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Quality yield measure for processes with asymmetric tolerances

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Process capability indices provide numerical measures on whether or not a process is able to produce products that meet prespecified quality targets and are often used by manufacturers to evaluate manufacturing performance. Although process yield is the primary focus of the performance criteria, a formula that combines the yield and the average process loss, called the quality yield index, has been developed. This index, the quality yield, can be viewed as the conventional process yield minus the truncated expected relative process loss within the specifications. Although cases with symmetric tolerances dominate in practical situations, cases with asymmetric tolerances can also occur. In this paper, we generalize the quality yield index for asymmetric tolerances. The generalization technique is justified, and some statistical properties of the estimated generalization are investigated. An application example on high-density light emitting diodes is also presented to illustrate the applicability of the generalization.

1. Introduction

Process capability indices (PCIs) are widely used in manufacturing industries, to provide a numerical measure on whether or not a process is capable of producing items that meet a preset quality requirement. Kane (1986) considered the two basic indices C_p and C_{pk}, and investigated some properties of their estimators. The two basic indices C_p and C_{pk} are defined as:

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

$$C_{pk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sigma}, \tag{2}$$

where USL and LSL are the upper and lower specification limits, μ and σ are the process mean and the standard deviation of the characteristic, respectively. Since the designs of C_p and C_{pk} are independent of the target value T, they can fail to account for process loss incurred by a departure from the target value. For this reason, two more advanced indices C_{pm} and C_{pmk} were developed by Chan *et al.* (1988), and Pearn *et al.* (1992), which are defined as:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

$$C_{pmk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{4}$$

For more details about PCIs, see the books by Kotz and Johnson (1993) and Kotz and Lovelace (1998) for a good overview of the literature. The recent review paper by Kotz and Johnson (2002) provides a compact survey with interpretations and comments on some 170 publications on PCIs, that were published during 1992–2000.

There are three measures that are of considerable interest to us and we will now highlight their properties.

Yield measure: Boyles (1991) noted that C_{pk} is a yield-based index. Yield, the proportion of conforming items, is a commonly accepted measurement criterion for the process capability. Suppose that the proportion of conforming items is the primary concern, then the natural measure is the proportion itself called the yield, which we refer to as Y and is defined as:

$$Y = \int_{LSL}^{USL} \mathrm{d}F_X(x), \tag{5}$$

*Corresponding author

where $F_X(x)$ is the cumulative distribution function of the measured characteristic X. The disadvantage of the yield measure is that it does not distinguish between the products that fall inside the specification limits. To remedy this disadvantage, the quadratic loss function can be used to distinguish between the products by increasing the penalty as the departure from the target value increases. However, the quadratic loss function does not provide any comparison with the specification limits.

Loss measure: To remedy this problem, Johnson (1992) developed the so-called relative expected loss L_e for the symmetric case, which is defined as the ratio of the expected quadratic loss to the square of the half-specification width:

$$L_e = \frac{\sigma^2 + (\mu - T)^2}{d^2}, \tag{6}$$

where $d = (USL - LSL)/2$ is the half-specification width. This measure has a direct relationship with C_{pm} because $L_e = (3C_{pm})^{-2}$. The advantage of L_e over C_{pm} is that the estimator of the former has better statistical properties than that of the latter, since the former does not involve a reciprocal transformation of the process mean and variance. The disadvantage of the L_e index is the difficulty in setting a standard for the measure since its value ranges from zero to infinity.

Quality yield measure: Tsui (1997) proposed the quality yield index to incorporate the average process loss obtained with the conventional yield measure. The quality yield index, which has been referred to as the Q-yield and denoted by Y_q is defined as:

$$Y_q = \int_{LSL}^{USL} \left[1 - \frac{(x-T)^2}{d^2}\right] \mathrm{d}F_X(x). \tag{7}$$

A process is said to have a symmetric tolerance if the target value T is set to be the midpoint of the specification interval [LSL,USL], i.e., $T = M = (USL + LSL)/2$. Most research in the quality assurance literature is focused on cases in which the manufacturing tolerance is symmetric. Examples include Kane (1986), Chan *et al.* (1988), Choi and Owen (1990), Boyles (1991), Pearn *et al.* (1992), Vännman (1995), Vännman and Kotz (1995), and Spiring (1997). Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances often occur in manufacturing industries.

In general, asymmetric tolerances simply reflect that deviations from the target value are less tolerable in one direction than in the other direction (Boyles, 1994; Vännman,1997; Wu and Tang, 1998). Asymmetric tolerances can also arise from a situation in which the tolerances are symmetric to begin with, but the process follows a non-normal distribution and the data are transformed to achieve approximate normality, as shown by Chou *et al.* (1998) who have used Johnson's curves to transform non-normal process data. Unfortunately, there has been comparatively little research published on cases with asymmetric tolerances. Exceptions include Boyles (1994), Vännman (1997), Chen (1998), Pearn and Chen (1998), Chen *et al.* (1999), and Pearn *et al.* (1999).

In this paper, we consider the quality yield index for processes with asymmetric tolerances. We consider an asymmetric loss function, and the corresponding truncated worth function to generalize the quality yield index. Comparisons among the yield, the quality yield, and some popular process capability indices are examined. Distributional properties of the estimated Y_q are also investigated. A confidence interval for Y_q is constructed to estimate the manufacturing capability. Finally, an application example using the index Y_q to assess the manufacturing capability of light emitting diodes is presented to illustrate the applicability of the proposed approach.

2. Quality yield with asymmetric tolerances

Yield is currently defined as the percentage of processed units that pass inspection. Therefore, the yield index Y can be defined mathematically as the expected value of the worth $W(X)$ where $W(x) = 1$ for $LSL < x < USL$ and $W(x) = 0$ for $x \le LSL$ or $x \ge USL$, that is, $Y = E[W(X)]$. The disadvantage of the yield measure is that it does not distinguish the worth of the products that fall inside the specification limits, i.e., they are equally good.

Taguchi championed the concept of the process loss (product's worth) when the quality characteristic departs from the customers' ideal value T. The cost of a characteristic X missing the target is often assumed to be well approximated by the symmetric squared error loss function (Hsiang and Taguchi, 1985):

$$L(x) = k(x - T)^2, \tag{8}$$

where k is a positive constant. A product has the maximal worth W_T when the corresponding characteristic X has the target value T (Johnson, 1992). Using the loss function given by Equation (8), the worth of the product with characteristic X is:

$$W(x) = W_T - k(x - T)^2. \tag{9}$$

Therefore, as the deviation of X from T increases, the worth becomes less, eventually becoming zero and then negative.

2.1. *Asymmetric loss function*

Now, for a process with the manufacturing specification (LSL, T, USL), we can redefine $W(x) = 0$ for $x \le LSL$ or $x \ge USL$, and $W(x) = W_T - k(x - T)^2$ for $LSL < x < USL$. Using $W(LSL) = 0$, we obtain $k = W_T/(d_l)^2$, where $d_l = T - LSL$. On the other hand, using $W(USL) = 0$, we obtain $k = W_T/(d_u)^2$, where $d_u = USL - T$. For the symmetric case, both of the values of k reduce to $W_T/(d)^2$. Without loss of generality, we can set $W_T = 1$. Therefore, for a process with a manufacturing specification of (LSL, T, USL), we can define a general truncated loss function of x as:

$$L(x) = \begin{cases} [(T-x)/d_l]^2 & LSL < x \le T, \\ [(x-T)/d_u]^2 & T \le x < USL, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Hence, the corresponding general truncated worth function of x becomes:

$$W(x) = \begin{cases} 1-[(T-x)/d_l]^2 & LSL < x \le T, \\ 1-[(x-T)/d_u]^2 & T \le x < USL, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then, the expected loss L_e, defined as $E[L(X)]$, can be expressed as:

$$\begin{aligned} L_e = \int_{-\infty}^{\infty} L(x)\mathrm{d}F_X(x) &= 1 + F_X(LSL) - F_X(USL) \\ &+ (d_l)^{-2}E[(T-X)^2|LSL < X \le T]P[LSL < X \le T] \\ &+ (d_u)^{-2}E[(X-T)^2|T \le X < USL]P[T \le X < USL]. \end{aligned} \tag{12}$$

Figure 1 is a plot of $L(x)$ for a process with an asymmetric manufacturing specification of (LSL, T, USL) = (10, 40, 50). Figure 2 is a plot of $W(x)$ for a process with an asymmetric tolerance of (LSL, T, USL) = (10, 40, 50). Now, using the worth function, we can distinguish between the product worths of products that fall inside of the specification limits.

Consider two items x_1 and x_2 with $x_1 > T$ and $x_2 < T$, satisfying the relationship $(x_1 - T)/d_u = (T - x_2)/d_l$ (equal departure ratios). In this case, the worth values given to items x_1 and x_2 are the same. For example, we note that for the midpoint of the left-hand side tolerance, $x_1 = (T + LSL)/2$, and the midpoint of the right-hand side tolerance, $x_2 = (T + USL)/2$, the corresponding worth can be calculated as:

$$\begin{aligned} W(x_1) &= 1 - [(T - x_1)/d_l]^2 \\ &= 1 - \{[T - (T + LSL)/2]/(T - LSL)\}^2 = 3/4, \\ W(x_2) &= 1 - [(x_2 - T)/d_u]^2 \\ &= 1 - \{[((T + USL)/2) - T]/(USL - T)\}^2 = 3/4. \end{aligned}$$

Obviously, the two points x_1 and x_2 have the same departure ratio (relative departure) $k = (T - x_1)/d_l = (x_2 - T)/d_u = 1/2$. Checking the process loss at x_1 and x_2, we have that $L(x_1) = L(x_2) = 1/4$ and the equal worth value is 3/4. In fact, $0 < W(x) < 1$ for $LSL < x < USL$ and $W(T) = 1$. On the other hand, $W(x) = 0$ if x falls outside the specification limits.

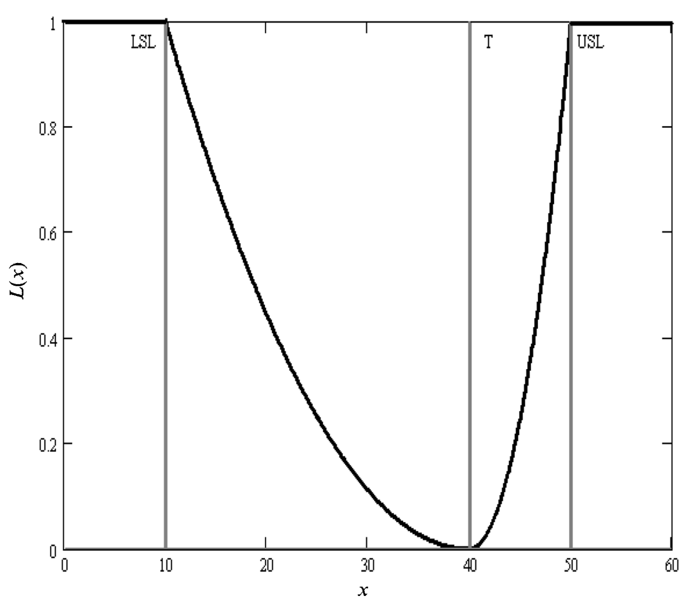

Fig. 1. The plot of $L(x)$, the loss function for an asymmetric specification (LSL, T, USL) = (10, 40, 50).

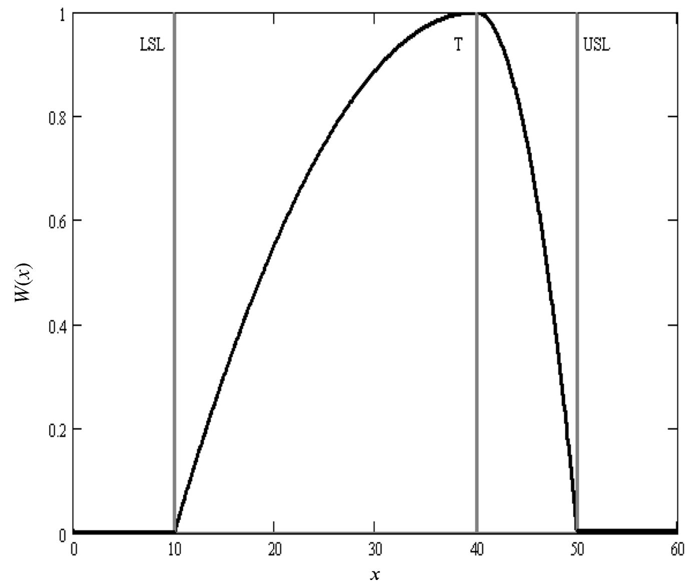

Fig. 2. The plot of $W(x)$, the worth function for an asymmetric specification (LSL, T, USL) = (10, 40, 50).

2.2. *Quality yield with asymmetric tolerances*

Suppose that a process characteristic X follows a distribution with the cumulative distribution function $F_X(x)$ and the probability density function $f_X(x)$. $F_W(w)$, the cumulative distribution function of $W(X)$, can be expressed as (see Appendix):

$$F_W(w) = 1 + F_X(T - d_l\sqrt{1-w}) - F_X(T + d_u\sqrt{1-w}), \quad 0 \le w \le 1. \tag{13}$$

Particularly, the fraction of nonconforming items, the probability of an item falling outside the specified tolerance limits, can be calculated as:

$$F_W(0) = 1 + F_X(LSL) - F_X(USL). \tag{14}$$

Table 1. Normal distribution with $\mu = T$ compared to $Y_q = 0.5(0.1)0.9$

Y_q (%)	Case 1	
	μ	σ
50	T	3.558 213
60	T	2.782 604
70	T	2.176 123
80	T	1.651 2655
90	T	1.121 61

Hence, $f_W(w)$, the probability density function for $W(X)$, can be expressed as:

$$f_W(w) = \frac{1}{2\sqrt{1-w}}\{d_l f_X(T - d_l\sqrt{1-w}) + d_u f_X(T + d_u\sqrt{1-w})\}, \quad 0 < w < 1. \quad (15)$$

The mean value and variance of $W(X)$ can be calculated as:

$$E[W(X)] = \int_0^1 w \mathrm{d}F_W(w), = \int_0^1 \frac{w}{2\sqrt{1-w}}\{d_l f_X(T - d_l\sqrt{1-w}) + d_u f_X(T + d_u\sqrt{1-w})\}\mathrm{d}w, \quad (16)$$

$$E[W(X)]^2 = \int_0^1 w^2 \mathrm{d}F_W(w), = \int_0^1 \frac{w^2}{2\sqrt{1-w}}\{d_l f_X(T - d_l\sqrt{1-w}) + d_u f_X(T + d_u\sqrt{1-w})\}\mathrm{d}w, \quad (17)$$

$$\mathrm{Var}[W(X)] = E[W(X)]^2 - E^2[W(X)]. \quad (18)$$

Now we can define the Q-yield as $E[W(X)]$, the expected value of the worth $W(X)$. The Q-yield will be between zero and one, and can be used as an index of the ability of a process when considering process yield and process loss. The Q-yield index Y_q can be interpreted as the proportion of "perfect" items whereas the yield index Y is the proportion of conforming items. As with the existing process capability indices, the Q-yield index Y_q also has the larger-the-better property.

Table 2. Normal distribution with μ shifted from T to USL by $d_u/6$, $d_u/4$, and $d_u/3$, respectively, compared to $Y_q = 0.5(0.1)0.9$

Y_q (%)	Case 2		Case 3		Case 4	
	μ	σ	μ	σ	μ	σ
50	$T + d_u/6$	3.593 474	$T + d_u/4$	3.551 352	$T + d_u/3$	3.465 2255
60	$T + d_u/6$	2.824 0045	$T + d_u/4$	2.767 893	$T + d_u/3$	2.651 555
70	$T + d_u/6$	2.221 167	$T + d_u/4$	2.144 3699	$T + d_u/3$	1.981 3995
80	$T + d_u/6$	1.690 9245	$T + d_u/4$	1.575 1335	$T + d_u/3$	1.316 363
90	$T + d_u/6$	1.111 1475	$T + d_u/4$	0.852 496	$T + d_u/3$	—

Table 3. Normal distribution with μ shifted from T to LSL by $d_l/6$, $d_l/4$, and $d_l/3$, respectively, compared to $Y_q = 0.5(0.1)0.9$

Y_q (%)	Case 5		Case 6		Case 7	
	μ	σ	μ	σ	μ	σ
50	$T - d_l/6$	3.440 189	$T - d_l/4$	3.345 944	$T - d_l/3$	3.221 025
60	$T - d_l/6$	2.630 8625	$T - d_l/4$	2.503 9585	$T - d_l/3$	2.326 2755
70	$T - d_l/6$	1.985 113	$T - d_l/4$	1.818 3015	$T - d_l/3$	1.576 054
80	$T - d_l/6$	1.419 7015	$T - d_l/4$	1.216 756	$T - d_l/3$	0.930 123
90	$T - d_l/6$	0.850 78	$T - d_l/4$	0.587 4915	$T - d_l/3$	—

This quality yield index differs from the expected relative worth index defined in Johnson (1992) in that it truncates the deviation outside the specifications. With this truncation, the quality yield index will be between zero and one and, thus it provides a standardized measure. Also, by relating it to the yield measure, which is widely accepted in manufacturing industries, it will be better understood and accepted as a capability measure. The advantage of the Y_q index over the L_e index is the value of the former goes from zero to one. Similar to the yield index Y, an ideal value of Y_q is one, which provides the user a clear guide about the standard. Similarly to the yield Y, the yield index Y_q requires no normality assumption.

To illustrate some basic properties of the quality yield Y_q compared to a normal distribution for various application cases, we consider the parameter settings listed in Tables 1–4. For a process with an asymmetric tolerance (LSL, T, USL) = (3, 0, 4.5) (so that $d_l = 3$, $d_u = 4.5$), five levels of Y_q, 0.5(0.1)0.9, are selected in each case. The studied cases are arranged in the following manner. In case 1, we set $\mu = T$ and calculated the corresponding σ for each Y_q level. In cases 2–4, μ is shifted from T toward USL by $d_u/6$, $d_u/4$, and $d_u/3$, respectively. We then solve for σ in each setting. In cases 5–7, μ is shifted from T toward LSL by $d_l s/6$, $d_l/4$, and $d_l/3$, respectively. We then solve for σ in each case. Finally, in cases 8–10, σ is fixed at three levels, 1/3, 1/2, and 1. The corresponding values of μ in each setting have again been computed.

Figures 3–6 display four selected normally distributed processes, which are $N(\mu = T, \sigma)$, $N(\mu = T + d_u/4, \sigma)$, $N(\mu = T - d_l/4, \sigma)$ and $N(\mu, \sigma = 1/2)$ respectively, with the quadratic loss function and five levels of quality yield (see cases 1, 3, 6, and 9). The quality yield could be treated

Table 4. Normal distribution with σ fixed in three levels, 1/3, 1/2, and 1, respectively, compared to $Y_q = 0.5(0.1)0.9$

Y_q (%)	Case 8		Case 9		Case 10	
	μ	σ	μ	σ	μ	σ
50	3.164 4764	1/3	3.143 2	0.5	3.076668	1
60	2.826 46245	1/3	2.801 8575	0.5	2.689652	1
70	2.442 1075	1/3	2.413 5035	0.5	2.2613755	1
80	1.984 6635	1/3	1.949 3365	0.5	1.744542	1
90	1.383 308	1/3	1.332 17	0.5	0.960625	1

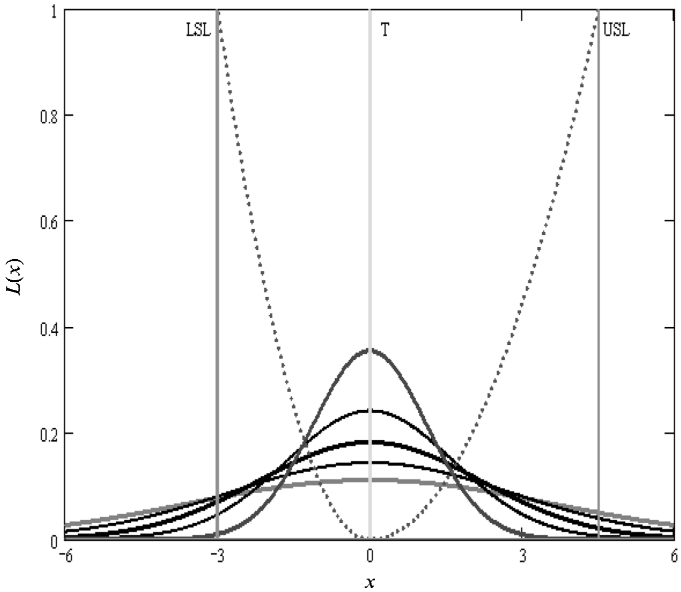

Fig. 3. Distribution plots of normal distribution $N(\mu = T, \sigma)$ with the loss function for various σ.

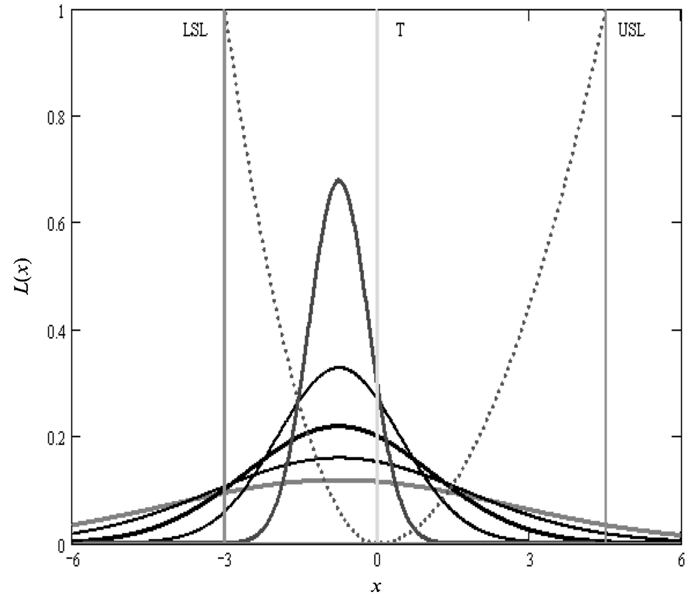

Fig. 5. Distribution plots of normal distribution $N(\mu = T - d_l/4, \sigma)$ with the loss function for various σ.

as the traditional yield minus the truncated expected relative loss within the specifications to quantify how well a process can reproduce product items to meet customer requirements. Whereas yield is the proportion of conforming products, Q-yield can be interpreted as the average degree of products reaching "perfect" or "on target" states.

3. Comparison of yield, Q-yield, and PCIs

To illustrate the basic differences between the yield Y, the quality yield Y_q, and the four well-known process capability indices C_p, C_{pk}, C_{pm} and C_{pmk}, we compare values measured on some processes based on the yield Y, quality yield Y_q, and the four indices.

3.1. *Comparison of Q-yield and yield*

Both the Q-yield index and the conventional yield index can be applied to processes with any distribution. The conventional yield, however, does not distinguish between the products that fall inside the specification tolerance. For example, if X follows the uniform distribution $U(LSL, USL)$

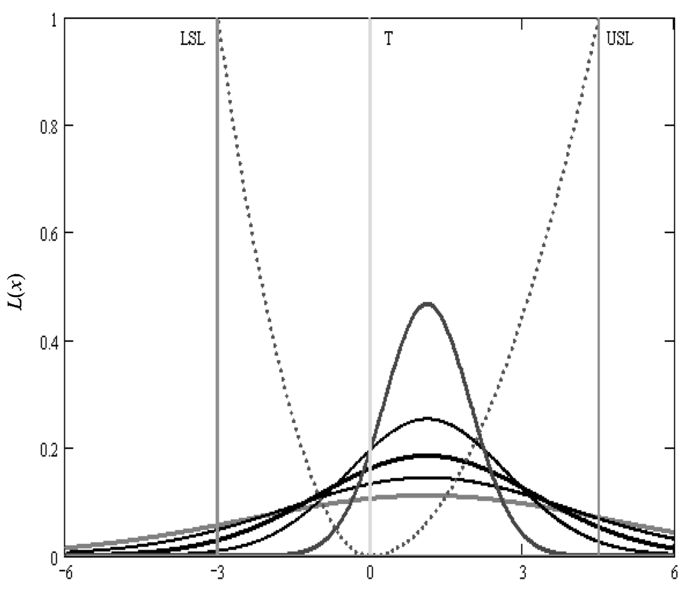

Fig. 4. Distribution plots of normal distribution $N(\mu = T + d_u/4, \sigma)$ with the loss function for various σ.

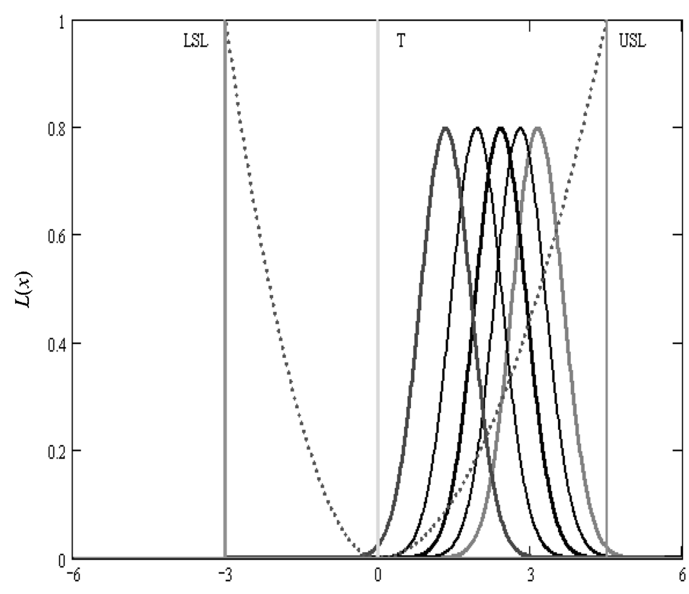

Fig. 6. Distribution plots of normal distribution $N(\mu, \sigma = 1/2)$ with the loss function for various μ.

with target T where $LSL < T < USL$, then yield $Y = 1.00$ and Q-yield $Y_q = 0.667$, respectively. From a manufacturing perspective all the produced units are good products, however, the consumer would consider the process to be of a low quality even though the yield $Y = 1.00$. To further demonstrate the difference between yield and Q-yield, we consider a set of triangular-distributed processes with $a < x < b$ and mode c. Table 5 lists the quality yield measure of those triangular-distributed processes with modes $c = 11(1)49$, $(a, b) = (LSL, USL) = (10, 50)$, and target value $T = 30(5)45$. For these processes, the yield value given to all processes is $Y = 1.00$. On the other hand, the Q-yield obtains its maximum of 0.833 (in bold type) not at

Table 5. Comparisons of the Q-yield measure for triangular processes with mode $c = 11(1)49$, $(LSL, USL) = (10, 50)$, and $T = 30(5)45$

c	μ	$T = 30$	$T = 35$	$T = 40$	$T = 45$
11	23.667	0.6828	0.6404	0.5979	0.5551
12	24.000	0.6981	0.6557	0.6118	0.5679
13	24.333	0.7129	0.6687	0.6250	0.5821
14	24.667	0.7259	0.6832	0.6388	0.5936
15	25.000	0.7396	0.6952	0.6515	0.6059
16	25.333	0.7517	0.7088	0.6627	0.6179
17	25.667	0.7611	0.7208	0.6760	0.6282
18	26.000	0.7733	0.7322	0.6877	0.6411
19	26.333	0.7834	0.7430	0.6990	0.6524
20	26.667	0.7899	0.7532	0.7100	0.6633
21	27.000	0.7992	0.7630	0.7203	0.6739
22	27.333	0.8073	0.7715	0.7304	0.6809
23	27.667	0.8128	0.7779	0.7399	0.6943
24	28.000	0.8180	0.7880	0.7489	0.7040
25	28.333	0.8229	0.7963	0.7591	0.7133
26	28.667	0.8257	0.8022	0.7665	0.7232
27	29.000	0.8297	0.8067	0.7740	0.7307
28	29.333	0.8314	0.8142	0.7829	0.7355
29	29.667	0.8328	0.8188	0.7887	0.7490
30	30.000	**0.8333**	0.8219	0.7930	0.7560
31	30.333	0.8328	0.8267	0.8018	0.7645
32	30.667	0.8314	0.8290	0.8080	0.7709
33	31.000	0.8297	0.8325	0.8130	0.7790
34	31.333	0.8257	0.8329	0.8163	0.7853
35	31.667	0.8229	**0.8333**	0.8222	0.7918
36	32.000	0.8180	0.8329	0.8260	0.7980
37	32.333	0.8128	0.8314	0.8284	0.8043
38	32.667	0.8073	0.8284	0.8314	0.8066
39	33.000	0.7992	0.8259	0.8331	0.8147
40	33.333	0.7899	0.8199	**0.8333**	0.8203
41	33.667	0.7834	0.8157	0.8326	0.8242
42	34.000	0.7733	0.8071	0.8305	0.8275
43	34.333	0.7611	0.8001	0.8272	0.8308
44	34.667	0.7517	0.7906	0.8226	0.8327
45	35.000	0.7396	0.7796	0.8147	**0.8333**
46	35.333	0.7259	0.7686	0.8065	0.8321
47	35.667	0.7129	0.7549	0.7958	0.8289
48	36.000	0.6981	0.7409	0.7824	0.8206
49	36.333	0.6828	0.7254	0.7674	0.8085

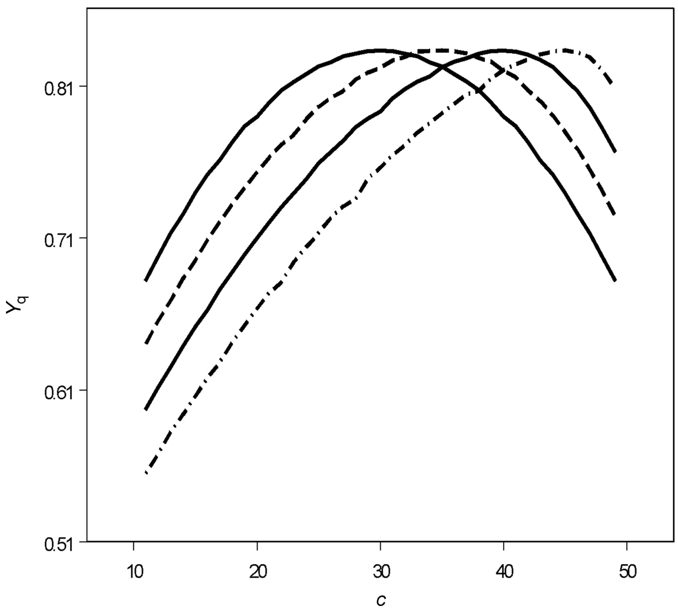

Fig. 7. Plots of Y_q for processes with $c = 11(1)49$, and $T = 30(5)45$ (left to right).

$\mu = T$ but at the mode $c = T$ for the triangular-distributed processes. The plots of Y_q versus mode $c = 11\ (1)49$ with $T = 30(5)45$ are displayed in Fig. 7. The figure shows that Y_q always attains its maximum value of 0.833 at the mode, as the target value moves from 30 to 45 in steps of five.

For a normally distributed process with mean μ and standard deviation σ, we can write $X \sim N(\mu, \sigma^2)$. Using Equation (13), the corresponding cumulative distribution function of $W(X)$ is:

$$F_W(w) = 1 + \Phi((T - \mu - d_l\sqrt{1-w})/\sigma) - \Phi((T - \mu + d_u\sqrt{1-w})/\sigma),\ 0 \leq w \leq 1, \quad (19)$$

where Φ is the cumulative distribution function of the standard normal distribution. The corresponding probability density function of $W(X)$ is:

$$f_W(w) = \frac{1}{2\sigma\sqrt{1-w}}\{d_l\phi((T - \mu - d_l\sqrt{1-w})/\sigma) + d_u\phi((T - \mu + d_u\sqrt{1-w})/\sigma)\}, \quad (20)$$

where $0 < w < 1$, and ϕ is the probability density function of the standard normal distribution. The corresponding Q-yield, defined as the expected value function of $W(X)$, therefore, can be expressed as:

$$Y_q = \int_0^1 \frac{w}{2\sigma\sqrt{1-w}}\{d_l\phi((T - \mu - d_l\sqrt{1-w})/\sigma) + d_u\phi((T - \mu + d_u\sqrt{1-w})/\sigma)\}\mathrm{d}w. \quad (21)$$

Table 6 is a comparison of the Q-yield for normally distributed processes with $\mu = 10(1)50$, $\sigma = 10/3$ and $20/3$ respectively, where $(LSL, USL) = (10, 50)$ and $T = 30(5)45$. For symmetric cases ($T = 30$), the maximal Y_q occurs at

Table 6. Comparisons of the Q-yield measure for normal processes with $\mu = 10(1)50$, $\sigma = 10/3$, $20/3$, $(LSL, USL) = (10, 50)$, and $T = 30(5)45$

	$\sigma = 10/3$				$\sigma = 20/3$			
μ	$T=30$	$T=35$	$T=40$	$T=45$	$T=30$	$T=35$	$T=40$	$T=45$
10	0.119	0.098	0.082	0.071	0.210	0.177	0.153	0.134
11	0.167	0.137	0.116	0.101	0.249	0.210	0.182	0.159
12	0.223	0.184	0.156	0.136	0.290	0.246	0.213	0.187
13	0.286	0.236	0.201	0.175	0.334	0.285	0.247	0.218
14	0.352	0.292	0.250	0.218	0.381	0.326	0.283	0.250
15	0.420	0.350	0.300	0.262	0.428	0.368	0.321	0.284
16	0.487	0.409	0.351	0.307	0.476	0.412	0.360	0.319
17	0.552	0.466	0.401	0.352	0.524	0.456	0.400	0.355
18	0.613	0.521	0.451	0.396	0.572	0.500	0.440	0.392
19	0.670	0.573	0.498	0.439	0.617	0.543	0.480	0.429
20	0.722	0.622	0.543	0.481	0.661	0.586	0.520	0.466
21	0.770	0.669	0.587	0.521	0.702	0.627	0.559	0.502
22	0.812	0.712	0.628	0.559	0.740	0.666	0.597	0.538
23	0.850	0.752	0.667	0.596	0.774	0.703	0.634	0.573
24	0.882	0.789	0.703	0.631	0.804	0.737	0.669	0.606
25	0.910	0.822	0.738	0.664	0.830	0.769	0.702	0.639
26	0.932	0.853	0.770	0.696	0.851	0.797	0.733	0.670
27	0.950	0.880	0.800	0.726	0.868	0.822	0.761	0.699
28	0.962	0.904	0.828	0.755	0.880	0.843	0.787	0.727
29	0.970	0.924	0.853	0.782	0.887	0.860	0.811	0.753
30	**0.972**	0.942	0.877	0.807	**0.890**	0.873	0.831	0.777
31	0.970	0.955	0.898	0.831	0.887	0.882	0.849	0.799
32	0.962	0.965	0.916	0.853	0.880	**0.887**	0.862	0.818
33	0.950	0.970	0.933	0.873	0.868	0.886	0.872	0.835
34	0.932	**0.971**	0.947	0.892	0.851	0.881	0.878	0.848
35	0.910	0.966	0.958	0.909	0.830	0.870	**0.879**	0.858
36	0.882	0.956	0.965	0.925	0.804	0.854	0.875	0.865
37	0.850	0.938	**0.968**	0.938	0.774	0.833	0.866	**0.866**
38	0.812	0.914	0.966	0.949	0.740	0.806	0.850	0.863
39	0.770	0.881	0.957	0.957	0.702	0.775	0.829	0.854
40	0.722	0.840	0.939	**0.961**	0.661	0.738	0.802	0.839
41	0.670	0.791	0.910	0.959	0.617	0.697	0.769	0.818
42	0.613	0.734	0.868	0.947	0.572	0.653	0.731	0.790
43	0.552	0.669	0.813	0.923	0.524	0.605	0.687	0.756
44	0.487	0.598	0.744	0.882	0.476	0.555	0.640	0.716
45	0.420	0.520	0.663	0.823	0.428	0.503	0.588	0.671
46	0.352	0.440	0.573	0.743	0.381	0.451	0.534	0.621
47	0.286	0.360	0.477	0.646	0.334	0.399	0.479	0.567
48	0.223	0.283	0.381	0.538	0.290	0.349	0.424	0.512
49	0.167	0.213	0.290	0.426	0.249	0.301	0.370	0.455
50	0.119	0.153	0.210	0.320	0.210	0.256	0.319	0.398

$\mu = T$. However for asymmetric cases ($T \neq 30$), the maximal Y_q occurs not at $\mu = T$, but at a value between the target value T and 30 (the center of the specification interval). This is reasonable, because the on-target process ($\mu{=}T$) has a larger proportion of low-quality products than the process with maximal Y_q value. For example, let's compare two processes A and B with $\mu_A = 40$, $\mu_B = 45$, $\sigma_A = \sigma_B = 10/3$, and $(LSL, T, USL) = (10, 45, 50)$. In Table 6, we have $Y_q = 0.961$ for process A and $Y_q = 0.823$ for process B; the result corresponds to the fact that on-target process B has a larger proportion of low-quality products than process A.

As we mentioned earlier, two items with equal departure ratios have equal worth. However, for two processes, A and B, with equal departure ratios $(\mu_A - T)/d_u = (T - \mu_B)/d_l$ and $\sigma_A = \sigma_B$, there are not equal average worths for the two samples produced in processes A and B. For example, normally distributed processes A and B with $\mu_A = USL$, $\mu_B = LSL$ and $\sigma_A = \sigma_B$ have an equal yield Y, with the proportions of conforming items being 50%, but the Y_q values given to processes A and B are different for asymmetric cases. In fact, Table 6 also displays that the Y_q value given to process B is less than that given to process A, since the average quality of products coming from process A is better than that coming from process B for cases in which $T > M$.

3.2. *Comparison of Q-yield and PCIs*

Most of the investigations performed on the existing PCIs, C_p, C_{pk}, C_{pm}, and C_{pmk}, depend heavily on the assumption of a normal variability. If the underlying distributions are non-normal, then the capability calculations are highly unreliable since the conventional estimator S^2 of σ^2 is sensitive to departures from normality, and estimators of those indices are calculated using S^2(Somerville and Montgomery, 1997). Table 7 displays comparisons of the six indices: the yield Y, Q-yield Y_q, C_p, C_{pk}, C_{pm}, and C_{pmk} using normal processes for various values of μ with fixed $\sigma = 20/3$, and $(LSL, T, USL) = (10, 30, 50)$. For the symmetric case, all the six indices obtain their maximum at $\mu = T$.

Figures 8 and 9 display plots of Y_q, Yagainst μ and the four PCIs C_p, C_{pk}, C_{pm}, and C_{pmk} against μ respectively. With a fixed $\sigma = 20/3$, and $(LSL, T, USL) = (10, 30, 50)$, μ is varied from 10 to 50 in unit steps to examine the sensitivity of these indices with respect to μ. For the symmetric case, all the six indices attain their maximum at $\mu = T = 30$ as can be easily seen in the plots. However, as μ departs from T, all (except C_p) decrease, as one may expect.

Tables 8 and 9 are comparisons of the six indices in normal processes for various values of μ with a fixed $\sigma = 10/3$ and $(LSL, T, USL) = (10, 40, 50)$ and a fixed $\sigma = 20/3$ and $(LSL, T, USL) = (10, 40, 50)$ respectively. For the asymmetric case, with a fixed $\sigma = 10/3$ and $(LSL, T, USL) = (10, 40, 50)$, Figs. 10 and 11 display plots of Y_q, Yagainst μ and four PCIs C_p, C_{pk}, C_{pm}, and C_{pmk}, against μ respectively. Similarly, Figs. 12 and 13 display plots of Y_q, Yagainst μ and the four PCIs C_p, C_{pk}, C_{pm} and C_{pmk}, against μ respectively. The specification limits are set to $(LSL, T, USL) = (10, 40, 50)$ and $\sigma = 20/3$ is fixed. In this setting, μ is varied from 10 to 50 in unit steps to examine the sensitivity of these indices with respect to μ, for processes with asymmetric tolerances.

For the asymmetric cases, none (except Y_q) among the six indices accurately reflects the process performance. In fact, the Y index only reflects the quantity and not the quality of the conforming items, C_p cannot reflect the shift of the process mean, C_{pk} being a yield-based index cannot reflect the departure of the process mean μ from the target value T. The index C_{pm} attains its maximum at $\mu = T$,

Table 7. Comparisons among the six indices for normal processes with various μ, fixed $\sigma = 20/3$, and $(LSL, T, USL) = (10, 30, 50)$

μ	Y	Y_q	C_p	C_{pk}	C_{pm}	C_{pmk}
10	0.500	0.210	**1.000**	0.000	0.316	0.000
11	0.560	0.249	**1.000**	0.050	0.331	0.017
12	0.618	0.290	**1.000**	0.100	0.347	0.035
13	0.674	0.334	**1.000**	0.150	0.365	0.055
14	0.726	0.381	**1.000**	0.200	0.385	0.077
15	0.773	0.428	**1.000**	0.250	0.406	0.102
16	0.816	0.476	**1.000**	0.300	0.430	0.129
17	0.853	0.524	**1.000**	0.350	0.456	0.160
18	0.885	0.572	**1.000**	0.400	0.486	0.194
19	0.912	0.617	**1.000**	0.450	0.518	0.233
20	0.933	0.661	**1.000**	0.500	0.555	0.277
21	0.951	0.702	**1.000**	0.550	0.595	0.327
22	0.964	0.740	**1.000**	0.600	0.640	0.384
23	0.974	0.774	**1.000**	0.650	0.690	0.449
24	0.982	0.804	**1.000**	0.700	0.743	0.520
25	0.988	0.830	**1.000**	0.750	0.800	0.600
26	0.992	0.851	**1.000**	0.800	0.857	0.686
27	0.994	0.868	**1.000**	0.850	0.912	0.775
28	0.996	0.880	**1.000**	0.900	0.958	0.862
29	**0.997**	0.887	**1.000**	0.950	0.989	0.939
30	**0.997**	**0.890**	**1.000**	**1.000**	**1.000**	**1.000**
31	**0.997**	0.887	**1.000**	0.950	0.989	0.939
32	0.996	0.880	**1.000**	0.900	0.958	0.862
33	0.994	0.868	**1.000**	0.850	0.912	0.775
34	0.992	0.851	**1.000**	0.800	0.857	0.686
35	0.988	0.830	**1.000**	0.750	0.800	0.600
36	0.982	0.804	**1.000**	0.700	0.743	0.520
37	0.974	0.774	**1.000**	0.650	0.690	0.449
38	0.964	0.740	**1.000**	0.600	0.640	0.384
39	0.951	0.702	**1.000**	0.550	0.595	0.327
40	0.933	0.661	**1.000**	0.500	0.555	0.277
41	0.912	0.617	**1.000**	0.450	0.518	0.233
42	0.885	0.572	**1.000**	0.400	0.486	0.194
43	0.853	0.524	**1.000**	0.350	0.456	0.160
44	0.816	0.476	**1.000**	0.300	0.430	0.129
45	0.773	0.428	**1.000**	0.250	0.406	0.102
46	0.726	0.381	**1.000**	0.200	0.385	0.077
47	0.674	0.334	**1.000**	0.150	0.365	0.055
48	0.618	0.290	**1.000**	0.100	0.347	0.035
49	0.560	0.249	**1.000**	0.050	0.331	0.017
50	0.500	0.210	**1.000**	0.000	0.316	0.000

but the corresponding on-target process is not the process with the best average quality (proportion of "perfect" items, when considering both process yield and loss) for asymmetric cases, as we pointed out earlier. The index C_{pmk} cannot accurately distinguish the average quality of items produced using different processes. For example, although the value of C_{pmk} is zero for both the A and B processes with $\mu_A = USL$, $\mu_B = LSL$ and $\sigma_A = \sigma_B$, the average quality (measured by Y_q) of items produced by process A is better than that produced by process B for cases where $T > M$ (as mentioned earlier).

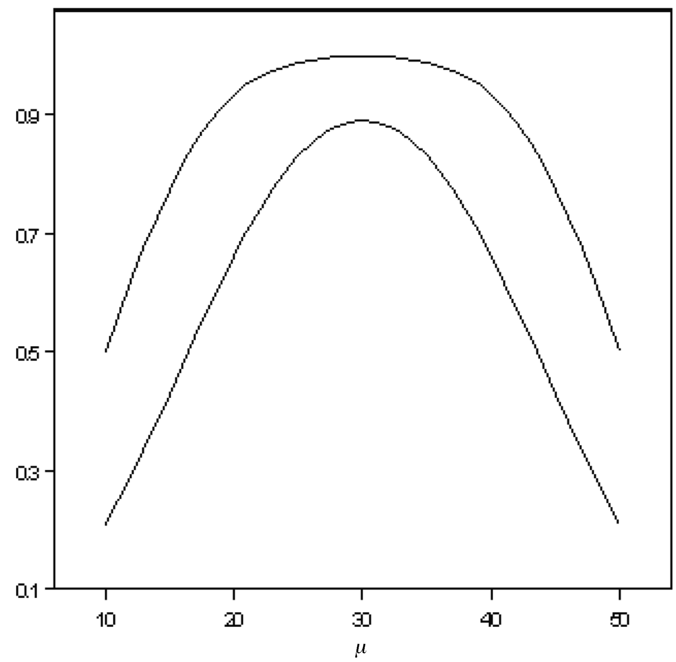

Fig. 8. Plots of Y and Y_q (top to bottom) versus $\mu = 10(1)50$ for processes with fixed $\sigma = 20/3$, and $(LSL, T, USL) = (10, 30, 50)$.

4. Distributional properties of the estimated Y_q

We now investigate some of the distributional properties of an estimator of Y_q. A confidence interval for Y_q is constructed. An approximate process performance testing procedure is also investigated.

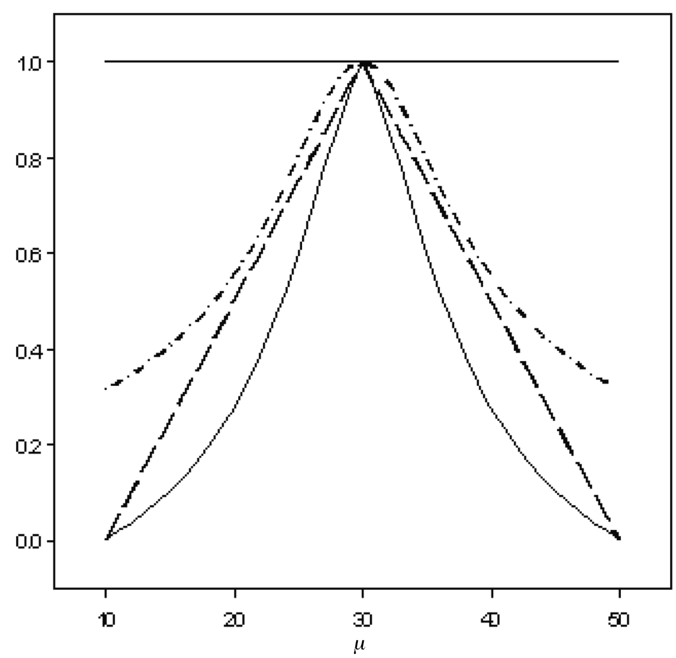

Fig. 9. Plots of C_p, C_{pk}, C_{pm} and C_{pmk} (top to bottom) versus $\mu = 10(1)50$ for processes with fixed $\sigma = 20/3$, $(LSL, T, USL) = (10, 30, 50)$.

Table 8. Comparisons among the six indices for normal processes with various μ, fixed $\sigma = 10/3$, and $(LSL, T, USL) = (10, 40, 50)$

μ	Y	Y_q	C_p	C_{pk}	C_{pm}	C_{pmk}
10	0.500	0.082	**2.000**	0.000	0.221	0.000
11	0.618	0.116	**2.000**	0.100	0.228	0.011
12	0.726	0.156	**2.000**	0.200	0.236	0.024
13	0.816	0.201	**2.000**	0.300	0.245	0.037
14	0.885	0.250	**2.000**	0.400	0.254	0.051
15	0.933	0.300	**2.000**	0.500	0.264	0.066
16	0.964	0.351	**2.000**	0.600	0.275	0.083
17	0.982	0.401	**2.000**	0.700	0.287	0.100
18	0.992	0.451	**2.000**	0.800	0.300	0.120
19	0.997	0.498	**2.000**	0.900	0.314	0.141
20	0.999	0.543	**2.000**	1.000	0.329	0.164
21	**1.000**	0.587	**2.000**	1.100	0.346	0.190
22	**1.000**	0.628	**2.000**	1.200	0.364	0.219
23	**1.000**	0.667	**2.000**	1.300	0.385	0.250
24	**1.000**	0.703	**2.000**	1.400	0.408	0.286
25	**1.000**	0.738	**2.000**	1.500	0.434	0.325
26	**1.000**	0.770	**2.000**	1.600	0.463	0.371
27	**1.000**	0.800	**2.000**	1.700	0.497	0.422
28	**1.000**	0.828	**2.000**	1.800	0.535	0.482
29	**1.000**	0.853	**2.000**	1.900	0.580	0.551
30	**1.000**	0.877	**2.000**	**2.000**	0.633	0.632
31	**1.000**	0.898	**2.000**	1.900	0.695	0.660
32	**1.000**	0.916	**2.000**	1.800	0.769	0.692
33	**1.000**	0.933	**2.000**	1.700	0.860	0.731
34	**1.000**	0.947	**2.000**	1.600	0.971	0.777
35	**1.000**	0.958	**2.000**	1.500	1.109	0.832
36	**1.000**	0.965	**2.000**	1.400	1.281	0.896
37	**1.000**	**0.968**	**2.000**	1.300	1.487	0.966
38	**1.000**	0.966	**2.000**	1.200	1.715	1.029
39	**1.000**	0.957	**2.000**	1.100	1.916	**1.054**
40	0.999	0.939	**2.000**	1.000	**2.000**	1.000
41	0.997	0.910	**2.000**	0.900	1.916	0.862
42	0.992	0.868	**2.000**	0.800	1.715	0.686
43	0.982	0.813	**2.000**	0.700	1.487	0.520
44	0.964	0.744	**2.000**	0.600	1.281	0.384
45	0.933	0.663	**2.000**	0.500	1.109	0.277
46	0.885	0.573	**2.000**	0.400	0.971	0.194
47	0.816	0.477	**2.000**	0.300	0.860	0.129
48	0.726	0.381	**2.000**	0.200	0.769	0.077
49	0.618	0.290	**2.000**	0.100	0.695	0.035
50	0.500	0.210	**2.000**	0.000	0.633	0.000

Table 9. Comparisons among the six indices for normal processes with various μ, fixed $\sigma = 20/3$, and $(LSL, T, USL) = (10, 40, 50)$

μ	Y	Y_q	C_p	C_{pk}	C_{pm}	C_{pmk}
10	0.500	0.153	**1.000**	0.000	0.217	0.000
11	0.560	0.182	**1.000**	0.050	0.224	0.011
12	0.618	0.213	**1.000**	0.100	0.232	0.023
13	0.674	0.247	**1.000**	0.150	0.240	0.036
14	0.726	0.283	**1.000**	0.200	0.248	0.050
15	0.773	0.321	**1.000**	0.250	0.258	0.064
16	0.816	0.360	**1.000**	0.300	0.268	0.080
17	0.853	0.400	**1.000**	0.350	0.278	0.097
18	0.885	0.440	**1.000**	0.400	0.290	0.116
19	0.912	0.480	**1.000**	0.450	0.303	0.136
20	0.933	0.520	**1.000**	0.500	0.316	0.158
21	0.951	0.559	**1.000**	0.550	0.331	0.182
22	0.964	0.597	**1.000**	0.600	0.347	0.208
23	0.974	0.634	**1.000**	0.650	0.365	0.237
24	0.982	0.669	**1.000**	0.700	0.385	0.269
25	0.988	0.702	**1.000**	0.750	0.406	0.305
26	0.992	0.733	**1.000**	0.800	0.430	0.344
27	0.994	0.761	**1.000**	0.850	0.456	0.388
28	0.996	0.787	**1.000**	0.900	0.486	0.437
29	**0.997**	0.811	**1.000**	0.950	0.518	0.492
30	**0.997**	0.831	**1.000**	**1.000**	0.555	0.555
31	**0.997**	0.849	**1.000**	0.950	0.595	0.566
32	0.996	0.862	**1.000**	0.900	0.640	0.576
33	0.994	0.872	**1.000**	0.850	0.690	0.587
34	0.992	0.878	**1.000**	0.800	0.743	0.595
35	0.988	**0.879**	**1.000**	0.750	0.800	**0.600**
36	0.982	0.875	**1.000**	0.700	0.857	**0.600**
37	0.974	0.866	**1.000**	0.650	0.912	0.593
38	0.964	0.850	**1.000**	0.600	0.958	0.575
39	0.951	0.829	**1.000**	0.550	0.989	0.544
40	0.933	0.802	**1.000**	0.500	**1.000**	0.500
41	0.912	0.769	**1.000**	0.450	0.989	0.445
42	0.885	0.731	**1.000**	0.400	0.958	0.383
43	0.853	0.687	**1.000**	0.350	0.912	0.319
44	0.816	0.640	**1.000**	0.300	0.857	0.257
45	0.773	0.588	**1.000**	0.250	0.800	0.200
46	0.726	0.534	**1.000**	0.200	0.743	0.149
47	0.674	0.479	**1.000**	0.150	0.690	0.104
48	0.618	0.424	**1.000**	0.100	0.640	0.064
49	0.560	0.370	**1.000**	0.050	0.595	0.030
50	0.500	0.319	**1.000**	0.000	0.555	0.000

4.1. *Estimation of the Q-yield*

If the process parameters μ and σ are unknown, then Y_q must be estimated from a sample. Let $X_1, X_2, \ldots, X_n$ be a random sample taken from the process, and $W_1, W_2, \ldots, W_n$ be the corresponding worth. To estimate the Q-yield, we can consider the following estimator:

$$\hat{Y}_q = \sum_{i=1}^{n} \frac{W_i}{n} = \bar{W}. \quad (22)$$

It is easy to verify that $E(\hat{Y}_q) = Y_q$. Therefore, $\hat{Y}_q$ is an unbiased estimator of Y_q with $\mathrm{Var}(\hat{Y}_q) = n^{-1}\mathrm{Var}(W_1)$. Use of the unbiased estimator $\hat{Y}_q$ does not require any knowledge of the process distribution. However, if the distribution of the characteristic X is given with cumulative distribution function F_X, then the cumulative distribution function of the corresponding worth F_W can be calculated, and the cumulative distribution function of $\hat{Y}_q$ can be expressed as the n-fold convolution of F_W:

$$F_{\hat{Y}_q}(y) = P(\hat{Y}_q \le y) = P\Big(\sum W_i \le ny\Big) = G(ny), \quad (23)$$

where G is the n-fold convolution of F_W. The complexity of the cumulative distribution function of $\hat{Y}_q$ comes from

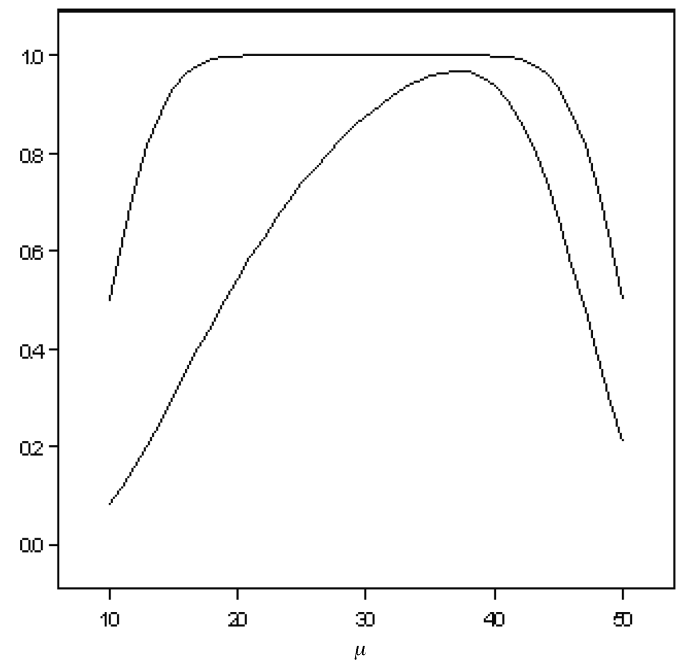

Fig. 10. Plots of Y (top) and Y_{q} (bottom) versus $\mu = 10(1)50$, for processes with $\sigma = 10/3$, $(LSL, T, USL) = (10, 40, 50)$.

the truncation property of the worth function. There is no analytic closed-form solution for $F_{\hat{Y}_{\text{q}}}(y)$. However, for a large sample size n, we can show that:

$$\frac{\sqrt{n}(\hat{Y}_{\text{q}} - Y_{\text{q}})}{S} \rightarrow N(0,1), \tag{24}$$

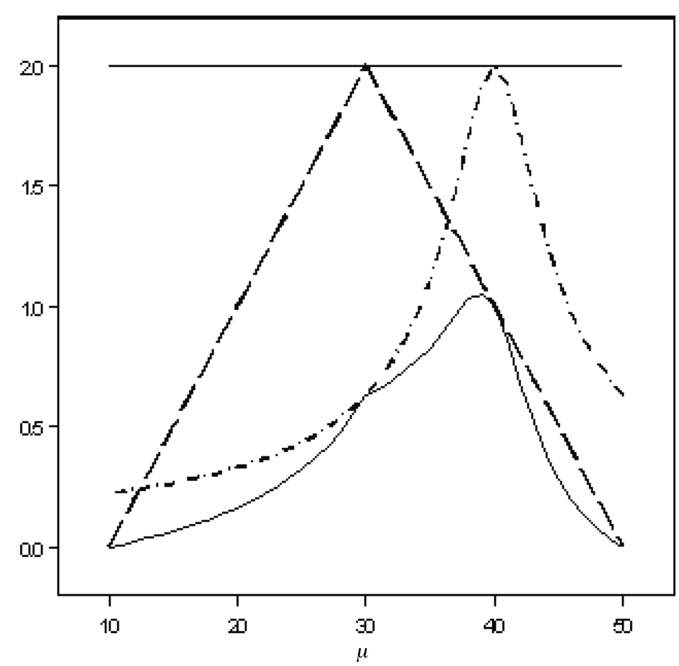

Fig. 11. Plots of C_p(top), C_{pk}(left), C_{pm} (right) and C_{pmk} (bottom) versus $\mu = 10(1)50$, for processes with $\sigma = 10/3$, $(LSL, T, USL) = (10, 40, 50)$.

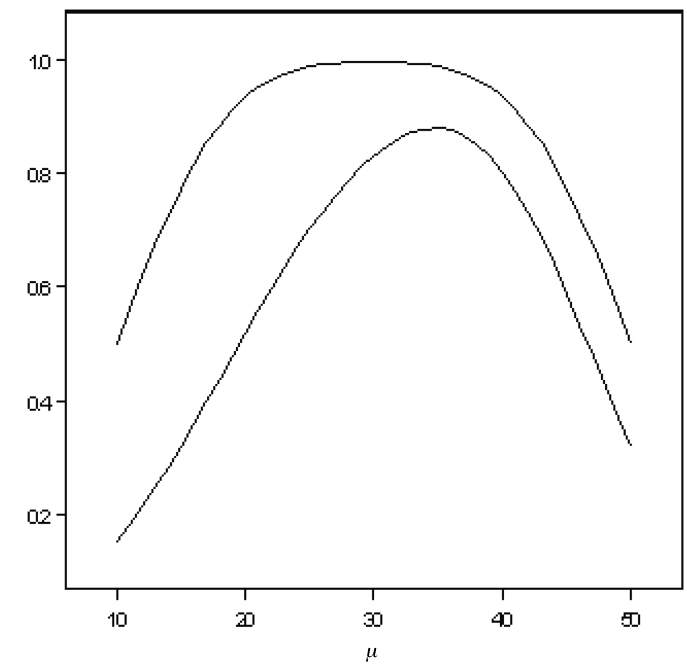

Fig. 12. Plots of Y (top) and Y_q (bottom) versus $\mu = 10(1)50$, for processes with $\sigma = 20/3$, and $(LSL, T, USL) = (10, 40, 50)$.

where the sample variance $S^2 = \sum (W_i - \bar{W})^2/(n-1)$. Consequently, an approximate $(1-\alpha)100\%$ confidence interval of Y_{q} can be established as:

$$(\hat{Y}_{\text{q}} - z_{1-\alpha/2}S/\sqrt{n},\ \hat{Y}_{\text{q}} + z_{1-\alpha/2}S/\sqrt{n}), \tag{25}$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile value of the standard normal distribution $N(0, 1)$. We note that a lower

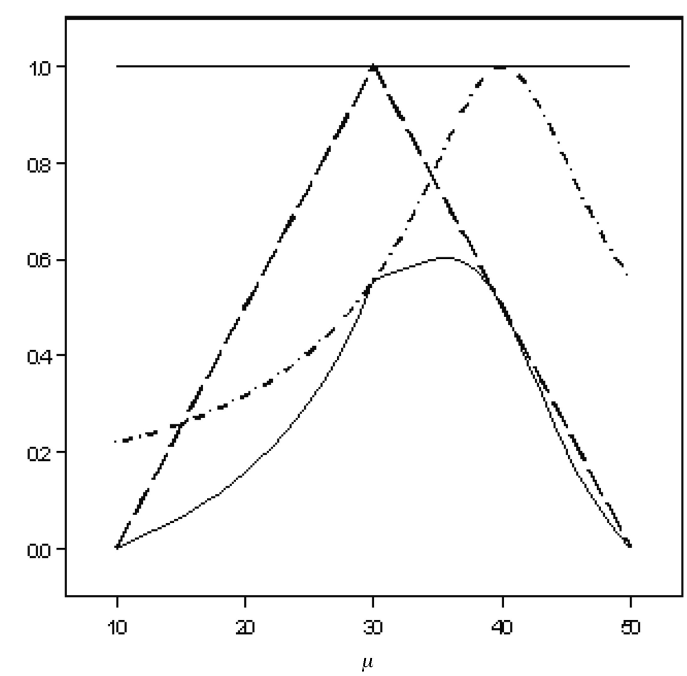

Fig. 13. Plots of C_p (top), C_{pk} (left), C_{pm} (right) and C_{pmk} (bottom) versus $\mu = 10(1)50$, $\sigma = 20/3$, $(LSL, T, USL) = (10, 40, 50)$.

$(1-\alpha)100\%$ confidence limit can be obtained from the lower (one-sided) confidence limit. If the calculated lower confidence limit is greater than the predetermined index value, then we would judge that the process is capable. Otherwise, the process is considered to be incapable, and some quality improvement activities must be initiated.

4.2. *Distribution plot of the Q-yield estimator*

Monte Carlo simulations were performed to investigate the behavior of the sampling distribution of the estimated Y_q, for several selected cases, where the underlying process distributions are normal, skewed, or heavy tailed. A true value of the quality yield $Y_q = 0.6$ is picked, with the underlying process distributions being set to:

1. A normal distribution $N(\mu, \sigma^2)$ with probability density function:

$$f(x) = (\sqrt{2\pi}\sigma)^{-1} \exp[-(x-\mu)^2/2\sigma^2], \quad (26)$$

with mean μ and variance σ^2, for $-\infty < x < \infty$.

2. A lognormal distribution $LN(\mu, \sigma^2)$ with probability density function:

$$f(x) = (x\sqrt{2\pi}\sigma)^{-1} \exp[-(\ln x-\mu)^2/2\sigma^2], \quad (27)$$

with mean $\exp(\mu+\sigma^2/2)$ and variance $\exp(2\mu+2\sigma^2) - \exp(2\mu+\sigma^2)$, for $x > 0$.

3. A Student's t distribution t_k with degree of freedom k, where the probability density function is:

$$f(x) = [\Gamma((k+1)/2)/\Gamma(k/2)](\sqrt{k\pi})^{-1} \times (1+x^2/k)^{-(k+1)/2}, \quad -\infty < x < \infty, \quad (28)$$

with mean $\mu = 0$, for $k > 1$ and variance $\sigma^2 = k/(k-2)$, for $k > 2$.

4. A chi-square distribution χ_k^2 with degree of freedom k, where the probability density function is:

$$f(x) = [1/\Gamma(k/2)](1/2)^{k/2} x^{k/2-1} e^{-x/2}, \ x > 0, \quad (29)$$

with mean $\mu = k$ and variance $\sigma^2 = 2k$, $k = 1, 2, \ldots$.

5. A Weibull distribution $W(\alpha, \beta)$ with probability density function:

$$f(x) = \alpha\beta x^{\beta-1} \exp(-\alpha x^{\beta}), \quad (30)$$

with mean $\mu = \alpha^{-1/\beta}\Gamma(1+\beta^{-1})$ and variance $\sigma^2 = \alpha^{-2/\beta}[\Gamma(1+2\beta^{-1}) - \Gamma^2(1+\beta^{-1})]$, for $x > 0$.

We randomly generated $N = 10\,000$ samples of sizes $n = 25$, 50, 75, 100, 150, 200, 250, and 300 for each distribution and then calculated the estimate value of Y_q for each sample. Figures 14–21 plot the distribution of $\hat{Y}_q$ for the eight levels of sample size with $Y_q = 0.6$, respectively. In each figure, five underlying process distributions including normal, lognormal, Student's t, chi-square, and Weibull are drawn with fixed sample size in order to investigate how the sample size affects the distribution of $\hat{Y}_q$. From those plots, one may observe that for a moderate sample size n (about 100) the distributions of the estimated Q-yield index all appear quite close to normal. Therefore, for practical purposes, normal approximations may be used for capability testing of Y_q.

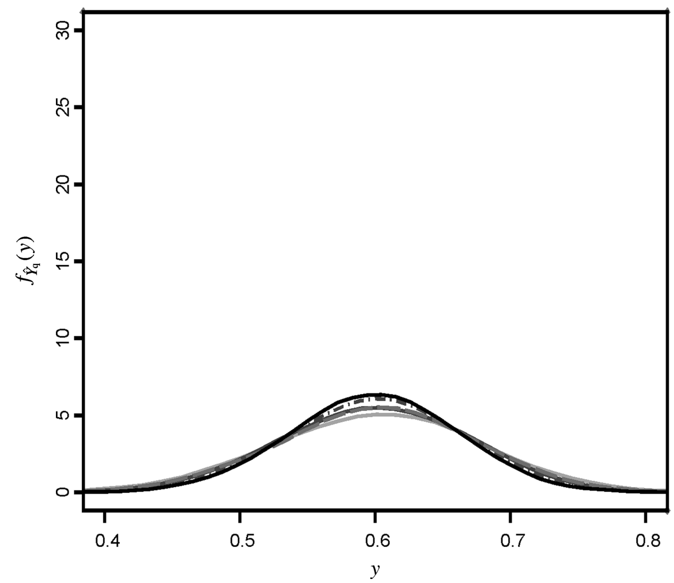

Fig. 14. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 25$.

5. An application example

We consider a case study for illustration purpose. The use of Light Emitting Diodes (LEDs) has rapidly expanded

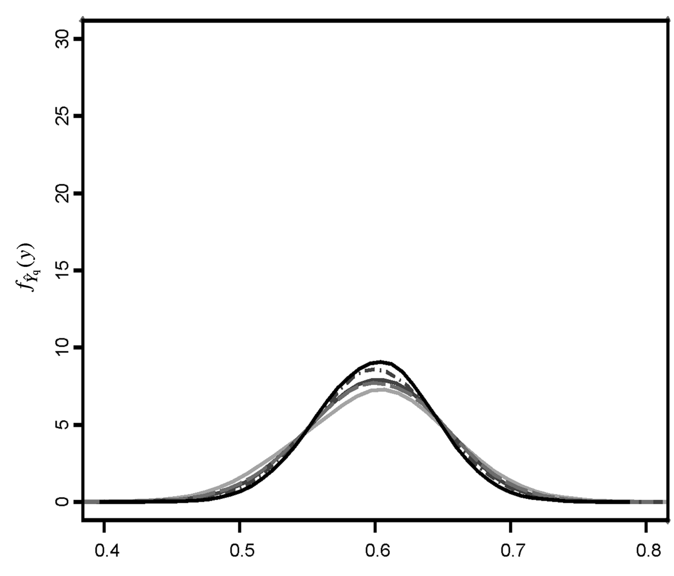

Fig. 15. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 50$.

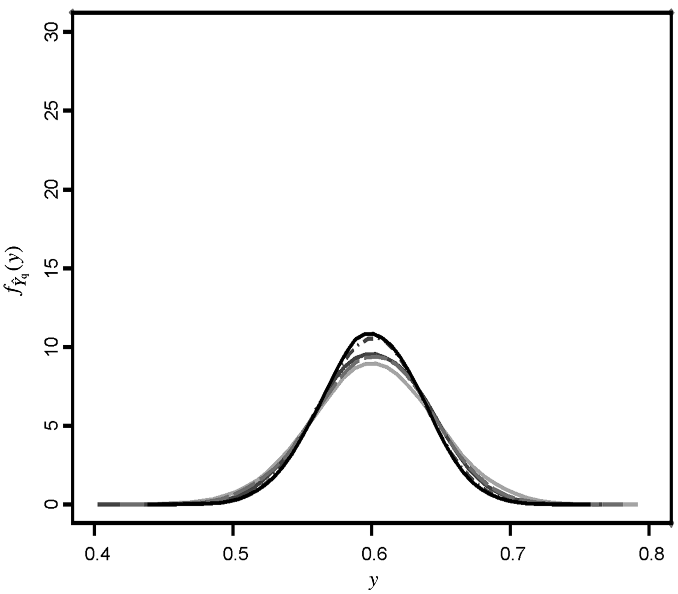

Fig. 16. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 75$.

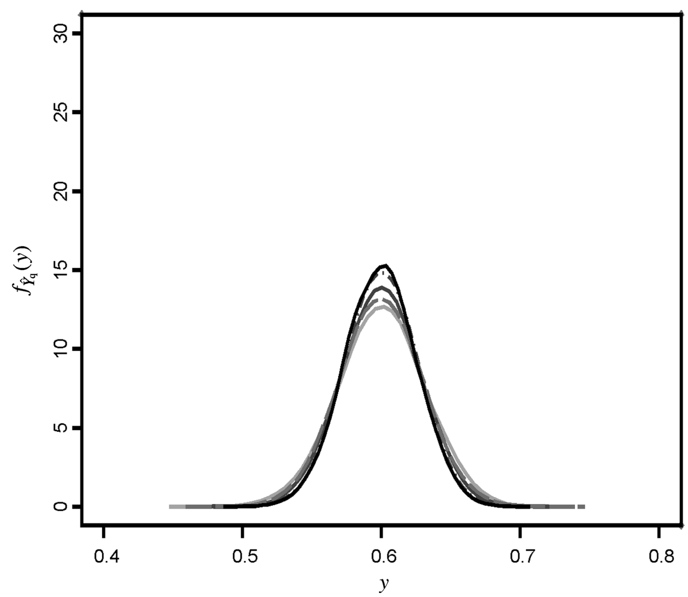

Fig. 18. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 150$.

since the development of high-intensity LEDs with a wide range of colors that has led to their application in a wide variety of areas. LEDs are considerably different from lamps in terms of their physical size, flux level, spectrum, and spatial intensity distribution. LED technology provides a number of benefits over incandescent bulbs. Some benefits of LEDs for instrument cluster lighting are:

1. LEDs have a lower power consumption: a LED instrument cluster uses approximately 1/5 of the electrical current of an incandescent instrument cluster.
2. LEDs generate less heat: interior thermal measurements within the instrument cluster case indicate that the LED design operates 10–15°C cooler than an incandescent light design. Interior thermal measurements within the cavity airspace indicate that the LED

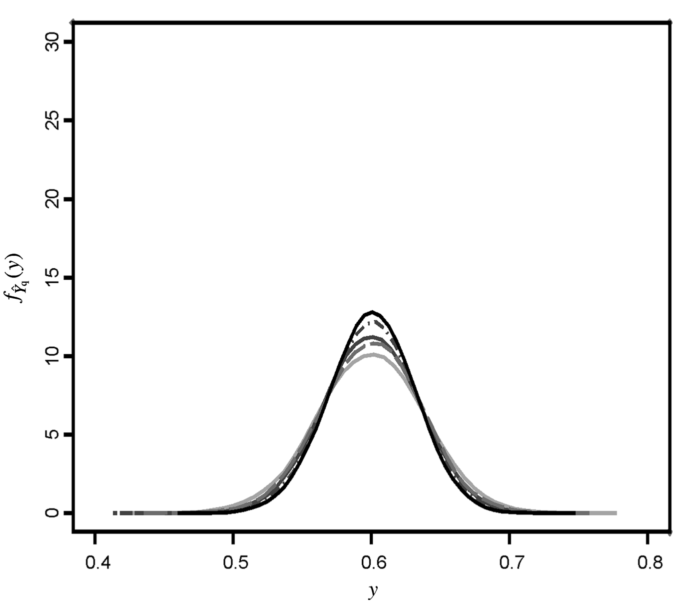

Fig. 17. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 100$.

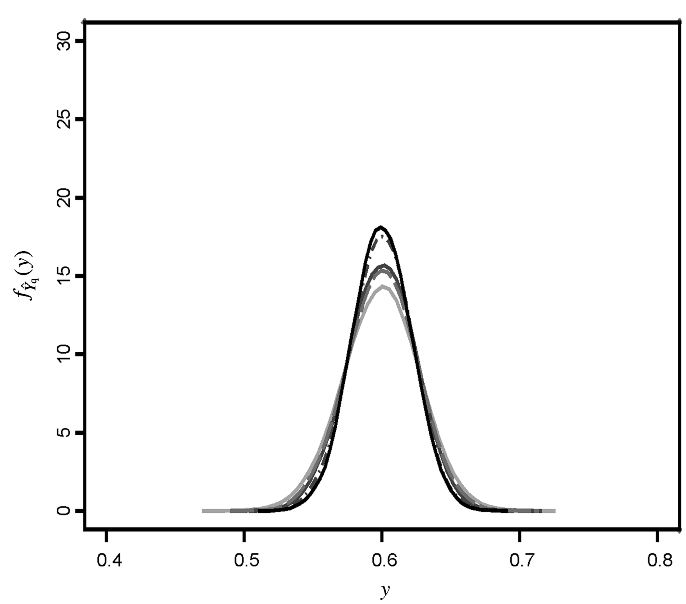

Fig. 19. Distribution plots of $\hat{Y}_q$ for $N(\mu, \sigma^2)$, χ_k^2, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 200$.

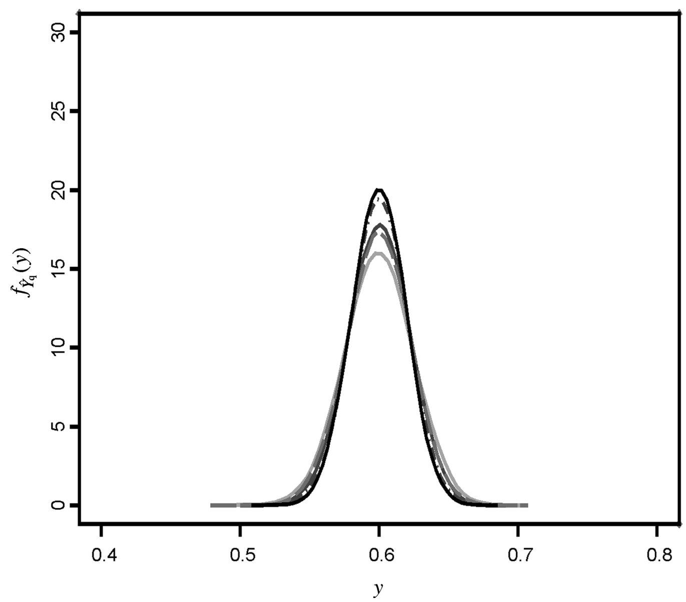

Fig. 20. Distribution plots of $\hat{Y}_{\mathrm{q}}$ for $N(\mu, \sigma^2)$, χ^2_k, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 250$.

design operates 25–50°C cooler than an incandescent design.

3. LEDs provide an equivalent or better lighting: some comparative performances are that red LEDs are 3x brighter and amber LEDs are 2x brighter.
4. LEDs have a better reliability: LEDs are capable of withstanding high degrees of mechanical shock and vibration without failure. LEDs are capable of withstanding over 1000 temperature cycles between 40/100°C, without failing.

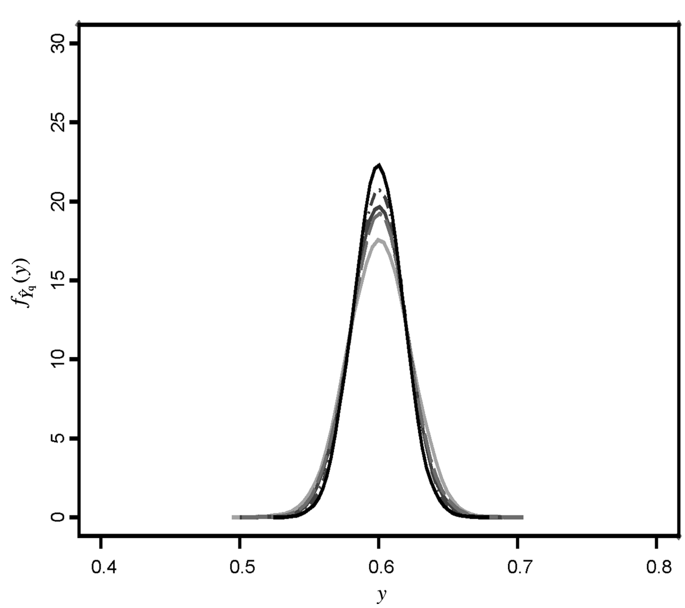

Fig. 21. Distribution plots of $\hat{Y}_{\mathrm{q}}$ for $N(\mu, \sigma^2)$, χ^2_k, t_k, $LN(\mu, \sigma^2)$, $W(\alpha, \beta)$ (bottom to top) with $n = 300$.

5. LEDs allow for smaller telltales: since LEDs are available in sizes less than 1/8″ in diameter, LED telltales can be placed on spacing of 0.25–0.30″, if desired.
6. LEDs are dimmable using a potentiometer: LEDs are normally wired in series with a current limiting resistor. In general, LEDs can be dimmed with a single potentiometer, as long as all series strings use the same number of LEDs. LEDs can also be dimmed through pulse width modulation. In this case, the number of lamps in each series string is not critical.
7. LEDs provide direct cost savings: potentially, LEDs allow for less expensive drive circuits. LEDs operate at lower currents (20 mA instead of 255 mA). Also, LEDs do not have a high inrush current when first turned on. In general, LEDs outperformed the incandescent bulbs for all gauge colors.

With a focus on a critical characteristic of the luminous intensity of LED sources, we examine a particular LED product model, with the upper and lower specification limits of luminous intensity being set to $USL = 90$ mcd and $LSL = 40$ mcd with the target value being set to $T = 60$ mcd. We note that this is an asymmetric tolerances case. The LED is said to be defective if the characteristic data does not fall within the specification limits (LSL, USL). To make the use of the methodology more convenient and accelerate the computation, an integrated S-PLUS computer program was developed (available from the authors) to calculate the lower confidence bounds. We only need to input the manufacturing specification limits, USL, LSL, target value T, and the collected sample data of size n. Then the estimated values $\hat{Y}$, $\hat{Y}_{\mathrm{q}}$ and the lower confidence bounds of $\hat{Y}_{\mathrm{q}}$ can be obtained easily. Thus, whether or not the process is capable may be determined.

A total of 150 observations were collected from a stable process in the factory, which are displayed in Fig. 22. Figure 23 is a histogram of the sample data. From Fig. 23, it is evident from the density line that the underlying process distribution is far from normal. Refering to the distribution plots of the Q-yield estimator, a random sample of size $n = 150$ seems to be large enough to apply the normal approximation approach to the capability testing of Y_{q}. Proceeding with the calculations by running the integrated S-PLUS program with a 95% confidence level,

55	59	46	68	50	43	58	50	70	56	51	57	78	47	54
61	65	44	52	57	60	43	58	55	59	54	50	59	43	53
52	58	46	52	44	45	58	56	49	43	57	85	46	53	59
64	60	46	65	66	50	66	48	68	58	53	48	72	51	57
51	48	64	52	61	59	47	61	54	59	65	57	57	45	47
61	41	43	62	62	61	46	61	51	55	56	72	69	57	55
88	62	57	60	69	54	61	56	55	45	72	45	60	49	82
52	43	62	45	60	45	61	59	49	56	47	77	46	53	56
65	53	68	45	66	62	52	66	71	73	70	52	58	56	81
52	42	57	64	56	63	63	61	70	53	47	62	53	55	59

Fig. 22. A sample of observations of size $n = 150$.

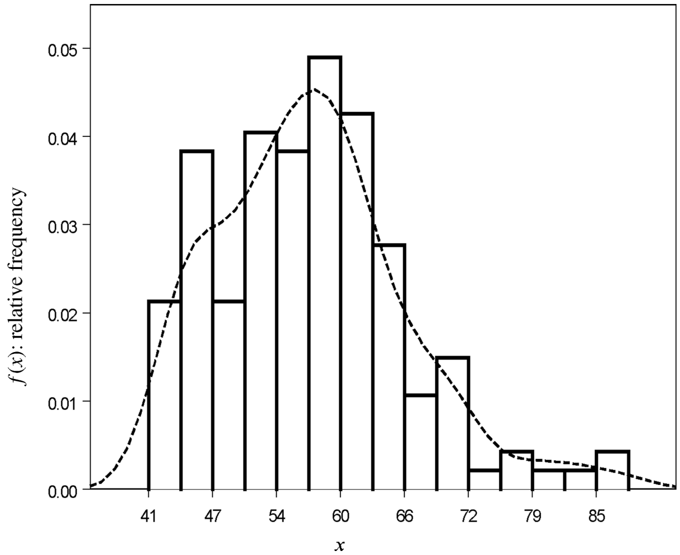

Fig. 23. Histogram of the sample data.

we obtain $\hat{Y}_q = 0.8082$ and the corresponding lower confidence bound as 0.7768. We note that the estimated $\hat{Y}_q$ index value is about 0.81. In fact, all 150 observations fall within the specification interval (*LSL*, *USL*) so that the sample estimator of yield is $\hat{Y} = 1$. From the producer's point of view, the proportion of conforming products is 100%. However, to quantify how well a process can meet customer requirements a lower confidence bound of $\hat{Y}_q$ of approximately 0.78 can be interpreted as a degree of satisfaction with the products, of at least 78%, with a 95% confidence level. From the corresponding lower confidence bound on Y_q, 0.7768, an example of capability testing is that if the Q-yield requirement preprint on the contract Y_q is set to 0.78, we may only conclude that the process is marginally capable, with a 95% confidence level.

6. Conclusions

In this paper, we first reviewed the Q-yield Y_q, proposed by Tsui (1997) for processes with symmetric tolerances. We then used the worth function to generalize the concept of the Q-yield to processes with asymmetric tolerances. The analysis and comparisons showed that the new generalization incorporates the asymmetry of the manufacturing tolerance (with an asymmetric loss function), which reflects process performance more accurately. We also proposed an unbiased estimator of Y_q to access the ability of the considered process, which does not require the assumption of normal variability. Some Monte Carlo simulations were conducted to investigate the behavior of the sampling distribution of the estimated Y_q. The result showed that for moderate sample size n of no greater than 300 the distributions of the estimated Y_q all appear to be normal. Therefore, normal approximation may be used to perform the capability testing.

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Appendix

Suppose a process characteristic X follows a distribution with the cumulative distribution function $F_X(x)$ and the probability density function $f_X(x)$. The fraction of nonconforming items, i.e., the probability of an item falling outside specified tolerance limits, can be derived as:

$$\begin{aligned} F_W(0) &= P[W(X)=0], \\ &= P[X \le LSL] + P[X \ge USL], \\ &= F_X(LSL) + 1 - F_X(USL). \end{aligned} \tag{A1}$$

For the case where $w > 0$, the cumulative distribution function of $W(X)$, can be obtained as:

$$\begin{aligned} F_W(w) &\\ &= P[W(X) \le w], \\ &= P[W(X)=0] \\ &\quad + P[(0 < W(X) \le w) \cap (LSL < X \le T)] \\ &\quad + P[(0 < W(X) \le w) \cap (T \le X < USL)], \\ &= P[W(X)=0] + P[(0 < 1 - [(T-X)/d_l]^2 \le w) \\ &\quad \cap (LSL < X \le T)] \\ &\quad + P[(0 < 1 - [(X-T)/d_u]^2 \le w) \\ &\quad \cap (T \le X < USL)], \\ &= P[W(X)=0] + P\big[\big(0 < d_l^2 - (T-X)^2 \le d_l^2 w\big) \\ &\quad \cap (LSL < X \le T)\big] + P\big[\big(0 < d_u^2 - (X-T)^2 \le d_u^2 w\big) \\ &\quad \cap (T \le X < USL)\big], \\ &= P[W(X)=0] + P[(d_l^2(1-w) \le (T-X)^2 < d_l^2) \\ &\quad \cap (LSL < X \le T)\big] + P\big[\big(d_u^2(1-w) \le (X-T)^2 < d_u^2\big) \\ &\quad \cap (T \le X < USL)\big], \\ &= P[W(X)=0] + P[(d_l\sqrt{1-w} \le (T-X) < d_l) \\ &\quad \cap (LSL < X \le T)] + P[(d_u\sqrt{1-w} \le (X-T) < d_u) \\ &\quad \cap (T \le X < USL)], \\ &= P[W(X)=0] + P[LSL < X \le T - d_l\sqrt{1-w}] \\ &\quad + P[T + d_u\sqrt{1-w} \le X < USL], \\ &= [F_X(LSL) + 1 - F_X(USL)] + [F_X(T - d_l\sqrt{1-w}) \\ &\quad - F_X(LSL)] + [F_X(USL) - F_X(T + d_u\sqrt{1-w})], \\ &= 1 + F_X(T - d_l\sqrt{1-w}) - F_X(T + d_u\sqrt{1-w}), \\ &\qquad 0 \le w \le 1. \end{aligned} \tag{A2}$$

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