an inflationary de Sitter solution in pure gravity models is expected to have one stable mode and one unstable mode for the system to undergo inflation with the help of the stable mode. Later on, the inflationary era will come to an end once the unstable mode takes over after a brief period of inflationary expansion. The method developed in Refs. [10,20] was shown to be a helpful way in choosing physically acceptable model for our universe. Our result indicates, however, that the unstable mode will also tamper the stability of the isotropic space. To be more specific, if the model has an unstable mode for the de Sitter background perturbation with respect to isotropic perturbation, this unstable mode will also be unstable with respect to any anisotropic perturbations.

In particular, we will show in this paper that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe. First of all, third order terms will be shown to determine the expansion rate H_0 for the inflationary de Sitter space. The quadratic terms will be shown to have nothing to do with the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different.

II. NONREDUNDANT FIELD EQUATION AND BIANCHI IDENTITY IN KS SPACE

Given the metric of the following form:

$$ds^2 = -dt^2 + c^2(t)dr^2 + a^2(t)(d^2\theta + f^2(\theta)d\varphi^2) \quad (1)$$

with $f(\theta) = (\theta, \sinh\theta, \sin\theta)$ denoting the flat, open and close anisotropic space known as Kantowski-Sachs type anisotropic spaces. To be more specific, Bianchi I (BI), III(BIII), and Kantowski-Sachs (KS) space correspond to the flat, open and closed model, respectively. One can instead write the metric as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right) + a_z^2(t)dz^2 \quad (2)$$

with r, θ, and z read as the polar coordinates and z coordinate for convenience. One writes it this way in order to make the comparison with the FRW metric easier. Note that $k = 0, 1 - 1$ stands for the flat, open and closed universes similar to the FRW space.

In addition, one can restore the g_{tt} component $b^2(t)$ for the purpose of deriving the nonredundant field equation associated with G_{tt} that will be shown shortly. As a result, one has

$$ds^2 = -b^2(t)dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right) + a_z^2(t)dz^2. \quad (3)$$

One can show that all nonvanishing spin connections read

$$\Gamma^t_{tt} = H_0 \equiv \frac{\dot{b}}{b} = -\frac{\dot{B}}{2B} \quad (4)$$

$$\Gamma^t_{ii} = BH_i g_{ii} \quad (5)$$

$$\Gamma^i_{ti} = H_i \quad (6)$$

$$\Gamma^r_{rr} = \frac{kr}{1-kr^2} \quad (7)$$

$$\Gamma^r_{\theta\theta} = -r(1-kr^2) \quad (8)$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}. \quad (9)$$

Here $B \equiv 1/b^2$ and $H_i \equiv (\dot{a}/a, \dot{a}/a, \dot{a}_z/a_z) \equiv (H_1, H_2 = H_1, H_z)$ for r, θ, and z component, respectively. One can also define $\Gamma_\mu \equiv \Gamma^\nu_{\mu\nu}$ and $\Gamma^\mu \equiv \Gamma^\mu_{\nu\alpha} g^{\nu\alpha}$ for convenience. As a result, one can compute all nonvanishing components, with $b = 1$ being reset to unity:

$$\Gamma^t = 3H \quad (10)$$

$$\Gamma^r = \frac{1}{ra^2} \quad (11)$$

$$\Gamma_t = 3H \quad (12)$$

$$\Gamma_r = \frac{1}{r(1-kr^2)}. \quad (13)$$

Writing $H_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}$, one can show that $D_\mu H^{\mu\nu} = 0$ following the Bianchi identity $D_\mu G^{\mu\nu} = 0$ and the energy momentum conservation $D_\mu T^{\mu\nu} = 0$ for any energy momentum tensor $T^{\mu\nu}$ coupled to the system. With the metric (2), one can show that the r component of the equation $D_\mu H^{\mu\nu} = 0$ implies that

$$H^r{}_r = H^\theta{}_\theta. \quad (14)$$

The result says that any matter coupled to the system has the property that $T^r{}_r = T^\theta{}_\theta$. In addition, the equations $D_\mu H^{\mu\theta} = 0$ and $D_\mu H^{\mu z} = 0$ both vanish identically all by itself irrelevant to the form of the energy momentum tensor. More interesting information comes from the t component of this equation. It says:

$$(\partial_t + 3H)H^t{}_t = 2H_1 H^r{}_r + H_z H^z{}_z. \quad (15)$$

This equation implies that (i) $H^t{}_t = 0$ implies that $H^r{}_r = H^z{}_z = 0$ and (ii) $H^r{}_r = H^z{}_z = 0$ only implies that $(\partial_t + 3H)H^t{}_t = 0$. Case (ii) can be solved to give $H^t{}_t = \text{constant} \times \exp[-a^2 a_z]$ which approaches zero when $a^2 a_z \to \infty$. Therefore, for the anisotropic system one is considering here, the metric contains two independent variables a and a_z while the Einstein field equations have three nonvanishing components: $H^t{}_t = 0$, $H^r{}_r = H^\theta{}_\theta = 0$ and $H^z{}_z = 0$. The Bianchi identity implies that the tt component is not redundant which needs to be reserved for complete analysis. One can freely ignore one of the rr or zz components without affecting the final result of the system. In short, the $H^t_t = 0$ equation, known as the generalized Friedmann equation, is a nonredundant field equation as compared to the $H^r_r = 0$ and $H^z_z = 0$ equations.

In principle, one can reduce the Lagrangian of the system from a functional of the metric $g_{\mu\nu}$, $\mathcal{L}(g_{\mu\nu})$, to a simpler function of $a(t)$ and $a_z(t)$, namely $L(t) \equiv a^2 a_z \mathcal{L}(g_{\mu\nu}(a(t), a_z(t)))$. The equation of motion should be reconstructed from the variation of the reduced Lagrangian $L(t)$ with respect to the variable a and a_z. The result is, however, incomplete because, the variation of a and a_z are related to the variation of g_{rr} and g_{zz} respectively. One can never derive the field equation for g_{tt} without restoring the variable $b(t)$ in advance. This is the reason why one needs to introduce the metric (3) such that the reduced Lagrangian $L(t) \equiv a^2 a_z \mathcal{L}(g_{\mu\nu}(b(t), a(t), a_z(t)))$ contains the nonredundant information of the $H^t_t = 0$ equation. One can reset $b = 1$

after the variation of $b(t)$ has been done. The wanted and nonredundant Friedman equation can hence be reproduced accordingly.

After some algebra, one can also compute all nonvanishing components of the curvature tensor:

$$R^{ti}{}_{tj} = [\tfrac{1}{2}\dot{B}H_i + B(\dot{H}_i + H_i^2)]\delta^i_j \tag{16}$$

$$R^{ij}{}_{kl} = BH_iH_j\epsilon^{ijm}\epsilon_{klm} + \frac{k}{a^2}\epsilon^{ijz}\epsilon_{klz}. \tag{17}$$

Given a Lagrangian $L = \sqrt{g}\mathcal{L} = L(b(t), a((t), a_z(t))$ one can show that

$$L = \frac{a^2 a_z}{\sqrt{B}}\mathcal{L}(R^{ti}{}_{tj}, R^{ij}{}_{kl}) = \frac{a^2 a_z}{\sqrt{B}}\mathcal{L}(H_i, \dot{H}_i, a^2). \tag{18}$$

The variational equations for this action can be shown to be

$$\mathcal{L} + H_i\Big(\frac{d}{dt} + 3H\Big)L^i = H_iL_i + \dot{H}_iL^i \tag{19}$$

$$\mathcal{L} + \Big(\frac{d}{dt} + 3H\Big)^2 L^i = \Big(\frac{d}{dt} + 3H\Big)L_i - a^2\frac{\delta\mathcal{L}}{\delta a^2}(1 - \delta_{iz}). \tag{20}$$

Here $L_i \equiv \delta\mathcal{L}/\delta H_i$, $L^i \equiv \delta\mathcal{L}/\delta\dot{H}_i$, and $3H \equiv \sum_i H_i$. For simplicity, we will write $\mathcal{L}$ as L from now on in this paper. As a result, the field equations can be written in a more comprehensive form:

$$L + H_i\Big(\frac{d}{dt} + 3H\Big)L^i = H_iL_i + \dot{H}_iL^i \tag{21}$$

$$L + \Big(\frac{d}{dt} + 3H\Big)^2 L^i = \Big(\frac{d}{dt} + 3H\Big)L_i - a^2\frac{\delta\mathcal{L}}{\delta a^2}(1 - \delta_{iz}). \tag{22}$$

III. FRW SPACE AS A STABLE FINAL STATE

For simplicity, one will start with the Einstein-Hilbert (EH) action and study its evolutionary process from an anisotropic space. It is known that our final universe is isotropic to a very high precision. Therefore, any physical model should carry our physical universe from an initially anisotropic space to an isotropic final space. Since the lowest order Einstein-Hilbert action is the most well-known popular model, one expects such isotropilized process should also be realized in this model. Any acceptable higher order terms being considered as corrections around this stable EH background should not affect its intention evolving toward the FRW space. Therefore, one will start with a simple EH action with a cosmological constant term given by

$$S_{\rm EH} = \int d^4x\sqrt{g}\mathcal{L} = \int d^4x\sqrt{g}[-R - 2\Lambda]. \tag{23}$$

One can show directly that the reduced Lagrangian L is given by

$$L = 4\dot{H}_1 + 2\dot{H}_z + 6H_1^2 + 2H_z^2 + 4H_1H_z + \frac{2k}{a^2} - 2\Lambda. \tag{24}$$

Therefore, one can show that the Friedmann equation (21) and z-equation (22) take the following form:

$$H_1^2 + 2H_1H_z + \frac{k}{a^2} = \Lambda, \tag{25}$$

$$2\dot{H}_1 + 3H_1^2 + \frac{k}{a^2} = \Lambda. \tag{26}$$

For convenience, one can also use Eqs. (25) and (26) to derive the following equation:

$$\dot{H}_1 + H_1^2 = H_1H_z. \tag{27}$$

Equation (27) can be shown to give

$$a_z = k_0\dot{a} \tag{28}$$

for some integration constant k_0. Note also that Eq. (26) can also be integrated as

$$H_1^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{k_1}{a^3} \tag{29}$$

for some integration constant k_1. For the case $k = 0$, one can show that $\ddot{A} = 9H_0^2A/4$ if we write $A = a^{3/2}$. Hence one can integrate this equation to obtain

$$a(t) = a(0)\bigg[\frac{\exp[3H_0t/2] + k_2\exp[-3H_0t/2]}{1 + k_2}\bigg]^{2/3} \tag{30}$$

for some constant k_2. Here $H_0^2 = \Lambda/3$ denotes the expansion factor. For the case where $k \neq 0$, a special solution with $k_1 = 0$ can be found to be

$$a = a_1\bigg[\exp[H_0t] + \frac{k}{4H_0^2a_1^2}\exp[-H_0t]\bigg]. \tag{31}$$

In fact, one can show that the evolutionary properties of these solutions can be obtained without knowing the exact solutions. Indeed, one will show in a moment that the inflationary solution will try to evolve to an isotropic FRW space as $t \to \infty$. In addition, one can also show that these solutions will remain isotropic from a stability analysis. One will study the evolution of the physical universe from an anisotropic initial state to an isotropic final state. Therefore, one will write the above equations in terms of the following variables:

$$3H = 2H_1 + H_z = \frac{\dot{V}}{V}, \tag{32}$$

$$\Delta \equiv (H_1 - H_z) \tag{33}$$

with Σ, $V = a^2a_z$, and Δ the total expansion rate, 3-volume, and the deviation function, respectively. One expects $\Delta \to 0$ as the physical universe evolves toward an isotropic final state. Indeed, one can show that the field equations can be written as

$$\dot{\Delta} + 3H\Delta = -\frac{k}{a^2}. \qquad (34)$$

Equation (34) can be rewritten as

$$\frac{d}{dt}(V\Delta) = -ka_z. \qquad (35)$$

Hence one has

$$\Delta = \Delta(0)\frac{V(0)}{V} - \frac{ka^2(0)}{a^2}\frac{\int_0^t a_z(t')dt'}{a_z}. \qquad (36)$$

If a_z eventually expands as $\exp[\Lambda_0 t]$, one can show that $\int_0^t a_z(t')dt'/a_z \to 1/\Lambda_0$, a small constant, as $t \to \infty$. Since the scaling factor $a^2(0)/a^2 \to 0$ for an expanding a solution, this equation implies that $\Delta \to 0$ as $t \to \infty$. Hence one shows that the field equations of the EH action will definitely take the anisotropic universe, either one of the KS type spaces, to the final FRW universe as $t \to \infty$.

One can also show that the final isotropic FRW universe is stable against any small perturbations $H_1 = H_0 + \delta H_1$ and $H_z = H_0 + \delta H_z$. For convenience, one can also use Eqs. (25) and (26) to derive the following equation:

$$\dot{H}_z + H_z^2 + 2H_1H_z = \Lambda. \qquad (37)$$

Applying these perturbations to Eqs. (27) and (37), one has

$$\delta\dot{H}_1 + H_0\delta H_1 - H_0\delta H_z = 0, \qquad (38)$$

$$\delta\dot{H}_z + 4H_0\delta H_z + 2H_0\delta H_1 = 0. \qquad (39)$$

Adding Eqs. (38) and (39) one can derive

$$(\delta\dot{H}_1 + 3H_0\delta H_1) + (\delta\dot{H}_z + 3H_0\delta H_z) = 0. \qquad (40)$$

Hence one has $\delta H_1 + \delta H_z = \text{constant}/a^3 \to 0$ as $a \to \infty$. Hence any physical perturbation against the FRW background would imply $\delta H_1 \to -\delta H_z \equiv \delta H$. Hence, Eq. (38) implies

$$\delta\dot{H} + 2H_0\delta H = 0. \qquad (41)$$

This equation can be integrated to obtain the result

$$\delta H_1 = \frac{\text{constant}}{a^2} \to 0 \qquad (42)$$

as $t \to \infty$. Hence one can again show that the final FRW space is stable against this anisotropic perturbation. Note that both $\delta H_i \to 1/a^2$ asymptotically, while their signs in the order of $O(a^{-2})$ are opposite to each other such that their sum $\delta H_1 + \delta H_z \to 1/a^3$.

IV. STABILITY OF HIGHER DERIVATIVE INFLATIONARY SOLUTION

One can then apply the perturbation, $H_i = H_{i0} + \delta H_i$, to the field equation with H_{i0} the background solution to the system. This perturbation will enable one to understand whether the background solution is stable or not. In particular, one would like to learn whether a KS $\to$ FRW (KS/FRW) type evolutionary solution is stable or not.

Note that our universe could start out anisotropic even if evidences indicate that our universe is isotropic to a very high degree of precision in the post-inflationary era. Therefore, one expects that any physical model should admit a stable KS/FRW solution. In particular, one will be interested in a de Sitter (dS) background solution with $H_{i0} = H_0$ for some constant Hubble expansion parameter. One will denote such solution as KS $\to$ de Sitter (KS/dS) type inflationary solution.

One can show that any FRW inflationary solution with a stable mode and an unstable mode is a negative result to our search for a stable inflationary model. In particular, any FRW inflationary solution with a stable mode and an unstable mode will provide a natural way for the inflationary universe to exit the inflationary phase. Such models will, however, also be unstable against the anisotropic perturbations. Therefore, such solution will be harmful for the system to settle from anisotropic space to FRW space once the graceful exit process is done. One will show in this section that the higher derivative gravity theory one considers here could also accommodate two stable modes with appropriately chosen coupling constants. In such case, the inflationary de Sitter solution H_0 will also be stable against anisotropic perturbations.

First of all, one can show that the first order perturbation equation from the nonredundant field equation (21), with $H_i \to H_0 + \delta H_i$, gives

$$\begin{aligned}\langle H_i L^{ij}\delta\ddot{H}_j\rangle + 3H\langle H_i L^{ij}\delta\dot{H}_j\rangle + \delta\langle H_i\dot{L}^i\rangle \\ + 3H\langle(H_i L^i_j + L^j)\delta H_j\rangle + \langle H_i L^i\rangle\delta(3H) \\ = \langle H_i L_{ij}\delta H_j\rangle + \langle\dot{H}_i\delta L^i\rangle \qquad (43)\end{aligned}$$

with all functions of H_i evaluated at some FRW background with $H_i = H_0$. The notation $\langle A_iB_i\rangle \equiv \sum_{i=1,z}A_iB_i$ denotes the summation over $i = 1$ and z for repeated indices. Note that we have absorbed the information of $i = 2$ into $i = 1$ since they contribute equally to the field equations in the KS type spaces. In addition, $L^i_j \equiv \delta^2 L/\delta\dot{H}_i\delta H_j$ and similarly for L_{ij} and L^{ij} with upper index ^{i} and lower index $_j$ denoting variation with respect to $\dot{H}_i$ and H_j respectively for convenience. In addition, perturbing equation (22) can also be shown to reproduce the Eq. (43) in the FRW limit [1].

Once we adopt the de Sitter solution with H_0 = constant, one has

$$\langle H_i L^{ij}\delta\ddot{H}_j\rangle + 3H\langle H_i L^{ij}\delta\dot{H}_j\rangle + 3H\langle (H_i L^i_j + L^j)\delta H_j\rangle + \langle H_i L^i\rangle\delta(3H) = \langle H_i L_{ij}\delta H_j\rangle \tag{44}$$

where all field variables are understood to be evaluated at the background de Sitter space where $H_i = H_0 =$ constant for all directions.

If the inflationary de Sitter solution has one stable mode and one unstable mode for the system, the unstable mode is expected to collapse the de Sitter phase. Then the inflationary era will come to an end once the unstable mode takes over. It was shown earlier to be a helpful way to select a physically acceptable model for our universe. Our result shown here indicates, however, that the unstable mode will also tamper the stability of the isotropic space. Indeed, if the model has an unstable mode for the de Sitter perturbation, this unstable mode will also be unstable against the anisotropic perturbation.

For example, one can show that the model [10]

$$\mathcal{L} = -R + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma R^{\mu\nu}{}_{\beta\gamma} R^{\beta\gamma}{}_{\sigma\rho} R^{\sigma\rho}{}_{\mu\nu} \tag{45}$$

admits an inflationary solution if $\gamma > 0$. Note that the γ term is the minimal consistent effective low-energy two-loop renormalizable Lagrangian for pure gravity theory [21]. In addition, the quadratic terms can be shown to be derivable from the matter effect of quantum fields. For simplicity, one can write

$$A = \dot{H}_1 + H_1^2, \tag{46}$$

$$B = H_1^2 + \frac{k}{a^2}, \tag{47}$$

$$C = H_1 H_z, \tag{48}$$

$$D = \dot{H}_z + H_z^2 \tag{49}$$

since all curvature tensor components and all field equations will be functions of the above combinations. This notation will be shown to very convenient in tracking the field equations for any complicated models such as the higher derivative models we are working on in this paper. Indeed, one can show that the Lagrangian reads

$$\begin{aligned} L = {} & 4A + 2B + 4C + 2D + 4\alpha[4A^2 + B^2 + 4C^2 + D^2 \\ & + 4AB + 8AC + 4AD + 4BC + 2BD + 4CD] \\ & + 2\beta[3A^2 + B^2 + 3C^2 + D^2 + 2AB + 2AC + 2AD \\ & + 2BC + 2CD] + 8\gamma[2A^3 + B^3 + 2C^3 + D^3]. \end{aligned} \tag{50}$$

This Lagrangian will reduce to the de Sitter models when we set $H_i \to H$ in the isotropic limit. Hence one can show that the generalized Friedmann equation (21) gives

$$H_0^4 = 1/4\gamma \tag{51}$$

in the isotropic limit for an inflationary solution with $\dot{a}^2 \gg 1$.

Note that the quadratic terms does not contribute to the Friedmann equation in the de Sitter limit with a constant inflationary phase. This can be shown to be a general property of the quadratic models. One will show later that these terms will, however, affect the stability conditions for the inflationary phase. Indeed, quadratic terms will be shown to affect the duration of the inflationary de Sitter phase which will be shown in a moment.

Note that one can also show that

$$\langle H_i L^{i1}\rangle = 2\langle H_i L^{iz}\rangle, \tag{52}$$

$$\langle H_i L^i_1\rangle = 2\langle H_i L^i_z\rangle, \tag{53}$$

$$L^1 = 2L^z, \tag{54}$$

$$\langle H_i L_{i1}\rangle = 2\langle H_i L_{iz}\rangle, \tag{55}$$

in the inflationary de Sitter background with $H_0 =$ constant. Therefore, the stability equations (44) can be greatly simplified. For convenience, one will define the operator $\mathcal{D}$ as

$$\mathcal{D}\delta H \equiv \langle H_i L^{i1}\rangle\delta\ddot{H} + 3H\langle H_i L^{i1}\rangle\delta\dot{H} + 3H\langle H_i L^i_1 + L^1\rangle\delta H + 2\langle H_i L^i\rangle\delta H - \langle H_i L_{i1}\rangle\delta H. \tag{56}$$

As a result, one can show that the stability equation (44) reads

$$\mathcal{D}(\delta H_1 + \delta H_z) = 0 \tag{57}$$

with $i = 1$ and $i = z$ components summed over. Since one can perturb the filed H_i from any directions, stability conditions must also hold for perturbation from any direction. Hence, one expects the stability conditions must hold and be identical from the perturbations in each direction. Indeed, the above result shows that both δH_1 and δH_z follow the same condition. Hence one needs to solve the following stability condition $\mathcal{D}\delta H_j = 0$ for both $i = 1$ and $i = z$. Hence one has

$$\langle H_i L^{i1}\rangle\delta\ddot{H}_j + 3H\langle H_i L^{i1}\rangle\delta\dot{H}_j + 3H\langle H_i L^i_1 + L^1\rangle\delta H_j + 2\langle H_i L^i\rangle\delta H_j = \langle H_i L_{i1}\rangle\delta H_j \tag{58}$$

to see if the system is stable or not against any small perturbation with respect to the de Sitter background.

In addition, the stability equation (58) for δH_i can be shown to be

$$[6\alpha + 2\beta + 12\gamma H_0^2][\delta\ddot{H}_i + 3H_0\delta\dot{H}_i] + (1 - 12\gamma H_0^4)\delta H_i = 0 \tag{59}$$

for such KS/dS solutions. Hence one has

$$\delta H_i = c_i \exp\left[\frac{-3H_0 t}{2}(1+\delta_1)\right] + d_i \exp\left[\frac{-3H_0 t}{2}(1-\delta_1)\right] \quad (60)$$

with

$$\delta_1 = \sqrt{1 + 8/[27 + 9(6\alpha + 2\beta)H_0^2]} \quad (61)$$

and some arbitrary constants c_i, d_i to be determined by the initial perturbations. It is easy to see that any small perturbation δH_i will be stable against the de Sitter background if both modes characterized by the exponents

$$\Delta_\pm \equiv -[3H_0 t/2][1 \pm \delta_1] \quad (62)$$

are all negative. This will happen if $\delta_1 < 1$. In such case, the inflationary de Sitter space will remain a stable background as the universe evolves.

On the other hand, one would have a stable mode and an unstable mode if $\delta_1 > 1$. This indicates that this model admits one stable mode and one unstable mode following the stability equation (58) for the inflationary de Sitter solution. It is shown to be a positive sign for an inflationary model that is capable of resolving the graceful exit problem in a natural manner.

Indeed, one expects any unstable mode for a model to be of the form $\delta H_i \sim \exp[lH_0 t]$, to the lowest order in $H_0 t$, in a de Sitter background with l some constant characterizing the stability property of the model. In such models, the inflationary phase will only remain stable for a period of the order $\Delta t \sim 1/lH_0$. The inflationary phase will start to collapse after this period of time. This means that the de Sitter background fails to be a good approximation when $t \gg \Delta t$.

As a result, the anisotropy will also grow according to $\delta H_i \to \delta H_i^0 \exp[lH_0 \Delta t]$ with δH_i^0 denoting the initial perturbation. Hence this model will have a problem remaining isotropic for a long period of time. Therefore, a pure gravity model of this sort will not solve the graceful exit problem. One will need, for example, the help of a certain scalar field to end the inflation in a consistent way. The unstable mode gives us, however, a hope that small anisotropy observed today can be generated by the initial inflationary instability for models with appropriate factor l.

The result shown in this paper shows that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe. First of all, the third order term characterized by the coupling constant γ will determine the expansion rate H_0, given by Eq. (51), for the inflationary de Sitter space. The quadratic terms characterized by the coefficients α and β will not affect the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase depending on the sign of the characteristic function $\Delta_\pm$. Both the third order term and quadratic terms are closely related to the quantum corrections of the quantum fields [15,21]. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different. They are however equivalently important in the higher derivative models.

V. CONCLUSION

We have tried to obtain a model-independent formula for the nonredundant field equations in the Kantowski-Sachs (KS) type anisotropic space. This equation is applied to obtain the stability conditions in pure gravity theories. It is also shown that the existence of a stable de Sitter background is closely related to the choices of the coupling constants. We first derive a stability equation which turns out to be identical to the stability equation for the existence of the inflationary de Sitter solution discussed in Refs. [10,20].

If the inflationary de Sitter solution in the pure gravity theory has one stable mode and one unstable mode for the system, the unstable mode is expected to collapse the de Sitter phase. Later on, the inflationary era will come to an end once the unstable mode takes over after a brief period of inflationary expansion. Our result indicates, however, that the unstable mode will also tamper the stability of the isotropic space.

To be more specific, if the model has an unstable mode for the de Sitter background perturbation with respect to isotropic perturbation, this unstable mode will also be unstable with respect to any anisotropic perturbations. In particular, we have shown in this paper that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe. First of all, the third order term is shown to determine the expansion rate H_0 for the inflationary de Sitter space. The quadratic terms are shown to have nothing to do with the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different.

In short, the result of this paper shows that graceful exit and stability of any de Sitter model cannot work along in a naive way. The physics behind the inflationary de Sitter models appears to be much more complicated than one expects. In another words, the phase transition during and after the inflationary phase deserves more attention and requires extraordinary care in order to resolve the problem lying ahead.

In particular, the result shown in this paper shows that the roles played by the higher derivative terms are dramatically different in the inflationary phase of our physical universe. First of all, the third order term characterized by the coupling constant γ determines the expansion rate H_0, given by Eq. (51), for the inflationary de Sitter space. The quadratic terms characterized by the coefficients α and

β will not affect the expansion rate of the background de Sitter space. They will however affect the stability condition of the de Sitter phase depending on the sign of the characteristic function $\Delta_\pm$. Both the third order term and quadratic terms are closely related to the quantum corrections of the quantum fields [15,21]. Their roles played in the existence and stability condition of the evolution of the de Sitter space are dramatically different. They are however equally important in the higher derivative models.

ACKNOWLEDGMENTS

This work is supported in part by the National Science Council of Taiwan.

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