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Bianchi type I expanding universe in Weyl-invariant massive gravity

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We will study the cosmological evolutions of a Weyl-invariant de Rham, Gabadadze, and Tolley (dRGT) massive gravity theory with a general fiducial metric. In the unitary gauge, this model is equivalent to a massive gauge field coupled to the dRGT massive gravity model. The massive gravity terms will serve as an effective cosmological constant for all metric spaces if the fiducial metric is treated as an auxiliary field. A further discussion will also be addressed on the bimetric theory. In particular, we will discuss the role played by the Weyl vector boson in Bianchi type I expanding space.

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I. INTRODUCTION

A linearized massive gravity theory introduced by Fierz and Pauli (FP) is known to be ghost free since 1939 [1]. This theory propagates 5 degrees of freedom associated with massive spin-2 graviton in the background of Minkowski space. Boulware and Deser showed, however, that the ghost degree of freedom will survive in the nonlinear level [2]. This additional negative energy degree of freedom has been known as the Boulware-Deser (BD) ghost.

Starting in 2009, de Rham, Gabadadze, and Tolley (dRGT) proposed a comprehensive nonlinear theory that can be shown to be free of the BD ghost [3,4]. The dRGT was first shown to be ghost free with the introduction of a flat reference (or fiducial) metric. [4] It was later shown to be free of the BD ghost in the fully nonlinear level in the presence of a general fiducial metric in 2012 [5,6]. Reference [7] published in 2012 provides a detailed and elegant review on the related progress concerning the ghost-free massive gravity theory.

Note that a consistent theory of massive gravity can be applied to accommodate the recent discovery of dark energy and the cosmological constant problem. Massive gravity is also a nice resolution to the quest of a generalized theory of gravity. Research activities studying all possible implications of this theory to the evolution of the early Universe have thus attracted lots of attention [5,6,8–23].

For example, Ref. [9] shows that the nonlinear massive gravity theory does not admit spatially flat homogeneous and isotropic cosmological solutions. It admits, however, a set of spatially homogeneous and isotropic cosmological solutions [10,11] in open space. In addition, anisotropic solutions [9,12,13] and inhomogeneous solutions [9,14,15] were also found. The application to black holes physics also has attracted lots of attention lately [16,17]. Later on, a ghost-free bimetric theory [20] known as bigravity or bimetric theory, was also proposed by Hassan and Rosen [21,22]. For example, ghost-free multimetric theories were also discussed in Ref. [23].

Note that results presented in Refs. [4–6,8–18,21–23] focus on the isotropic reference metric space along with the physical metric space being generalized to the anisotropic metric spaces. We will thus propose to study a more general result of the nonlinear massive gravity theory with a more general reference metric. The reference metric will be treated first as an auxiliary field here. The variational equation then chooses the most probable solution to the reference metric that is compatible with the physical metric. As a result, we can show that the massive terms serve as an effective cosmological constant for all possible metric spaces.

A specific set of fiducial metric solutions can be solved accordingly for this model. The result is derived from the fact that the massive terms depend only on the trace of the field K. As a result, the fiducial metric equations depend only on the eigenvalues of K. We can therefore diagonalize K in Jordan normal form without affecting the generic property of the massive interaction terms. As an interesting example, we will present a general solution in Bianchi type I space [22,24–26]. There are also activities exploring all possible applications of massive gravity theory.[27]

In addition, local scale-invariant (or Weyl-invariant) theory is a successful model acting as an effective theory of our physical Universe [28–30]. Evidences also indicate that scale symmetry should play some important role in many physical applications of interest before the Higgs mechanism breaks the scale symmetry [31,32]. Moreover, the Weyl gauge field or Weyl vector boson has also been proposed as a possible candidate of dark matter [33–37].

There have been successful attempts to incorporate the Weyl symmetry as an alternative theory for massive gravity with the introduction of higher curvature terms [38]. We will try, however, to generalize the Weyl-invariant massive gravity theory with the introduction of a Weyl vector meson S_μ to the dRGT model. It will be shown that the intro-duction of Weyl symmetry will not affect the ghost-free picture of the massive gravity theory in the unitary gauge

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by choosing the scalar field as a constant. The resulting theory can be shown to be equivalent to a dRGT model coupled to a massive $U(1)$ gauge field. In addition, we will also show that the Weyl vector meson in fact plays little role in Bianchi type I (BI) metric space.

Note that the Weyl transformation is a local scale transformation relating all physical fields in different length scales. The transformation property of each field is determined by its corresponding conformal dimension. For example, the scalar field ϕ has conformal dimension one. The metric field $g_{\mu\nu}$ acts as a field with -2 conformal dimension. Therefore, they transform, respectively, as [32,39]

$$\phi \to \phi^\Omega = \Omega^{-1}\phi, \tag{1.1}$$

$$g_{\mu\nu} \to g^\Omega_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{1.2}$$

In order to preserve Weyl symmetry, the ordinary derivative ∂_μ must be replaced by a Weyl-covariant derivative ∇_μ in order to make sure that the transformation property of $\nabla_\mu\phi$ is the same as the scalar field ϕ. To be more specific, the Weyl-covariant derivative of a scalar field ϕ and $g_{\mu\nu}$ are defined as

$$\nabla_\mu\phi = (\partial_\mu - S_\mu)\phi, \tag{1.3}$$

$$\tilde{\partial}_\alpha g_{\mu\nu} = (\partial_\alpha + 2S_\alpha)g_{\mu\nu} \tag{1.4}$$

with the introduction of a Weyl gauge field or Weyl vector meson S_μ. As a result,

$$\nabla_\mu\phi \to (\nabla_\mu\phi)^\Omega = \Omega^{-1}\nabla_\mu\phi, \tag{1.5}$$

$$(\tilde{\partial}_\alpha g_{\mu\nu})^\Omega = \Omega^2\tilde{\partial}_\alpha g_{\mu\nu} \tag{1.6}$$

if the Weyl gauge field transforms as

$$S_\mu \to S^\Omega_\mu = S_\mu - \partial_\mu \ln\Omega. \tag{1.7}$$

As a result, the Weyl-invariant generalization of the spin connection

$$\tilde{\Gamma}^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\tilde{\partial}_\mu g_{\nu\beta} + \tilde{\partial}_\nu g_{\mu\beta} - \tilde{\partial}_\beta g_{\mu\nu}) \tag{1.8}$$

can be shown to be invariant. Indeed, we can show that

$$(\tilde{\Gamma}^\alpha{}_{\mu\nu})^\Omega = \tilde{\Gamma}^\alpha{}_{\mu\nu} \tag{1.9}$$

under the local scale transformation. Therefore the following generalization of gravitational theory can be shown to be Weyl invariant [5–7,39]:

$$S = \int d^4x\sqrt{g}\left(\frac{\epsilon}{2}\phi^2\tilde{R} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{4}H^2 - \frac{\lambda}{4}\phi^4\right), \tag{1.10}$$

with $\epsilon\phi^2$ and $\lambda\phi^4/4$ serving as dynamical coupling constants M_p^2 and $m_g^2 M_p^2/2$, respectively. Note that the S_μ field tensor

$$H_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu \tag{1.11}$$

is Weyl invariant by itself. In addition, the Weyl-invariant Ricci curvature tensor $\tilde{R}_{\mu\nu}$ can be defined as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}(\partial_\alpha g_{\beta\gamma} \to \tilde{\partial}_\alpha g_{\beta\gamma}). \tag{1.12}$$

Consequently, it can be shown that

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - (D_\mu S_\nu + D_\nu S_\mu) - D_\beta S^\beta g_{\mu\nu} + 2(S_\mu S_\nu - S_\beta S^\beta g_{\mu\nu}). \tag{1.13}$$

In addition, the Weyl-invariant scalar curvature

$$\tilde{R} = \tilde{R}^\mu{}_\mu \tag{1.14}$$

is defined as the trace of the Weyl-invariant Ricci curvature tensor. Similarly, the Weyl-invariant generalization of matter fields can also be introduced following a similar method. In particular, when a charged fermion field ψ is coupled to the system, the action becomes

$$S_0 = \int d^4x\sqrt{g}\left[\frac{\epsilon}{2}\phi^2\tilde{R} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi) + \bar{\psi}[i\gamma^\mu\nabla_\mu - \phi]\psi - \frac{1}{4}H^{\mu\nu}H_{\mu\nu} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]. \tag{1.15}$$

Here $V(\phi)$ is the scalar field potential. $F_{\mu\nu}$ is the field tensor of the electromagnetic vector field A_μ defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.16}$$

Note that the gauge field A_μ does not transform under scale transformation. In addition, the covariant derivative of the fermion field ψ is defined as

$$\nabla_\mu\psi = [\partial_\mu - \sigma_{ab}e^{b\nu}D_\mu e^a_\nu/2 - iA_\mu]\psi \tag{1.17}$$

with vielbein $e^b{}_\mu$ satisfying $e^a{}_\mu e_{a\nu} = g_{\mu\nu}$.

Note that the conformal dimension of the fermion field ψ is $3/2$. Therefore, it will transform as

$$\psi \to \psi^\Omega = \Omega^{-2/3}\psi \tag{1.18}$$

under Weyl transformation. In addition, $e^a{}_\mu$ and $\nabla_\mu\psi$ also transform as

$$e^a{}_\mu \to (e^a{}_\mu)^\Omega = \Omega e^a{}_\mu, \tag{1.19}$$

$$\nabla_\mu \psi \to (\nabla_\mu \psi)^\Omega = \Omega^{-3/2} \nabla_\mu \psi \tag{1.20}$$

under Weyl transformation. This is a very special property of the covariant derivative $\nabla_\mu \psi$ and vielbein. The general covariant structure with vielbein defined in Eq. (1.17) is Weyl covariant by itself without the introduction of a Weyl vector meson. See, for example, Ref. [32] for a brief review. Therefore, the general covariant fermion action is Weyl invariant by itself.

The scale of the Weyl-invariant theory can be introduced, for example, by the Higgs mechanism with a spontaneously symmetry-breaking potential of the form

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2. \tag{1.21}$$

Alternatively, the scale symmetry can also be broken by a dynamical approach if $V \propto \phi^4$ is scale invariant initially. As a result, an induced symmetry-breaking potential can be derived from the radiative corrections [40].

As a result, the vacuum of the theory, with $\phi^2 = \phi_0^2 =$ const, will set the scale of the theory. To be more specific, the vacuum solution of the theory will take the following form:

$$S_0 = \int d^4x \sqrt{g} \left[\frac{\epsilon}{2} \phi_0^2 \tilde{R} - \frac{\phi_0^2}{2} S_\mu S^\mu + i\bar{\psi}\gamma^\mu \nabla_\mu \psi - \phi_0 \bar{\psi}\psi - \frac{1}{4} H^{\mu\nu} H_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]. \tag{1.22}$$

Consequently, the Newtonian constant will be set by the coupling $M_p^2 = \epsilon\phi_0^2$. In addition, the Weyl vector meson and the fermion field become massive once the vacuum solution dominates.

This paper is organized as follows: In Sec. I, we briefly review the motivation of this research, and a more detailed review of the dRGT theory will be introduced in Sec. II. For heuristic reasons, a detailed derivation of the field equations will also be shown in this section. In Sec. III the analysis of the universal properties associated with a general reference metric will be presented. In particular, we will show that the massive terms serve as an effective cosmological constant in this section. In Sec. IV specific solutions to the effective cosmological constant will be presented when $\mathcal{K}$ is brought to its Jordan normal form. In Sec. V the fiducial metric equation will be presented for this model written as a functional of $\mathcal{K}$. Some useful properties of this equation will also be shown in this section. In Sec. VI a general Weyl-invariant dRGT model will be introduced along with some of its generic properties. In Sec. VII a general constraint derived from the conservation law will be presented. In Sec. VIII, some solutions of this model will be shown in a Bianchi type I physical space. Finally, concluding remarks and discussions will be given in Sec. IX. The recurrence relation of the massive Lagrangian will be given in the Appendix.

II. THE STÜCKELBERG FORMULATION

In this section, we will briefly review the physical origin of the dRGT model. We can expand the physical metric $g_{\mu\nu}$ around a reference (or fiducial) metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

with $h_{\mu\nu}$ the well-known linearized spin-2 field. $h_{\mu\nu}$ is, however, not covariant under diffeomorphism. In order to restore the generic diffeomorphism, we can expand the metric $g_{\mu\nu}$ alternatively as [4,8,9]

$$g_{\mu\nu} = Z_{\mu\nu} + H_{\mu\nu} \tag{2.2}$$

with respect to a background metric

$$Z_{\mu\nu} \equiv f_{ab} \partial_\mu \phi^a \partial_\nu \phi^b. \tag{2.3}$$

Here ϕ^a $(a = 0, 1, 2, 3)$ are the Stückelberg fields. The Roman letters a, b, c denote flat space indices to be raised or lowered by the Minkowski metric η^{ab} and η_{ab}, respectively. On the other hand, the Greek letters μ, ν, α denote curved space indices to be raised or lowered by the physical metric $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively. In addition, $\phi^a = x^a + \pi^a$ is the linear expansion of the Stückelberg field around the *unitary gauge* $\phi^a = x^a$. This is the *unitary gauge* associated with the diffeomorphism, in contrast to the unitary gauge of the Weyl symmetry that turns the scalar field as a constant. As a result, $Z_{\mu\nu}$ and $H_{\mu\nu}$ are both covariant under diffeomorphism with ϕ^a introduced as scalar fields under diffeomorphism.

The massive Lagrangian can therefore be written as a functional of the tensor field $\mathcal{K}^\mu{}_\nu$ defined by [4,8,9]

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - M^\mu{}_\nu, \tag{2.4}$$

with [41]

$$Z^\mu{}_\nu \equiv g^{\mu\alpha} Z_{\alpha\nu} \equiv M^\mu{}_\rho M^\rho{}_\nu. \tag{2.5}$$

These equations can also be written as matrix equations

$$\mathcal{K} = \delta - M, \tag{2.6}$$

$$M^2 = g^{-1} Z \tag{2.7}$$

with δ the unit matrix. Note that a tensor $A^\mu{}_\nu$ and the corresponding components of the 4×4 matrix A is related by $(A)_{\mu\nu} \equiv A^\mu{}_\nu$. Consequently, the multiplication of two matrices is defined as $(AB)_{\mu\nu} = (A)_{\mu\alpha}(B)_{\alpha\nu} = A^\mu{}_\alpha B^\alpha{}_\nu$.

As a result, the most general action of the ghost-free massive dRGT theory can be shown to be [4–6,8–18,21–23]

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{g}\{R + m_g^2(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4)\}, \quad (2.8)$$

with M_p the Planck mass, Λ the cosmological constant, and m_g the graviton mass. In addition, α_3, α_4 are free parameters. $g \equiv -\det g_{\mu\nu}$ is the determinant of the physical metric $g_{\mu\nu}$. The massive terms $\mathcal{L}_i$ $(i = 2-4)$ are defined as

$$\mathcal{L}_2 = \frac{1}{2}[\mathcal{K}]^2 - \frac{1}{2}[\mathcal{K}^2], \quad (2.9)$$

$$\mathcal{L}_3 = \frac{1}{6}[\mathcal{K}]^3 - \frac{1}{2}[\mathcal{K}][\mathcal{K}^2] + \frac{1}{3}[\mathcal{K}^3], \quad (2.10)$$

$$\mathcal{L}_4 = \frac{1}{24}[\mathcal{K}]^4 - \frac{1}{4}[\mathcal{K}]^2[\mathcal{K}^2] + \frac{1}{8}[\mathcal{K}^2]^2 + \frac{1}{3}[\mathcal{K}][\mathcal{K}^3] - \frac{1}{4}[\mathcal{K}^4]. \quad (2.11)$$

The bracket notation $[A] \equiv \mathrm{tr}A = \sum_i A^i{}_i$ denotes the trace of any matrix A [4,8,9].

The variational equation of the physical metric $g_{\mu\nu}$ can be shown to be

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + m_g^2(X_{\mu\nu} + \alpha_4 Y_{\mu\nu}) = 8\pi G T_{\mu\nu}, \quad (2.12)$$

with $X_{\mu\nu}$ and $Y_{\mu\nu}$ defined as

$$\begin{aligned} X_{\mu\nu} &= \mathcal{K}_{\mu\nu} - [\mathcal{K}]g_{\mu\nu} \\ &\quad - (\alpha_3 + 1)\left\{\mathcal{K}^2_{\mu\nu} - [\mathcal{K}]\mathcal{K}_{\mu\nu} + \frac{[\mathcal{K}]^2 - [\mathcal{K}^2]}{2}g_{\mu\nu}\right\} \\ &\quad + (\alpha_3 + \alpha_4)\left\{\mathcal{K}^3_{\mu\nu} - [\mathcal{K}]\mathcal{K}^2_{\mu\nu} + \frac{1}{2}\mathcal{K}_{\mu\nu}\{[\mathcal{K}]^2 - [\mathcal{K}^2]\}\right\} \\ &\quad - \frac{\alpha_3 + \alpha_4}{6}\{[\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\}g_{\mu\nu}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} Y_{\mu\nu} &= -\frac{\mathcal{L}_4}{2}g_{\mu\nu} + \frac{1}{6}[\mathcal{K}]^3\mathcal{K}_{\mu\nu} - \frac{1}{2}[\mathcal{K}][\mathcal{K}^2]\mathcal{K}_{\mu\nu} + \frac{1}{3}[\mathcal{K}^3]\mathcal{K}_{\mu\nu} \\ &\quad - \frac{1}{2}[\mathcal{K}]^2\mathcal{K}^2_{\mu\nu} + \frac{1}{2}[\mathcal{K}^2]\mathcal{K}^2_{\mu\nu} + [\mathcal{K}]\mathcal{K}^3_{\mu\nu} - \mathcal{K}^4_{\mu\nu}. \end{aligned} \quad (2.14)$$

Note that the massive terms have been shown to remain ghost free in the presence of a more general fiducial metric f_{ab}. We will therefore consider the effect of the dRGT model with a more general fiducial metric in this paper.

As a result, an additional set of field variables f_{ab} and hence more degrees of freedom are introduced to the system. The resulting massive Lagrangian is now a functional of the physical metric $g_{\mu\nu}$, the Stückelberg fields ϕ^a, and the fiducial metric f_{ab}. The Stückelberg fields ϕ^a and the fiducial metric f_{ab} will be treated equally as field variables here. Note also that when a kinetic term of f_{ab} are introduced, the theory becomes a bimetric theory.

For simplicity, we will not introduce the kinetic terms for the fiducial metric for the moment. The fiducial metric will be treated as an auxiliary field until we return to the bimetric theory later in this paper. Consequently, a complete set of field equations can be derived from the variation of $g_{\mu\nu}$, ϕ^a, and f_{ab}.

Note, however, that any change of the Stückelberg fields ϕ^a can be thought of as a coordinate transformation of the fiducial metric f_{ab}

$$f_{\mu\nu} = f_{ab}d\phi^a d\phi^b = f'_{ab}d\phi'^a d\phi'^b. \quad (2.15)$$

Therefore, we can simply take the *unitary gauge* $\phi^a = x^a$ for simplicity. The dynamics of the Stückelberg fields ϕ^a can thus be absorbed as parts of the dynamics represented by the general fiducial metric f_{ab}. As a result,

$$Z_{\mu\nu} = f_{ab}\delta^a{}_\mu\delta^b{}_\nu = f_{\mu\nu}. \quad (2.16)$$

For convenience and economy of notation, we will simply write $Z_{\mu\nu}$ as $f_{\mu\nu}$ from now on. In contrast to the bimetric theory with $\hat{R}(f_{\mu\nu}) \neq 0$, the auxiliary field treated here does not have a kinetic term. Therefore, a simple constraint can be obtained from the variational equation of $f_{\mu\nu}$. Consequently, this constraint will force the massive terms to behave as an effective cosmological constant.

The reference metric can be shown to act compatibly with the physical metric in order to satisfy the constraint equation of $f_{\mu\nu}$. Indeed, we can show that [39,42]

$$\begin{aligned} \delta(\sqrt{g}\mathcal{L}_M) &= \frac{1}{2}\sqrt{g}\mathcal{L}_M g^{\mu\nu}\delta g_{\mu\nu} + \sqrt{g}\frac{\delta\mathcal{L}_M}{\delta\mathcal{K}^\mu{}_\nu}\delta\mathcal{K}^\mu{}_\nu \\ &= \frac{1}{2}\sqrt{g}\mathcal{L}_M[g^{-1}\delta g] - \sqrt{g}[M^{-1}A\delta M], \end{aligned} \quad (2.17)$$

with $A^\mu{}_\nu \equiv M^\mu{}_\alpha\delta\mathcal{L}_M/\delta\mathcal{K}^\nu{}_\alpha$ and $\mathcal{L}_M = \mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4$. The equation

$$(\delta M)M + M\delta M = (\delta g^{-1})f + g^{-1}\delta f \quad (2.18)$$

is thus the direct result of the definition $M^2 = g^{-1}f$. Therefore, we have

$$\begin{aligned} &M^{-1}A\delta M + M^{-1}AM(\delta M)M^{-1} \\ &\quad = M^{-1}A(\delta g^{-1})fM^{-1} + M^{-1}Ag^{-1}(\delta f)M^{-1}. \end{aligned} \quad (2.19)$$

The trace of Eq. (2.19) gives

$$[M^{-1}A\delta M] = -\frac{1}{2}[Ag^{-1}\delta g] + \frac{1}{2}[Af^{-1}\delta f] = -[t\delta g] - [\hat{t}\delta f], \tag{2.20}$$

with the effect of the commuting properties $[A, M] = 0$ included. Consequently, we have

$$\frac{1}{\sqrt{g}}\delta(\sqrt{g}\mathcal{L}_M) = \frac{1}{2}\mathcal{L}_M[g^{-1}\delta g] + [t\delta g] + [\hat{t}\delta f] \tag{2.21}$$

with t the symmetric part of $Ag^{-1}/2$, and $\hat{t}$ the symmetric part of $-Af^{-1}/2$ given explicitly by

$$t = \frac{1}{4}(Ag^{-1} + g^{-1}A^T); \tag{2.22}$$

$$\hat{t} = -\frac{1}{4}(Af^{-1} + f^{-1}A^T). \tag{2.23}$$

For convenience, we can lower the tensor indices with the help of g and f, respectively, to define

$$t^\mu{}_\nu \equiv (tg)^\mu{}_\nu = \frac{1}{4}(A + g^{-1}A^T g), \tag{2.24}$$

$$\hat{t}^\mu{}_\nu \equiv (tf)^\mu{}_\nu = \frac{1}{4}(A + f^{-1}A^T f) = -t^\mu{}_\nu. \tag{2.25}$$

Note that the identity

$$\hat{t}^\mu{}_\nu = -t^\mu{}_\nu \tag{2.26}$$

can be proved by showing that $g^{-1}A^T g = f^{-1}A^T f$, or equivalently,

$$gAg^{-1} = fAf^{-1}. \tag{2.27}$$

Equation (2.27) is true because of the fact that $[A, M^2] = 0$ and the definition of $M^2 = g^{-1}f$. Indeed, the above identity follows from the fact that $gAg^{-1}f = gAM^2 = gM^2A = fA$. Hence we can show that $\hat{t}^\mu{}_\nu = -t^\mu{}_\nu$. Note again that the field $A^\mu{}_\nu$ is defined as

$$A^\mu{}_\nu \equiv M^\mu{}_\alpha \frac{\delta\mathcal{L}_M}{\delta\mathcal{K}^\nu{}_\alpha}. \tag{2.28}$$

III. THE $\hat{t} = 0$ EQUATION AND THE EFFECTIVE COSMOLOGICAL CONSTANT

If we treat the most general fiducial metric f_{ab} as auxiliary fields, the variational equation of the fiducial field is simply $\hat{t}^\mu{}_\nu = 0$. Since we have shown that $\hat{t}^\mu{}_\nu = -t^\mu{}_\nu$ in Eq. (2.26), the vanishing of the fiducial metric equation $\hat{t}^\mu{}_\nu = 0$ implies the vanishing of part of the energy-momentum tensor $t^\mu{}_\nu = 0$. Therefore, we will be lead to the important result that $\mathcal{L}_M =$ constant.

Indeed, the metric field equation can be shown to be

$$\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) = m_g^2\left[\frac{1}{2}\mathcal{L}_M g^{\mu\nu} + t^{\mu\nu}\right]. \tag{3.1}$$

Therefore, $t^\mu{}_\nu = 0$ implies that the energy-momentum tensor derived from the massive terms, $m_g^2\mathcal{L}_M g^{\mu\nu}/2$, acts as an effective cosmological constant $\Lambda_e = -m_g^2\mathcal{L}_M/2$. Indeed, the Bianchi identity assures that the conservation law

$$\partial^\nu\mathcal{L}_M = -2D_\mu t^{\mu\nu} = 0 \tag{3.2}$$

is obeyed. Hence the massive terms do act as an effective constant. What we need to do now is to compute the explicit value of the effective cosmological constant.

Recalling that any arbitrary $n \times n$ square matrix $\mathcal{K}$ can be brought to the Jordan normal form via a similarity transformation

$$\tilde{\mathcal{K}} = S^{-1}\mathcal{K}S, \tag{3.3}$$

with $\tilde{\mathcal{K}}$ a square matrix in block-diagonal form

$$\tilde{\mathcal{K}} \equiv \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & \cdots \end{pmatrix}. \tag{3.4}$$

For example, a 3×3 Jordan matrix takes the form

$$J_i \equiv \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix} \tag{3.5}$$

with λ_i the degenerated eigenvalue of the matrix $\mathcal{K}$ defined by

$$\mathcal{K}u = \lambda_i u. \tag{3.6}$$

Since we will focus on four-dimensional metric space, the number of distinct eigenvalues of the matrix $\mathcal{K}$ is less or equal to 4. In addition, for convenience, a matrix $\tilde{\mathcal{K}}$ will be referred to as a Jordan matrix or a matrix in Jordan coordinate when a matrix $\mathcal{K}$ is block diagonalized as $\tilde{\mathcal{K}} = S^{-1}\mathcal{K}S$ in Jordan normal form. From simplicity, we will name $\lambda_i = a, b, c, d$ as four possible equal of unequal eigenvalues of $\mathcal{K}$.

Note that the massive Lagrangian only has to do with the trace of $\mathcal{K}^n$. It is also easy to show that

$$[\mathcal{K}^n] = [\tilde{\mathcal{K}}^n] \tag{3.7}$$

for matrices related by a similarity transformation. Therefore, the field equations $\hat{t}_{ab} = 0$ of the fiducial metric

f_{ab} can be solved when $\mathcal{K}$ is brought to the Jordan normal form $\tilde{\mathcal{K}}$ given by Eq. (3.4).

Moreover, the trace of the Jordan matrix

$$[\tilde{\mathcal{K}}] \equiv \mathrm{Tr}\begin{pmatrix} a & \cdot & \cdot & \cdot \\ 0 & b & \cdot & \cdot \\ 0 & 0 & c & \cdot \\ 0 & 0 & 0 & d \end{pmatrix} = a + b + c + d \tag{3.8}$$

has nothing to do with the off-diagonal component of $\tilde{\mathcal{K}}$. In addition, it is straightforward to show that

$$[\tilde{\mathcal{K}}^n] \equiv \mathrm{Tr}\begin{pmatrix} a^n & \cdot & \cdot & \cdot \\ 0 & b^n & \cdot & \cdot \\ 0 & 0 & c^n & \cdot \\ 0 & 0 & 0 & d^n \end{pmatrix} = a^n + b^n + c^n + d^n. \tag{3.9}$$

Here we have represented all the irrelevant upper block as $\cdot$ in Eq. (3.9). Therefore, the derivation of the fiducial equation is related only to the eigenvalues of the matrix equation (3.6). Note again that we have named $\lambda_i = a, b, c, d$ as the eigenvalues of the matrix equation (3.6). It is also possible that some of the eigenvalues are identical to each other because the multiplicity of the eigenvalue equation.

A. The interaction terms

Note that even there are 10 degrees of freedom of a reference metric $f_{\mu\nu}$ that seem to be involved in the interaction with the physical metric. There are in fact at most 4 of them, via the eigenvalues of $\mathcal{K}$, interacting with the physical metric through the massive term $\mathcal{L}_M$. Therefore, only the diagonal part of the Jordan matrix $\mathcal{K}$ is relevant to the interaction. Note that we have written $\tilde{\mathcal{K}}$ as $\mathcal{K}$ for simplicity by assuming we have block diagonalized all matrices of interest.

The other components of the Jordan matrix $\mathcal{K}$ have nothing to do with our theory. Therefore, we can freely assume that $\hat{t}^\mu{}_\mu = 0$ are the only constraint equations. As a result,

$$\hat{t}^\mu{}_\mu = \frac{1}{4}(A^\mu{}_\mu + f^{\mu\mu}A^\mu{}_\mu f_{\mu\mu}) = \frac{1}{4}(1 + f^{\mu\mu}f_{\mu\mu})A^\mu{}_\mu = 0. \tag{3.10}$$

Note that μ is an open index here. Hence the constraint equations $\hat{t}^\mu{}_\mu = 0$ are equivalent to

$$\frac{\delta\mathcal{L}_M}{\delta\mathcal{K}^\mu{}_\nu} = 0. \tag{3.11}$$

We have kept ν open in Eq. (3.11); even the $\nu \neq \mu$ components do not contribute to the field equations. Once the variational equation is obtained, these extra constraints will automatically disappear.

Note that the constraint equations put a strong constraint on the relation of f and g via the definition of the matrix $M^2 = g^{-1}f$. It is known that M^2 can always be brought to the Jordan normal form. The relation of the physical and fiducial metrics is, however, a little bit more complicated. Indeed, we can write $g = fM^{-2}$ to express the physical metric g. Even we can write M^2 as a Jordan matrix by a similarity transformation such that $M_d = S^{-1}MS$, the corresponding metrics $S^{-1}gS$ and $S^{-1}fS$ may not be symmetric any more. Therefore, the block-diagonalized process cannot be done with a single transformation matrix S.

The resolution exists only when the metrics g and f are compatible with the solutions M. To be more specific, $g = fM^{-2} = fSM_d^{-2}S^{-1}$ has to remain symmetric with some appropriate choice of f. For example, we can start out with arbitrary diagonal metrics g and f. As a result, M^2 is a diagonal matrix without carrying out any similarity transformation. It is, however, very difficult to find a symmetric f for our purpose with a given arbitrary M^2. This is the main reason that most solutions to the bimetric theory can only be found by assuming that f and g are both in diagonal form. For the same reason, in the later part of this paper, we will focus on the metric space with a diagonal physical metric.

It will be interesting to note that some of the identities discussed in this paper will be valid for arbitrary g and f. Therefore, we will focus on the properties related to the most arbitrary metric g and f before we return to the bimetric theory.

B. Constraint equations and the effective cosmological constant

For heuristic reasons and for completeness, we will present a brief review of the derivation of the fiducial metric via the variational equation of $\mathcal{K}$. We will study some of the interesting properties associated with the fiducial metric equation in this form in the next section.

Recall that $\mathcal{L}_M = \mathcal{L}_2(\mathcal{K}) + \alpha_3\mathcal{L}_3(\mathcal{K}) + \alpha_4\mathcal{L}_4(\mathcal{K})$ with $\mathcal{L}_i$ defined by Eqs. (2.9)–(2.11). In addition, we will write

$$\{A, B, C\} = A + \alpha_3 B + \alpha_4 B \tag{3.12}$$

for convenience. Our task now is to solve the fiducial equation $\hat{t}_{ab} = 0$ with $\mathcal{K}$ given by the Jordan matrix shown in Eq. (3.4), or simply ignore the off-diagonal components

$$\mathcal{K} \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{3.13}$$

Note that the invertible requirement of $g_{\mu\nu}$ and $f_{\mu\nu}$ puts a constraint on the possible values of a, b, c, d, namely, $a \neq 1$, $b \neq 1$, $c \neq 1$, and $d \neq 1$. It is easy to show that the fiducial equation can also be shown as [42]

$$2\hat{t}^{\mu}{}_{\nu} = \{[\mathcal{K}] - \mathcal{K}, \mathcal{L}_2 - ([\mathcal{K}] - \mathcal{K})\mathcal{K}, \mathcal{L}_3 - \mathcal{K}\mathcal{L}_2 + ([\mathcal{K}] - \mathcal{K})\mathcal{K}^2\} = 0. \tag{3.14}$$

The massive terms are related by the recurrence relations (see appendix A for a brief review) [42]

$$\frac{\delta\mathcal{L}_n}{\delta\mathcal{K}} = \mathcal{L}_{n-1}\delta - \frac{\delta\mathcal{L}_{n-1}}{\delta\mathcal{K}}\mathcal{K}, \tag{3.15}$$

with $n \leq 4$, $\mathcal{L}_0 = 2$, $\mathcal{L}_1 = 2[\mathcal{K}]$, and δ the unit matrix. In addition, the recurrence relation ends with

$$\mathcal{L}_4\delta - \frac{\delta\mathcal{L}_4}{\delta\mathcal{K}}\mathcal{K} = 0 \tag{3.16}$$

from the fact that $\mathcal{L}_5 = 0$. In fact, the fiducial equations can also be derived from the variation of $\mathcal{L}_M$ with respect to $\mathcal{K}^i{}_i$ or equivalently a, b, c, d. First of all, we can show that

$$\mathcal{L}_2 = ab + ac + ad + bc + bd + cd, \tag{3.17}$$

$$\mathcal{L}_3 = abc + abd + bcd + acd, \tag{3.18}$$

$$\mathcal{L}_4 = abcd. \tag{3.19}$$

Therefore, the variational equations of the massive Lagrangian are simply

$$\{b + c + d, bc + bd + cd, bcd\} = 0, \tag{3.20}$$

$$\{a + c + d, ac + ad + cd, acd\} = 0, \tag{3.21}$$

$$\{a + b + d, ab + ad + bd, abd\} = 0, \tag{3.22}$$

$$\{a + b + c, ab + ac + bc, abc\} = 0. \tag{3.23}$$

It is clear that the field equations do respect the permutation symmetries among the eigenvalues a, b, c, d. Note that there are five distinct forms of Jordan matrix $\tilde{\mathcal{K}}$, or $\mathcal{K}$, with (i) $a = b = c = d$, (ii) $a = b = c \neq d$, (iii) $a = b \neq c = d$, (iv) $a \neq b \neq c = d$, and (v) $a \neq b \neq c \neq d$.

Subtracting any two of the above equations will lead to, for example,

$$(a - b)\{1, c + d, cd\} = 0, \tag{3.24}$$

$$(a - c)\{1, b + d, bd\} = 0, \tag{3.25}$$

$$(a - d)\{1, b + c, bc\} = 0, \tag{3.26}$$

and three other distinct permutations. Detailed analysis of the solutions will be presented in the following section.

IV. SOME SOLUTIONS TO THE EFFECTIVE COSMOLOGICAL CONSTANT

In order to solve the fiducial metric equation for the effective cosmological constant, we will try to extract some useful relations between the eigenvalues of $\mathcal{K}$ and the possible effective cosmological constants in this section. If $a \neq b$, we have

$$\{1, c + d, cd\} = 0. \tag{4.1}$$

This equation is equivalent to the expression

$$c = -\frac{1 + \alpha_3 d}{\alpha_3 + \alpha_4 d} = f(d). \tag{4.2}$$

The symmetry between c and d implies

$$c = f(d) = f(f(c)). \tag{4.3}$$

Similarly, for any combination of $\lambda_i \neq \lambda_j$, the other two eigenvalues λ_k, λ_l will also be related by the equation

$$\lambda_k = f(\lambda_l) = f(f(\lambda_k)). \tag{4.4}$$

Here we have assumed, $i \neq j \neq k \neq l$. Note that $a\{1, c + d, cd\} = 0$ and $\{a + c + d, ac + ad + cd, acd\} = 0$ imply immediately $\{c + d, cd, 0\} = 0$. Hence $a\{1, c + d, cd\} = 0$ and $\{c + d, cd, 0\} = 0$ can be solved to give

$$c \neq d = \lambda_{\pm} = \frac{\alpha_3 \pm \sqrt{4\alpha_4 - 3\alpha_3^2}}{2(\alpha_4 - \alpha_3^2)}. \tag{4.5}$$

Note that this set of solutions exists only when $4\alpha_4 \geq 3\alpha_3^2$ and $\alpha_4 \neq \alpha_3^2$. In particular, this set of solutions becomes

$$c = d = -\frac{2}{\alpha_3} \tag{4.6}$$

in the limit $4\alpha_4 = 3\alpha_3^2$. Note also that, the condition $\alpha_4 = \alpha_3^2$ will lead to a contradiction in the field equation. Hence no solution can be found in this limit. This is, in fact, the direct result of the condition $a \neq b$.

A. $a = b = c = d$

To demonstrate how to use the results derived above, we will analyze all possible combinations of solutions case by case in this subsection. First of all, we will focus on a very special combination of choice when (A) $a = b = c = d$. In this case, the field equation simply gives

$$a(3 + 3\alpha_3 a + \alpha_4 a^2) = 0. \tag{4.7}$$

Therefore, we can show that

$$a = \frac{-3\alpha_3 \pm \sqrt{9\alpha_3^2 - 12\alpha_4}}{2\alpha_4}. \tag{4.8}$$

Consequently, the effective cosmological constant $\Lambda_M = -m_g^2\mathcal{L}_M$ with

$$\mathcal{L}_M = a^2[6+4\alpha_3 a+\alpha_4 a^2] = a^2[3+\alpha_3 a], \\ = -\frac{3}{2\alpha_4^3}\left[9\alpha_3^4+6\alpha_4^2-18\alpha_3^2\alpha_4 \mp \alpha_3(3\alpha_3^2-4\alpha_4)\sqrt{3(3\alpha_3^2-4\alpha_4)}\right]. \quad (4.9)$$

Hence the effective cosmological constant is

$$\Lambda_M = \frac{3m_g^2}{2\alpha_4^3}\left[9\alpha_3^4+6\alpha_4^2-18\alpha_3^2\alpha_4 \mp \alpha_3(3\alpha_3^2-4\alpha_4)\sqrt{3(3\alpha_3^2-4\alpha_4)}\right]. \quad (4.10)$$

Note that solution exists only when $3\alpha_3^2 \geq 4\alpha_4$. In particular, in the limit $3\alpha_3^2 = 4\alpha_4$, the effective cosmological constant will become

$$\Lambda_M = 4\frac{m_g^2}{\alpha_3^2}. \quad (4.11)$$

B. $a = b = c \neq d$

For case (B) $a = b = c \neq d$, Eq. (3.23) implies

$$a = b = c = -\frac{2}{\alpha_3}; \quad 4\alpha_4 = 3\alpha_3^2. \quad (4.12)$$

On the other hand, $c \neq d$ implies that $(c,d) = (\lambda_+,\lambda_-)$, or (λ_-,λ_+). This immediately implies that $c = d = 2/\alpha_3$ in the limit $4\alpha_4 = 3\alpha_3^2$. Therefore the case (B) $a = b = c \neq d$ solution never exists.

C. $a = b \neq c = d$

For the case (C) $a = b \neq c = d$, it is easy to show that $a = b = \lambda_\pm, c = d = \lambda_\mp$ are two possible combinations. Note also that we have shown earlier that solution exists only when $3\alpha_3^2 \leq 4\alpha_4$. Indeed,

$$\mathcal{L}_M = (\mathcal{L}_2+\alpha_3\mathcal{L}_3+\alpha_4\mathcal{L}_4) = -\frac{1}{\alpha_4-\alpha_3^2}. \quad (4.13)$$

Note further that solution exists only when $3\alpha_3^2 \leq 4\alpha_4$. In particular, in the limit $3\alpha_3^2 = 4\alpha_4$, b equals c. As a result, case (C) will be identical to case (A). Indeed, we can also show that the effective cosmological constant becomes

$$\Lambda_M = 4\frac{m_g^2}{\alpha_3^2} \quad (4.14)$$

when $3\alpha_3^2 = 4\alpha_4$ is applied to Eq. (4.13). This also shows that solutions found in both case (A) and case (C) are consistent in the limit $3\alpha_3^2 = 4\alpha_4$. In fact, this asserts that only the case (A) solution will exist if $3\alpha_3^2 = 4\alpha_4$.

D. $a \neq b \neq c = d$ and $a \neq b \neq c \neq d$

For the case (D) $a \neq b \neq c = d$, and case (E) $a \neq b \neq c \neq d$, it is easy to show that we have only two distinct eigenvalues $\lambda_\pm$ for the equations derived from $\lambda_i \neq \lambda_j$. Therefore, there are not enough distinct eigenvalues to accommodate the solutions in both case (D) and (E).

In conclusion, the solutions to the fiducial metric equations depend only on the four distinct eigenvalues of the $\mathcal{K}$ matrix. The results shown in this section further point out that the positivity of the coupling constants $k_\alpha \equiv 3\alpha_3^2 - 4\alpha_4$ determines the possible solutions to the reference metric equation. Indeed, when $k_\alpha \geq 0$, the only solution to the $t = 0$ equation is the case $a = b = c = d$ solution give by Eq. (4.8):

$$a = b = c = d = \frac{-3\alpha_3 \pm \sqrt{9\alpha_3^2-12\alpha_4}}{2\alpha_4}.$$

If $k_\alpha \leq 0$, the only solution to the $t = 0$ equation is the case $a = b \neq c = d$ solution given by $a = b = \lambda_\pm, c = d = \lambda_\mp$. In particular, the limiting case with $k_\alpha = 0$ will lead both cases to the same result with $a = b = c = d = -2\alpha_3$. This result is for the fiducial metric models without kinetic term.

E. Existence of isotropic solution

Our result shows that the most favorable solution to the fiducial metric equation is the one satisfying the variational equation $\hat{t}^\mu{}_\nu = 0$. Hence, $f_{\mu\nu}$ has to tune itself to induce the desired solution $\hat{t}^\mu{}_\nu = 0$. As a result, the only role played by the fiducial metric is to induce an effective cosmological constant.

Note that it was shown in Ref. [9] that a spatially flat homogeneous and isotropic solution does not exist for massive gravity theory if the reference metric is assumed to be a flat Minkowski metric $f_{ab} = \eta_{ab}$. The result is completely different from our approach because the choice of flat reference metric puts a strong constraint on $Z_{\mu\nu}$,

$$Z_{\mu\nu} = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \quad (4.15)$$

For example, in the gauge $\phi^a = (f(t), x^i)$ [9],

$$Z_{\mu\nu} = \mathrm{diag}(\dot{f}^2, 1, 1, 1). \quad (4.16)$$

A similar constraint also exists for the open Fiedmann-Robertson-Walker (FRW) metric case [10] with a Stückelberg field chosen as $\phi^a = f(t)((1+x^2+y^2+z^2)^{1/2}, x^i)$. The constraint becomes

$$Z_{\mu\nu}dx^\mu dx^\nu = -(\dot{f})^2dt^2 + f^2 g_{ij}dx^i dx^j \quad (4.17)$$

with g_{ij} the spatial FRW metric. As a result, Z_{00} is related to Z_{ij} and cannot tune itself to accommodate a more general solution for the massive gravity theory. That is the main reason that the expanding solution only exits for open FRW space. Therefore, in contrast to the general reference metric result here, the chosen fiducial metric in some of the literatures may not be compatible with the physical metric $g_{\mu\nu}$. In summary, treating the reference metric as a dynamic or auxiliary field, extra degrees of freedom are introduced for $f_{\mu\nu}$ to accommodate a compatible solution. Therefore, we will try to explore possible anisotropic solutions in this paper. In particular, for a simple demonstration, we will take the BI metric space as an example.

V. SYMMETRIC FIELD EQUATIONS AND THE EFFECTIVE COSMOLOGICAL CONSTANT

Although the metric $\mathcal{K}$ can be nonsymmetric to start with, Eq. (3.14) derived from $\delta\mathcal{L}_M/\delta\mathcal{K}$ is symmetric by itself. In fact, there is a hidden constraint that forces a compatible fiducial metric solution to make $\mathcal{K}$ a diagonal matrix. For heuristic reasons, we will formally demonstrate this result by deriving the field equation explicitly.

Indeed, we can show that the variational equation of $\mathcal{L}_M$, Eq. (3.14), can be written formally as

$$\delta\mathcal{L}_M = \{\delta\mathcal{L}_2, \delta\mathcal{L}_3, \delta\mathcal{L}_4\} \tag{5.1}$$

with $(\delta\mathcal{L}_i)^\mu{}_\nu \equiv \delta\mathcal{L}_i/\delta\mathcal{K}^\nu{}_\mu$. Hence we can write

$$\delta\mathcal{L}_4 = -\frac{1}{\alpha_4}\delta\mathcal{L}_2 - \frac{\alpha_3}{\alpha_4}\delta\mathcal{L}_3. \tag{5.2}$$

With the recurrence relation (3.15), we can also show that

$$\mathcal{K}\delta\mathcal{L}_M = \mathcal{L}_M - \{\delta\mathcal{L}_3, \delta\mathcal{L}_4, 0\} = 0. \tag{5.3}$$

Hence we have the expression for the fiducial metric equation

$$\mathcal{L}_M\delta = \left(1 - \frac{\alpha_3^2}{\alpha_4}\right)\delta\mathcal{L}_3 - \frac{\alpha_3}{\alpha_4}\delta\mathcal{L}_2 \tag{5.4}$$

with δ denoting the unit matrix. Moreover, we can write Eq. (5.4) explicitly as

$$\left(\mathcal{L}_M - \frac{\alpha_4-\alpha_3^2}{2\alpha_4}([\mathcal{K}]^2 - [\mathcal{K}^2]) + \frac{\alpha_3}{\alpha_4}[\mathcal{K}]\right)\delta = -\frac{\alpha_4-\alpha_3^2}{\alpha_4}([\mathcal{K}]\mathcal{K} - \mathcal{K}^2) + \frac{\alpha_3}{\alpha_4}\mathcal{K}. \tag{5.5}$$

This implies that the diagonal components of the on-shell equation

$$\alpha_3\mathcal{K} - (\alpha_4 - \alpha_3^2)([\mathcal{K}]\mathcal{K} - \mathcal{K}^2) \tag{5.6}$$

are proportional to a unit matrix. All diagonal components are equal, and there is no nonvanishing off-diagonal component on shell. This indicates that the on-shell constraint will force the matrix $\mathcal{K}$ to be a symmetric matrix even it is not written in a Jordan coordinate.

For convenience, we will define

$$\lambda_i = \frac{\alpha_i}{\alpha_4 - \alpha_3^2} \tag{5.7}$$

for $i = 3, 4$. As a result, Eq. (5.4) can be written as

$$\left(\mathcal{K} + \frac{\lambda_3 - [\mathcal{K}]}{2}\right)^2 = \left(\lambda_4\mathcal{L}_M + \frac{1}{2}[\mathcal{K}^2] - \frac{1}{4}[\mathcal{K}]^2 + \frac{\lambda_3}{2}[\mathcal{K}] + \frac{\lambda_3^2}{4}\right)\delta \equiv A^2\delta. \tag{5.8}$$

Hence the solution of $\mathcal{K}$ to Eq. (5.8) is

$$\mathcal{K} = \frac{1}{2}([\mathcal{K}] - \lambda_3) + AJ, \tag{5.9}$$

with matrix J the solution to the matrix equation $J^2 = \delta$. Note that the matrix J is a diagonal matrix $J = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$, with $\kappa_i = \pm 1$ the eigenvalues of J. We can take the trace of Eq. (5.9) and obtain the result

$$[\mathcal{K}] = 2\lambda_3 - A[J]. \tag{5.10}$$

Therefore, we have

$$\mathcal{K} = \frac{1}{2}([\mathcal{K}] - \lambda_3) + \frac{1}{[J]}(2\lambda_3 - [\mathcal{K}])J \tag{5.11}$$

if $[J] \neq 0$. This completes the proof that the variational equation of $\mathcal{K}$ implies that $\mathcal{K}$ is a diagonal matrix.

We can further extract the information of $[\mathcal{K}^2]$ and derive the following expression for $\mathcal{L}_M$

$$\mathcal{L}_M = \frac{1}{\lambda_4}\left\{\left(\frac{1}{4} - \frac{1}{[J]^2}\right)([\mathcal{K}] - 2\lambda_3)^2 - \frac{3}{4}\lambda_3^2\right\}. \tag{5.12}$$

For example, we can show that $J = \delta$ corresponds to the case $a = b = c = d$ in Sec. IVA. And it also leads to the correct effective cosmological constant shown in Eq. (4.10).

On the other hand, if $[J] = 0$, then $[\mathcal{K}] = 2\lambda_3$. Hence we can show that

$$\mathcal{K} = \frac{1}{2}\lambda_3 + AJ. \tag{5.13}$$

Therefore, we have

$$A = \frac{\sqrt{4\alpha_4 - 3\alpha_3^2}}{2(\alpha_4 - \alpha_3^2)}. \tag{5.14}$$

This agrees with the result in Sec. IV C for the case $a = b \neq c = d$ with

$$\mathcal{L}_M = (\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) = -\frac{1}{\alpha_4 - \alpha_3^2}. \tag{5.15}$$

VI. WEYL-INVARIANT MASSIVE BIGRAVITY

Note that the Weyl transformation is a local scale transformation relating all physical fields in different length scales. The transformation property of each field will be determined by its corresponding conformal dimension. Therefore the following generalization of the dRGT theory can be shown to be Weyl invariant [5–7,39]:

$$S = \int d^4x \left\{ \sqrt{g}\left(\frac{\epsilon}{2}\phi^2\tilde{R}(g_{\mu\nu}) - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{4}H^2 - \frac{\lambda}{4}\phi^4\mathcal{L}_M\right) + \sqrt{f}\frac{1}{2}\epsilon_1\phi^2\tilde{R}(f_{\mu\nu})\right\}. \tag{6.1}$$

Here $\epsilon\phi^2$ and $\lambda\phi^4/4$ serve as M_p^2 and $m_g^2 M_p^2/2$ dynamical coupling constants, respectively, in the dRGT theory. Note that the term $\tilde{R}(f)$ denotes the kinetic curvature term of the fiducial metric $f_{\mu\nu}$. In addition, hatted notation, e.g. $\hat{R}$, will be used, when necessary, to denote the physical field evaluated solely as a functional of $f_{\mu\nu}$.

Similar to Eq. (1.12), the explicit expression of the Ricci curvatures can be shown as

$$\tilde{R}_{\mu\nu}(g) = R_{\mu\nu}(g) - (D_\mu S_\nu + D_\nu S_\mu) - D_\beta S^\beta g_{\mu\nu} + 2(S_\mu S_\nu - S_\beta S^\beta g_{\mu\nu}), \tag{6.2}$$

$$\tilde{R}_{\mu\nu}(f) = \hat{R}_{\mu\nu}(f) - (\hat{D}_\mu S_\nu + \hat{D}_\nu S_\mu) - \hat{D}_\beta S^\beta f_{\mu\nu} + 2(S_\mu S_\nu - S_\beta S^\beta f_{\mu\nu}). \tag{6.3}$$

Note that the Weyl transformation of the fiducial metric is given by

$$f_{\mu\nu} \to f^\Omega_{\mu\nu} = \Omega^2 f_{\mu\nu}. \tag{6.4}$$

We would like to remark here that there is no change in the massive terms $\mathcal{L}_M$ required for the Weyl symmetry. The massive terms respect the local scale transformation by itself because the coupling of the metric through the interaction of the form $M^2 = g^{-1}f$. This is in fact a very interesting built-in property of the dRGT theory that deserves more attention.

We will now show that the Weyl-invariant bimetric theory is in fact equivalent to the bimetric generalization of dRGT theory coupled to a massive gauge field. Indeed, we can take the unitary gauge of the Weyl symmetry, $\phi = \phi_0$, by choosing a gauge parameter $\Omega = \phi/\phi_0$. As a result, the Weyl-invariant bimetric theory will be equivalent to the following effective theory with $S = \int d^4x\sqrt{g}\mathcal{L}_1$:

$$S = \int d^4x\sqrt{g}\left\{\frac{\epsilon}{2}\phi_0^2 R - \frac{1}{2}\kappa^2 S_\mu S^\mu - \frac{1}{4}H^2 - \frac{\lambda}{4}\phi_0^4\mathcal{L}_M + \frac{\sqrt{f}}{\sqrt{g}}\left[\frac{\epsilon_1}{2}\phi_0^2\hat{R}(f) - \frac{1}{2}\kappa_1^2\hat{S}_\mu\hat{S}^\mu\right]\right\}, \tag{6.5}$$

or equivalently,

$$S = \int d^4x\sqrt{g}\left\{\frac{\epsilon'}{2}R - \frac{1}{2}\kappa^2 S_\mu S^\mu - \frac{1}{4}H^2 - \frac{\lambda'}{4}\mathcal{L}_M + \frac{\sqrt{f}}{\sqrt{g}}\left[\frac{\epsilon_1'}{2}\hat{R}(f) - \frac{1}{2}\kappa_1^2\hat{S}_\mu\hat{S}^\mu\right]\right\}. \tag{6.6}$$

Here $\epsilon' \equiv \epsilon\phi_0^2$, $\epsilon_1' \equiv \epsilon_1\phi_0^2$, $\lambda' \equiv \lambda\phi_0^4$, $\kappa^2 \equiv (1+6\epsilon)\phi_0^2$ and $\kappa_1^2 \equiv 6\epsilon_1\phi_0^2$.

Note that the hatted tensor fields, such as $\hat{S}^\mu = f^{\mu\nu}S_\nu$, are to be raised or lowered by $f^{\mu\nu}$ and $f_{\mu\nu}$, respectively. On the other hand, the unhatted tensor fields, such as $S^\mu = g^{\mu\nu}S_\nu$, will be raised or lowered by $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively.

The generic structure of the dRGT theory does not change very much in the unitary gauge. The resulting theory in the unitary gauge is identical to a theory with a massive gauge field. As a result, the results and analysis shown in this paper are also true for the conventional theory with a massive gauge field. We will therefore focus on the interesting applications of this model in the BI expanding Universe.

The S_μ equation is

$$D_\mu H^{\mu\nu} = \kappa^2 S^\nu + \frac{\sqrt{f}}{\sqrt{g}}\kappa_1^2\hat{S}^\nu \tag{6.7}$$

with a consequent constraint equation derived from applying a derivative to Eq. (6.7):

$$\kappa^2 D_\mu S^\mu + \kappa_1^2 D_\mu\left(\frac{\sqrt{f}}{\sqrt{g}}\hat{S}^\mu\right) = 0. \tag{6.8}$$

The metric field equations can be shown to be

$$\epsilon' G^{\mu\nu} = \epsilon'\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) = \frac{\lambda'}{2}t^{\mu\nu} + \left(\frac{1}{2}\kappa^2 S_\alpha S^\alpha + \frac{1}{4}H^2 + \frac{\lambda'}{4}\mathcal{L}_M\right)g^{\mu\nu} - \kappa^2 S^\mu S^\nu - H^{\mu\alpha}H^\nu{}_\alpha, \tag{6.9}$$

$$\epsilon_1'\hat{G}^{\mu\nu} = \epsilon_1\left(\frac{1}{2}f^{\mu\nu}\hat{R} - \hat{R}^{\mu\nu}\right) = \frac{\lambda'\sqrt{g}}{2\sqrt{f}}\hat{t}^{\mu\nu} - \kappa_1^2\hat{S}^\mu\hat{S}^\nu + \frac{1}{2}\kappa_1^2\hat{S}_\alpha\hat{S}^\alpha g^{\mu\nu}, \tag{6.10}$$

with the energy-momentum tensor given by

$$t^{\mu\nu} = \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}}, \tag{6.11}$$

$$\hat{t}^{\mu\nu} = \frac{\delta \mathcal{L}_M}{\delta f_{\mu\nu}}. \tag{6.12}$$

Recalling that we have shown that $\hat{t}^\mu{}_\nu = -t^\mu{}_\nu$ earlier with $t^\mu{}_\nu = t^{\mu\alpha} g_{\alpha\nu}$ and $\hat{t}^\mu{}_\nu = \hat{t}^{\mu\alpha} f_{\alpha\nu}$. Alternatively, the metric equation can also be written as

$$\epsilon' R_{\mu\nu} = \left(\frac{\lambda'}{4}\mathcal{L}_M - \frac{1}{4}H^2\right) g_{\mu\nu} + \kappa^2 S_\mu S_\nu + H_{\mu\alpha} H_\nu{}^\alpha - \frac{\lambda'}{2} t_{\mu\nu} + \frac{\lambda'}{4} t g_{\mu\nu} \tag{6.13}$$

by eliminating the trace R from the metric equation (6.9).

Note that the metric equations are not completely independent. They are known to be related by the Bianchi identities $D_\mu G^\mu{}_\nu = 0$ and $\hat{D}_\mu \hat{G}^\mu{}_\nu = 0$. The resulting constraint equations known as the energy-momentum conservation laws are in fact the direct result of the field equations. We have two independent metric fields here in the bimetric theory. Therefore, it will be interesting to study the relations between these two seemly independent conservation laws. The result should be intuitive and they should both lead to the same constraint equations reflecting the conservation properties of the on-shell solutions. For heuristic reasons, we will provide a simple proof for the consistent conservation laws. This is also an alternative way to make sure no error is committed during the derivation of the field equations.

Indeed, the Bianchi identity $D_\mu G^\mu{}_\nu = 0$ implies the energy-momentum conservation law of the following form:

$$\frac{\lambda'}{2}\left(D_\mu t^\mu{}_\nu + \frac{1}{2}\partial_\nu \mathcal{L}_M\right) = -\frac{\sqrt{f}}{\sqrt{g}}\kappa_1^2 (\hat{D}_\mu \hat{S}^\mu S_\nu + \hat{S}^\mu H_{\mu\nu}). \tag{6.14}$$

On the other hand, the Bianchi identity $\hat{D}_\mu \hat{G}^\mu{}_\nu = 0$ leads the energy-momentum conservation law of the following form:

$$\frac{\lambda'}{2}\hat{D}_\mu\left(\frac{\sqrt{g}}{\sqrt{f}}\hat{t}^\mu{}_\nu\right) = \kappa_1^2 (\hat{D}_\mu \hat{S}^\mu S_\nu + \hat{S}^\mu H_{\mu\nu}), \tag{6.15}$$

with the fact that $\hat{H}_{\mu\nu} = H_{\mu\nu}$. Therefore, the Bianchi identities $D_\mu G^\mu{}_\nu = 0$ and $\hat{D}_\mu \hat{G}^\mu{}_\nu = 0$ lead to the existence of a seemingly new constraint equation of the following form:

$$\left(\frac{\sqrt{g}}{\sqrt{f}}\right)\left(D_\mu t^\mu{}_\nu + \frac{1}{2}\partial_\nu \mathcal{L}_M\right) = -\hat{D}_\mu\left(\frac{\sqrt{g}}{\sqrt{f}}\hat{t}^\mu{}_\nu\right). \tag{6.16}$$

This constraint equation is in fact the direct result of the field equations that has to do with the structure of the $g^{-1}f$ interacting pattern in the massive Lagrangian $\mathcal{L}_M$.

Indeed, we can show, from the definition $f = gM^2$ that

$$\hat{D}_\mu\left(\frac{\sqrt{g}}{\sqrt{f}} t^\mu{}_\nu\right) = \left(\frac{\sqrt{g}}{\sqrt{f}}\right)(D_\mu t^\mu{}_\nu + (\Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu}) t^\mu{}_\alpha). \tag{6.17}$$

We can also chain-differentiate the massive Lagrangian $\mathcal{L}_M$ to show that

$$\begin{aligned} \partial_\nu \mathcal{L}_M &= \frac{\delta \mathcal{L}_M}{\delta K^\alpha{}_\beta}\partial_\nu K^\alpha{}_\beta = t^{\alpha\beta}\partial_\nu g_{\alpha\beta} + \hat{t}^{\alpha\beta}\partial_\nu f_{\alpha\beta} \\ &= 2(\Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu}) t^\mu{}_\alpha. \end{aligned} \tag{6.18}$$

As a result, we reach the following identity:

$$\hat{D}_\mu\left(\frac{\sqrt{g}}{\sqrt{f}} t^\mu{}_\nu\right) = \left(\frac{\sqrt{g}}{\sqrt{f}}\right)\left(D_\mu t^\mu{}_\nu + \frac{1}{2}\partial_\nu \mathcal{L}_M\right). \tag{6.19}$$

This completes the proof that two seemingly independent Bianchi identities lead to the same conservation law. This result also indicates that the massive interaction term $\mathcal{L}_M$ couples the metrics g and f in a consistent way without offering any other extra constraint to the field equations. The result can also be put in a short statement: "$D_\mu T^\mu{}_\nu = 0$ if and only if $\hat{D}_\mu \hat{T}^\mu{}_\nu = 0$." Here $T^\mu{}_\nu$ and $\hat{T}^\mu{}_\nu$ are the generalized energy-momentum tensors derived with respect to the variation of the metric $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. This property also has to do with the fact that there are at most four different eigenvalues of $\mathcal{K}$, related by f and g in the massive interaction Lagrangian. And the result remains true with or without the presence of the S_μ field.

Note that in the absence of the dynamical term $\hat{R}$, only the diagonal part of the matrix M, for example in the Jordan coordinate, is relevant to the physics of the massive gravity theory. In the presence of the dynamical term $\hat{R}$, some off-diagonal terms of M will also play a role in the bigravity theory. But the rest of the f metric in fact acts as a free field. Therefore, they will not affect any physics of the system. These terms can be thought of as irrelevant to the interaction terms that are present in $\mathcal{L}_M$. Therefore, for simplicity, we can assume that $g_{\mu\nu}$ is a general metric, and only parts of the fiducial metric $f_{\mu\nu}$ contributing to the diagonal part of M is relevant to our theory.

An observation also quickly finds it difficult to extract the physically relevant part of the f metric. This is because the matrix S to bring $\hat{M} = SMS^{-1}$ into a Jordan matrix cannot, in general, diagonalize f and g matrices at the same time unless they are all commuting with each other. Even worse, the new matrix $\hat{g} = SgS^{-1}$ and $\hat{f} = SfS^{-1}$ cannot remain symmetric unless $S^t = S^{-1}$.

Therefore, for simplicity, we will assume that $\hat{M}$ is a diagonal matrix and M commutes with f and g. We can, however, show that the only possibility is that $\hat{f}$ and $\hat{g}$ are both diagonal matrices. Also for convenience, we will be working on the Jordan coordinate and remove the hatted notation from now on. Therefore, the g, f, and $P = M^2$ will be assumed to be $g_{\mu\nu} = \mathrm{diag}(g_0^2, g_1^2, g_2^2, g_3^2)$; $f_{\mu\nu} = \mathrm{diag}(f_0^2, f_1^2, f_2^2, f_3^2)$; $P^\mu{}_\nu = \mathrm{diag}(P_0, P_1, P_2, P_3)$, respectively. As a result, $P_\mu = f_\mu/g_\mu$ from the definition $P \equiv M^2 = g^{-1}f$.

With the t matrix taking the following form

$$t^\mu{}_\nu = t_\nu \delta^\mu{}_\nu, \tag{6.20}$$

the conservation equation can be reduced to

$$\partial_\nu \mathcal{L}_M = 2(\Gamma^\mu_{\nu\mu} - \hat{\Gamma}^\mu_{\nu\mu})t_\mu = -\sum_\mu (\partial_\nu \ln P_\mu) t_\mu \tag{6.21}$$

for diagonal metric f and g. It is apparent that $\mathcal{L}_M$ is an effective cosmological *constant* if and only if $\sum_\mu (\partial_\nu \ln P_\mu) t_\mu = 0$. For example, in two different cases with (a) $t_\mu = 0 \forall \mu$ or (b) $P_\mu = \text{constant}\ \forall \mu$, both lead to the same conclusion that $\mathcal{L}_M = \text{constant}$. We are unable to find a systematic way to solve the bimetric equations. The only exception is the solutions under the condition $\hat{t}^\mu{}_\nu = 0$. Therefore, for the remainder of this paper, we will focus on this special set of solutions to the system that is more or less equivalent to the model without a dynamical term $\hat{R}$.

VII. CONSERVATION LAWS OF THE WEYL-INVARIANT MASSIVE GRAVITY

It is clear that the dynamics of the f metric in the bigravity theory is generally very complicated. Therefore, we will focus on the local scale-invariant theory in the limit $\epsilon_1 \to 0$ for the moment. To be more specific, we will focus on the theory with $\hat{R} = 0$. Recall that the action of this model is

$$S = \int d^4x \sqrt{g} \left(\frac{\epsilon}{2} \phi^2 \tilde{R} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} H^2 - \frac{\lambda}{4} \phi^4 \mathcal{L}_M \right). \tag{7.1}$$

We can derive the variational equation of S_μ as

$$2\nabla_\nu H^{\nu\mu} + (1 + 6\epsilon) \nabla^\mu \phi^2 = 0. \tag{7.2}$$

Hence a further covariant derivative ∇_μ to Eq. (7.2) leads to the constraint equation

$$\nabla^2 \phi^2 = 0. \tag{7.3}$$

This equation imposes a strong constraint on S_μ and ϕ. Moreover, we can also show that the metric equations take the following form:

$$\mathcal{L} g_{\mu\nu} - \epsilon \phi^2 \tilde{R}_{\mu\nu} + \frac{\epsilon}{2} (\nabla_\mu \nabla_\nu \phi^2 + \nabla_\nu \nabla_\mu \phi^2) + \nabla_\mu \phi \nabla_\nu \phi + H_{\mu\alpha} H_\nu{}^\alpha = 0 \tag{7.4}$$

with the result shown in Eq. (7.3) included. Note that we have also taken into account the fact that $t_{\mu\nu} = 0$, which is a direct result of the Stückelberg field equation $\hat{t}_{\mu\nu} = 0$. In addition, the variational equations of ϕ can be shown to be

$$\epsilon \phi^2 \tilde{R} = \lambda \phi^4 \mathcal{L}_M - \phi \nabla^2 \phi. \tag{7.5}$$

Note that the constraint $\nabla^2 \phi^2 = 0$ has also been included in deriving Eq. (7.5). Eliminating the trace of Eq. (7.4), or equivalently Eq. (7.5), we obtain the following equivalent form of the metric equation:

$$\tilde{R}_{\mu\nu} = \left(\frac{\lambda}{4\epsilon} \phi^2 \mathcal{L}_M - \frac{H^2}{4\epsilon \phi^2} \right) g_{\mu\nu} + \frac{1}{2\phi^2} (\nabla_\mu \nabla_\nu \phi^2 + \nabla_\nu \nabla_\mu \phi^2) + \frac{1}{4\epsilon \phi^4} \nabla_\mu \phi^2 \nabla_\nu \phi^2 + \frac{1}{\epsilon \phi^2} H_{\mu\alpha} H_\nu{}^\alpha. \tag{7.6}$$

Note that the metric equation (7.4) can be written as

$$\left(\frac{1}{2} \tilde{R} g^{\mu\nu} - \tilde{R}^{\mu\nu} - H^{\mu\nu} \right) + \mathcal{L}_m g^{\mu\nu} + \frac{1}{\phi^2} \nabla^\nu \nabla^\mu \phi^2 + \frac{1}{4\epsilon \phi^4} \nabla^\mu \phi^2 \nabla^\nu \phi^2 + \frac{1}{\epsilon \phi^2} H^{\mu\alpha} H^\nu{}_\alpha = 0 \tag{7.7}$$

with the help of the identity

$$[\nabla_\mu, \nabla_\nu] \phi^2 = -2\phi^2 H_{\mu\nu} \tag{7.8}$$

and the new definition of $\mathcal{L}_m$

$$\mathcal{L}_m = -\frac{1}{8\epsilon \phi^4} \nabla_\mu \phi^2 \nabla^\mu \phi^2 - \frac{1}{4\epsilon \phi^2} H^2 - \frac{\lambda}{4\epsilon} \phi^2 \mathcal{L}_M. \tag{7.9}$$

Recall that the equation $\hat{t}^\mu{}_\nu = 0$ implies that $t^\mu{}_\nu = 0$. As a result, the energy-momentum conservation law implies that $\mathcal{L}_M$ acts as an effective cosmological constant in the absence of the Weyl-invariant terms.

We can also show that the energy-momentum conservation law also enforces $\mathcal{L}_M$ to act as an effective cosmological constant in the Weyl-invariant model. The reason is quite obvious by observing the structure of the effective potential term $-\lambda \phi^4 \mathcal{L}_M/4$. Indeed, we can show that the conservation law holds if $\mathcal{L}_M = \text{constant}$ in the Weyl-invariant model with or without the massive gravity terms.

The proof is quite straightforward. For heuristic reasons, we will provide a brief but detailed proof of the conservation law. First of all, we can show that

$$\nabla_\mu \left(\frac{1}{2} \tilde{R} g^{\mu\nu} - \tilde{R}^{\mu\nu} - H^{\mu\nu} \right) = 0. \tag{7.10}$$

Hence the energy-momentum conservation law becomes

$$\nabla_\mu\left(\mathcal{L}_m g^{\mu\nu}+\frac{1}{\phi^2}\nabla^\nu\nabla^\mu\phi^2+\frac{1}{4\epsilon\phi^4}\nabla^\mu\phi^2\nabla^\nu\phi^2+\frac{1}{\epsilon\phi^2}H^{\mu\alpha}H^\nu{}_\alpha\right)$$
$$-\frac{\lambda}{4\epsilon}\phi^2\partial^\nu\mathcal{L}_M=0. \tag{7.11}$$

Furthermore, we can use the Jacobi identity

$$\nabla_\mu H_{\nu\alpha}+\nabla_\alpha H_{\mu\nu}+\nabla_\nu H_{\alpha\mu}=0 \tag{7.12}$$

to show that

$$\nabla_\mu\left(\frac{1}{\epsilon\phi^2}H^{\mu\alpha}H^\nu{}_\alpha-\frac{1}{4\epsilon\phi^2}H^2g^{\mu\nu}\right)$$
$$=\frac{1}{4\epsilon\phi^4}H^2\nabla^\nu\phi^2-\frac{1}{\epsilon\phi^4}\nabla_\mu\phi^2H^{\mu\alpha}H^\nu{}_\alpha+\frac{1}{\epsilon\phi^2}\nabla_\mu H^{\mu\alpha}H^\nu{}_\alpha. \tag{7.13}$$

With the help of the following identity

$$\nabla_\mu\nabla^\nu\nabla^\mu\phi^2=(\tilde{R}^{\mu\nu}-2H^{\mu\nu})\nabla_\mu\phi^2, \tag{7.14}$$

we can show that the energy conservation law becomes

$$\partial^\nu\mathcal{L}_M=0. \tag{7.15}$$

Note that, in deriving these equations, we have also used an identity and a definition of the equation

$$\nabla_\mu\nabla_\nu\phi^2=(D_\mu-3S_\mu)\nabla_\nu\phi^2-S_\nu\nabla_\mu\phi^2+g_{\mu\nu}S_\alpha\nabla^\alpha\phi^2, \tag{7.16}$$

$$\nabla_\mu\tilde{R}^{\mu\nu}=(D_\mu+2S_\mu)\tilde{R}^{\mu\nu}-S^\nu\tilde{R}. \tag{7.17}$$

Hence this result is consistent with the Stückelberg field equation $\hat{t}_{\mu\nu}=0$. This shows that the Weyl-invariant extension of massive gravity is a self-consistent model. To be more specific, we have shown that the condition $t_{\mu\nu}=0$ and $\mathcal{L}_M=$ constant is also compatible in the Weyl-invariant model. The Weyl-invariant massive gravity theory will then be identical to the conventional Weyl-invariant theory given by

$$S=\int d^4x\sqrt{g}\left\{\frac{\epsilon}{2}\phi^2\tilde{R}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{1}{4}H^2-\frac{\lambda'}{4}\phi^4\right\}, \tag{7.18}$$

with $\lambda'\equiv\mathcal{L}_M$.

VIII. BIANCHI TYPE I SPACE IN THE $\epsilon_1=0$ LIMIT

We can take the unitary gauge of the Weyl symmetry, $\phi=\phi_0$, by choosing a gauge parameter $\Omega=\phi/\phi_0$. Here ϕ_0 is a constant parameter. Once a solution is found in the unitary gauge, we can obtain a whole class of a gauge equivalent set of solutions related by gauge transformations. Therefore, we will focus on the solution in the unitary gauge from now on.

The Weyl-invariant theory can be shown to be equivalent to the following effective theory:

$$S=\int d^4x\sqrt{g}\mathcal{L}_1$$
$$=\int d^4x\sqrt{g}\left\{\frac{\epsilon}{2}\phi_0^2R-\frac{1}{2}\kappa^2S_\mu S^\mu-\frac{1}{4}H^2-\frac{\lambda}{4}\phi_0^4\mathcal{L}_M\right\}, \tag{8.1}$$

with $\kappa^2\equiv(1+6\epsilon)\phi_0^2$ in the unitary gauge.

Note also that the choice of the parameter ϕ_0 in the unitary gauge $\Omega=\phi/\phi_0$ is expected to be the symmetry-breaking scale ϕ_0 shown in Sec. I, for example, by the dynamical symmetry-breaking potential introduced in Ref. [40]. As a result, the solutions for the metric field $g_{\mu\nu}$ in this section are to be interpreted as the physical metric from the point particle point of view. In particular, a different choice of $\phi=\phi_1$ only scales the metric field by a constant scale $g'_{\mu\nu}=g_{\mu\nu}(\phi_0/\phi_1)^2$. Here $\phi_1\neq\phi_0$ represents another constant scale.

It was shown earlier that the fiducial equation $\hat{t}=0$ naturally leads to the result $t=0$. This set of consistent solutions will hence imply the result $\mathcal{L}_M=$ constant. As a result, the contribution from the massive terms simply acts as an effective cosmological constant. Therefore, we only need to compute the exact value of the effective cosmological constant $\Lambda_e=-m_g^2\mathcal{L}_M/2$. The most general solutions and resulting $\mathcal{L}_M$ have been listed in Sec. V. We will hence turn our attention to the metric equation and the Weyl vector meson equation.

The metric equation is

$$\mathcal{L}_1g_{\mu\nu}-\epsilon\phi_0^2R_{\mu\nu}+\kappa^2S_\mu S_\nu+H_{\mu\alpha}H_\nu{}^\alpha=0. \tag{8.2}$$

In particular, we are interested in the role of the Weyl vector meson in anisotropically expanding universes. For a simple demonstration, we will focus on the solutions in the presence of a physical metric space such that $R^\mu{}_\nu=R^\mu{}_\mu\delta^\mu{}_\nu$. Note that μ,ν are both open indices in this expression. Bianchi type I (BI) metric spaces, with the metric given by

$$ds^2=-dt^2+\exp[2\alpha_1]dx^2+\exp[2\alpha_2]dy^2+\exp[2\alpha_3]dz^2$$
$$=-dt^2+\exp[2\alpha-4\sigma_+]dx^2$$
$$+\exp[2\alpha+2\sigma_++2\sqrt{3}\sigma_-]dy^2$$
$$+\exp[2\alpha+2\sigma_+-2\sqrt{3}\sigma_-]dz^2, \tag{8.3}$$

is a simple example. Here $\alpha(t),\alpha_i(t),\sigma_\pm(t)$ are functions of time only. In fact, it can be shown that BI space is the only Bianchi type metric space with the property $R^\mu{}_\nu=R^\mu{}_\mu\delta^\mu{}_\nu$.

Another famous example is the spherically symmetric metric spaces with the metric given by

$$ds^2 = -\exp[2A(r)]dt^2 + \exp[2B(r)]dr^2 + r^2 d\Omega. \tag{8.4}$$

For all metric spaces that $R^\mu{}_\nu = R^\mu{}_\mu \delta^\mu{}_\nu$, the metric equation becomes

$$\kappa^2 S_\mu S_\nu + H_{\mu\alpha} H_\nu{}^\alpha = 0, \quad \forall\ \mu \neq \nu. \tag{8.5}$$

In addition, the S_μ equation is

$$D_\mu H^{\mu\nu} = \kappa^2 S^\nu. \tag{8.6}$$

The $\nu = 0$ component equation of Eq. (8.6) gives the constraint

$$(\partial_1 + \Gamma_1) H^{10} = \kappa^2 S^0 \tag{8.7}$$

for all metric spaces with $R^\mu{}_\nu = R^\mu{}_\mu \delta^\mu{}_\nu$. This equation implies immediately that

$$S_0 = 0. \tag{8.8}$$

Therefore, we are left with three-component equations

$$D_\mu H^{\mu i} = \kappa^2 S^i. \tag{8.9}$$

This equation is equivalent to

$$\partial_\mu(\sqrt{g} H^{\mu i}) = \sqrt{g}\kappa^2 S^i. \tag{8.10}$$

The result is hence

$$\partial_t(\sqrt{g} H^{0i}) = \sqrt{g}\kappa^2 S^i. \tag{8.11}$$

Note that a further derivative to Eq. (8.6) leads to the constraint equation $D_\mu S^\mu = D_i S^i = 0$. Consequently, we have

$$\partial_i(\sqrt{g} S^i) = 0. \tag{8.12}$$

For BI metric space, Eq. (8.12) is automatically satisfied if $S_\mu = S_\mu(t)$ is a function of time only.

A. Bianchi type I physical metric space

The homogeneity, isotropy, and flatness of the observed Universe have led to the proposal of the inflationary scenarios [43]. Most models acquire a positive cosmological constant to induce a fast expansion of the cosmic scale factor. To be more specific, the field equations of a gravitational system with a cosmological constant Λ can be represented as

$$G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{8.13}$$

Here $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor.

Gibbons and Hawking [44], Hawking and Moss [45] had conjectured that all models with a positive cosmological constant will approach a late time de Sitter space. This conjecture has been known as the cosmic no-hair theorem for Einstein gravity. One of the important progresses was an analytic proof to support this conjecture given by Wald in Ref. [46]. It was shown that any model with a positive cosmological constant will drive, at least locally, the late-time evolution towards the de Sitter spacetime. It was shown that this result remains valid for all nontype-IX Bianchi spaces if the matter sources obey (i) the dominant energy condition

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \tag{8.14}$$

and (ii) the strong-energy condition

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) t^\mu t^\nu \geq 0 \tag{8.15}$$

for any timelike vector t^μ [46]. Note that T denotes the trace of the energy-momentum tensor of all possible fields coupled to the gravitational system. The proof also holds for the type IX Bianchi space if Λ is sufficiently large [46].

Many literatures favored the existence of some constraints on the field parameters for its occurrence [46–55]. It is also known, however, that counterexamples do exist where these energy conditions do not hold exactly [56–58]. Some of these solutions had later been shown to be unstable [51,59–61]. It is therefore important to pay more attention to the effect of this conjecture. It was shown that the Weyl-invariant massive gravity with a general reference metric can be shown to act as a Weyl-invariant theory with a cosmological constant. The resulting theory also acts as a gravitational theory with a massive Weyl vector meson in the unitary gauge. Therefore, it will be interesting to study whether the no-hair conjecture holds for this model. In particular, we will focus on the effect of the Weyl vector meson in the presence of the BI metric space in this subsection.

Note that all nonvanishing components of the Riemann curvature tensor for BI metric space can be written as

$$R^{ti}{}_{ti} = \dot{H}_i + H_i^2, \tag{8.16}$$

$$R^{ij}{}_{ij} = H_i H_j, \quad \forall\ i \neq j, \tag{8.17}$$

with $H_i \equiv \dot{\alpha}_i$. Therefore all nonvanishing components of the Ricci tensor $R^\mu{}_\nu$ and scalar curvature R can be shown as

$$R^t{}_t = 3\dot{H} + \sum_i H_i^2, \tag{8.18}$$

$$R^i{}_i = \dot{H}_i + 3HH_i, \tag{8.19}$$

$$R = 6\dot{H} + 9H^2 + \sum_i H_i^2, \tag{8.20}$$

respectively, with $3H \equiv H_1 + H_2 + H_3$. Finally, the Einstein tensor defined by $G^\mu{}_\nu = Rg^\mu{}_\nu/2 - R^\mu{}_\nu$ becomes

$$G^t{}_t = H_1H_2 + H_2H_3 + H_3H_1, \tag{8.21}$$

$$G^1{}_1 = \dot{H}_2 + \dot{H}_3 + H_2H_3 + H_2^2 + H_3^2, \tag{8.22}$$

$$G^2{}_2 = \dot{H}_1 + \dot{H}_3 + H_1H_3 + H_1^2 + H_3^2, \tag{8.23}$$

$$G^3{}_3 = \dot{H}_2 + \dot{H}_1 + H_2H_1 + H_1^2 + H_2^2. \tag{8.24}$$

It is apparent that off-diagonal components of $R^\mu{}_\nu$ vanish for the BI metric space. Therefore, we have the constraint that

$$(\partial_t + 3\dot{\alpha})H^{0i} = \kappa^2 S^i, \quad \text{and} \tag{8.25}$$

$$\kappa^2 S_\mu S_\nu + H_{\mu\alpha}H_\nu{}^\alpha = 0, \quad \forall\ \mu \neq \nu. \tag{8.26}$$

Assuming that $S_\mu = S_\mu(t)$ only, we can show immediately that $H_{ij} = 0$. Hence the $(\mu,\nu) = (0,i)$ component of Eq. (8.26) becomes

$$S_0 S_i = 0. \tag{8.27}$$

Since we already know that $S_0 = 0$ holds for the BI metric space, Eq. (8.26) is satisfied automatically. Note that the $(\mu,\nu) = (i,j)$ component of Eq. (8.26) implies

$$\kappa^2 S_i S_j = \dot{S}_i \dot{S}_j \tag{8.28}$$

for all $i \neq j$. Therefore we obtain the result

$$S_i = (k_1, k_2, k_3)\exp[\pm\kappa t] \tag{8.29}$$

if $S_1S_2S_3 \neq 0$. Here k_i represents some nonzero integration constants.

On the other hand, the solutions will be different if some components of S_i vanish. In order to discuss all possible solutions to Eq. (8.26), we can classify the solutions into four different cases: (i) $S_1S_2S_3 \neq 0$, (ii) $S_1S_2 \neq 0$ and $S_3 = 0$, (iii) $S_1 \neq 0$ and $S_2 = S_3 = 0$, and (iv) $S_1 = S_2 = S_3 = 0$. We would like to make a remark on the choice of the nonvanishing components here. For example, we can choose S_2 as the nonvanishing component for case (iii). The result will be the same if we rename g_{22} as g_{11} and g_{11} as g_{22}. As a result, the physics will not be affected by the choice of the nonvanishing components. Therefore, all possible solutions can be classified as (i) all S_i are nonvanishing, (ii) two of the S_i are nonvanishing, (iii) one of the S_i is nonvanishing, and (iv) $S_i = 0$ for all i.

B. Solution for the case (i) $S_1S_2S_3 \neq 0$

Let us focus on the first case $S_1S_2S_3 \neq 0$ for the moment. The S_μ equation $D_\mu H^{\mu i} = (\partial_t + 3\dot{\alpha})H^{0i} = \kappa^2 S^i$ implies

$$(3\dot{\alpha} - 2\dot{\alpha}_i \pm 2\kappa)S_i = 0 \tag{8.30}$$

for all i. If $S_1S_2S_3 \neq 0$, this equation implies that $2\dot{\alpha}_i = 3\dot{\alpha} \pm 2\kappa$ for all i. This is equivalent to the solution $2(\alpha_i - \alpha_i(0)) - 3(\alpha - \alpha(0)) = \pm 2\kappa t$. Therefore we have

$$\alpha_i = \alpha_i(0) \mp 2\kappa t. \tag{8.31}$$

This solution indicates that the condition $S_1S_2S_3 \neq 0$ will force the BI metric space to evolve isotropically. Hence this is not an interesting solution. In addition, the other components of the Einstein equation will also relate the coupling constants κ and $\mathcal{L}_M$ under a strong constraint.

Indeed, the metric equation can be written as

$$\mathcal{L}_1 g_{\mu\nu} - \epsilon' R_{\mu\nu} + \kappa^2 S_\mu S_\nu + H_{\mu\alpha}H_\nu{}^\alpha = 0 \tag{8.32}$$

with

$$\mathcal{L}_1 = \frac{\epsilon'}{2}R - \frac{1}{2}\kappa^2 S_\mu S^\mu - \frac{1}{4}H^2 - \Lambda'_e \tag{8.33}$$

and $\Lambda'_e = \Lambda_e \epsilon \phi^2 = \lambda\phi_0^4 \mathcal{L}_M/4$.

First of all, we can show that

$$\frac{1}{2}\kappa^2 S_\mu S^\mu + \frac{1}{4}H^2 = 0 \tag{8.34}$$

for S_i given by Eq. (8.29). Hence the metric equation (8.32) becomes

$$\epsilon' G_{\mu\nu} + \kappa^2 S_\mu S_\nu + H_{\mu\alpha}H_\nu{}^\alpha - \Lambda'_e g_{\mu\nu} = 0. \tag{8.35}$$

Indeed, the (0,0) component of Eq. (8.35) gives

$$3\epsilon' H^2 + \kappa^2 g^{ii}(0)k_i k_i \exp[-6\kappa t] = \Lambda'_e. \tag{8.36}$$

Therefore the only consistent solution to Eq. (8.36) is the solution with $k_1 = k_2 = k_3 = 0$. This contradicts the assumption. Hence case (i) cannot be true because of the nontrivial constraint hidden in the field equations.

C. Solution for the case (ii) $S_1S_2 \neq 0$ and $S_3 = 0$

For the case (ii) $S_1S_2 \neq 0$ and $S_3 = 0$, the S_μ equation becomes

$$\ddot{S}_i + (3\dot{\alpha} - 2\dot{\alpha}_i)\dot{S}_i + \kappa^2 S_i = 0 \tag{8.37}$$

for $i = 1, 2$. In addition, Eq. (8.26) gives

$$\dot{S}_1\dot{S}_2 = \kappa^2 S_1 S_2. \tag{8.38}$$

Therefore, we can eliminate the second derivative terms and obtain

$$a + b = -\dot{\alpha}_3, \tag{8.39}$$

$$ab = \kappa^2 \tag{8.40}$$

with $a \equiv \dot{S}_1/S_1$ and $b \equiv \dot{S}_2/S_2$. a and b can thus be solved to give

$$a, b = A_\pm \equiv \frac{1}{2}(-\dot{\alpha}_3 \pm \sqrt{\dot{\alpha}_3^2 - 4\kappa^2}). \tag{8.41}$$

Note that both solutions indicate that $S_\mu \to 0$ for all expanding solutions. This agrees with the prediction of the no-hair conjecture [44,45].

D. Solution for the case (iii) $S_1 \neq 0$ and $S_2 = S_3 = 0$

We will now focus on case (iii) with $S_1 \neq 0$ and $S_2 = S_3 = 0$. As a result, the S_1 equation gives

$$\ddot{S}_1 + (3\dot{\alpha} - 2\dot{\alpha}_1)\dot{S}_1 + \kappa^2 S_1 = 0. \tag{8.42}$$

In addition, we can show that

$$\mathcal{L}_S \equiv -\frac{1}{2}\kappa^2 S_\mu S^\mu - \frac{1}{4}H^2 = \frac{1}{2}g^{11}\dot{S}_1^2 - \frac{1}{2}g^{11}\kappa^2 S_1^2 \tag{8.43}$$

with $\mathcal{L}_S$ the S_μ-related parts of the Lagrangian. Therefore, the metric equations become

$$\epsilon' G^0{}_0 - \frac{1}{2}g^{11}\dot{S}_1^2 - \frac{1}{2}g^{11}\kappa^2 S_1^2 = \Lambda_e', \tag{8.44}$$

$$\epsilon' G^1{}_1 - \frac{1}{2}g^{11}\dot{S}_1^2 + \frac{1}{2}g^{11}\kappa^2 S_1^2 = \Lambda_e', \tag{8.45}$$

$$\epsilon' G^2{}_2 + \frac{1}{2}g^{11}\dot{S}_1^2 - \frac{1}{2}g^{11}\kappa^2 S_1^2 = \Lambda_e', \tag{8.46}$$

$$\epsilon' G^3{}_3 + \frac{1}{2}g^{11}\dot{S}_1^2 - \frac{1}{2}g^{11}\kappa^2 S_1^2 = \Lambda_e'. \tag{8.47}$$

The last two equations indicate that H_2 and H_3 obey the same equation. Hence these equations can be used to obtain the solution

$$H_3 - H_2 = h_2 \exp[-3\alpha] \tag{8.48}$$

with h_2 some integration constant. This indicates that the difference between H_2 and H_3 vanishes when $t \to \infty$ for all expanding solutions.

In addition, adding the second and third or fourth equation gives

$$3\dot{H} + \dot{H}_3 + 3HH_3 + H_1^2 + H_2^2 + H_3^2 = 2\Lambda_e, \tag{8.49}$$

$$3\dot{H} + \dot{H}_2 + 3HH_2 + H_1^2 + H_2^2 + H_3^2 = 2\Lambda_e. \tag{8.50}$$

Moreover, the first and second equation can be shown to be

$$H_1H_2 + H_2H_3 + H_3H_1 = \Lambda_e + \frac{1}{2\epsilon'}g^{11}(\dot{S}_1^2 + \kappa^2 S_1^2), \tag{8.51}$$

$$\dot{H}_2 + \dot{H}_3 + H_2^2 + H_3^2 + H_2H_3 = \Lambda_e + \frac{1}{2\epsilon'}g^{11}(\dot{S}_1^2 - \kappa^2 S_1^2). \tag{8.52}$$

For simplicity, we will stick to the solution with $H_2 = H_3$. In this case, the field equations become

$$3\dot{H} + \dot{H}_2 + 3HH_2 + H_1^2 + 2H_2^2 = 2\Lambda_e, \tag{8.53}$$

$$2H_1H_2 + H_2^2 = \Lambda_e + \frac{1}{2\epsilon'}g^{11}(\dot{S}_1^2 + \kappa^2 S_1^2), \tag{8.54}$$

$$2\dot{H}_2 + 3H_2^2 = \Lambda_e + \frac{1}{2\epsilon'}g^{11}(\dot{S}_1^2 - \kappa^2 S_1^2). \tag{8.55}$$

In addition, we can also obtain the following equation from the combination of the Einstein tensor components $2G^1{}_1 - G^2{}_2 - G^3{}_3$

$$3(\partial_t + 3\dot{\alpha})\dot{\sigma}_+ = \frac{1}{\epsilon'}g^{11}(\dot{S}_1^2 - \kappa^2 S_1^2). \tag{8.56}$$

Moreover, we can show that

$$\partial_t(\exp[\alpha + 4\sigma_+]S_1\dot{S}_1) = \exp[\alpha + 4\sigma_+](\dot{S}_1^2 - \kappa^2 S_1^2) \tag{8.57}$$

from Eq. (8.42). Therefore we can write

$$\dot{\sigma}_+ = \frac{1}{3\epsilon'}\exp[-2\alpha + 4\sigma_+]S_1\dot{S}_1 + k_+\exp[-3\alpha] \tag{8.58}$$

with some constant of integration k_+.

By computing the combination of the $3G^t{}_t + G^i{}_i$ component equations, we can obtain the following equation:

$$6\ddot{\alpha} + 18\dot{\alpha}^2 = 6\Lambda_e + \frac{1}{\epsilon'}g^{11}(\dot{S}_1^2 + 2\kappa^2 S_1^2). \tag{8.59}$$

Note that the right-hand side of Eq. (8.59) is always positive for a model with a large cosmological constant $\Lambda_e \gg 1$. Therefore, all expanding solutions are strong expanding solutions with $\ddot{V} > 0$ if there is a large cosmological constant. Note that $V \equiv \exp[3\alpha]$. Moreover, we can also express S_1 and $\dot{S}_1$ as functions of the metric fields

$$\dot{S}_1^2 = \epsilon' g_{11}[2A_0 + 2\Sigma_0], \tag{8.60}$$

$$\kappa^2 S_1^2 = \epsilon' g_{11}[2A_0 - \Sigma_0]. \tag{8.61}$$

Here $A_0 \equiv \ddot{\alpha} + 3\dot{\alpha}^2 - \Lambda_e$ and $\Sigma_0 \equiv \ddot{\sigma}_+ + 3\dot{\alpha}\sigma_+$. As a result, we can also write $\dot{\sigma}_+$ as

$$\dot{\sigma}_+ = \frac{1}{3\kappa}[2A_0 + 2\Sigma_0]^{1/2}[2A_0 - \Sigma_0]^{1/2} + k_+ \exp[-3\alpha]. \quad (8.62)$$

All other combinations of field equations can be derived from Eqs. (8.60)–(8.62).

1. The evolution of σ_+ when $\dot{\alpha}_e \equiv \alpha + 4\sigma_+ \gg 1$

We wish to know whether $\dot{\sigma}_+$ will grow large or not as time evolves. To be more specific, we are interested in the asymptotic behavior of $g^{11} S_1 \dot{S}_1$, following Eq. (8.58), when $t \to \infty$. In fact, we will show that this factor does go to zero when $t \to \infty$ for all expanding solutions. Hence Eq. (8.58) implies that the anisotropy tends to vanish when $t \to \infty$ for all expanding solutions. This agrees with the prediction of the no-hair conjecture [44,45].

Let us focus first on the strong anisotropically expanding solutions with $\dot{\alpha}_i \gg 1$ and $\dot{\alpha} \gg |\dot{\sigma}_\pm|$. In other words, we are interested in the case where $\dot{\alpha}_e \equiv \alpha + 4\sigma_+ \gg 1$. Note that the S_1 field equation can also be written as

$$\partial_t(\dot{S}_1^2 + \kappa^2 S_1^2) = -2(\dot{\alpha} + 4\dot{\sigma}_+)\dot{S}_1^2, \quad (8.63)$$

$$\partial_t(\exp[\alpha + 4\sigma_+]\dot{S}_1) = -\kappa^2 S_1 \exp[\alpha + 4\sigma_+]. \quad (8.64)$$

Hence Eq. (8.63) implies that the combination $\dot{S}_1^2 + \kappa^2 S_1^2$ is always decreasing. As a result, the values of $|S_1|$ and $|\dot{S}_1|$ both tend to converge to small values provided that $\dot{\alpha}_e > 0$ when $t \to \infty$. Therefore, for strong expanding solutions, the anisotropy will tend to vanish when $t \to \infty$.

2. Contour method

We are, however, unable to solve the S_1 equation analytically. Nevertheless, we can show by a contour analysis that $|\dot{S}_1 S_1|$ tends to converge to a stationary point in the phase space of ($p = \dot{S}_1$, $q = \kappa S_1$). Indeed, the contour evolution of p and q described by Eq. (8.57) is very helpful to understand the convergent behavior of p and q when $t \to \infty$.

Note that Eq. (8.42) can be written as

$$\ddot{S}_1 = -[\dot{\alpha}_e \dot{S}_1 + \kappa^2 S_1]. \quad (8.65)$$

In addition to the phase diagram shown in Fig. 1(a), we can also draw a curve L ($Q \equiv \dot{\alpha}_e p + \kappa q = 0$) on the phase diagram. To be more specific, the coordinate of the point on L is $(q, p) = (-\dot{\alpha}_e/\kappa, p)$. Assuming that the contour C starts at point $A = (1, 1)$ in quadrant I, q will increase as long as $p > 0$. Let $a = (3, 1)$, not shown in Fig. 1, be the corresponding starting point of A on curve L.

The dotted curve L in Fig. 1(a) is described by the equation $\dot{\alpha}_e p + \kappa q = 0$. The solid red curves are the contour evolution of the numerical solutions to Eq. (8.65). The dotted line starts from quadrant I, then enters quadrant II and IV, and finally oscillates with a close-to-constant slope between quadrants II and IV. The constants and initial conditions are chosen as $\kappa = 1$, $\Lambda_e = 3$, $p(0) = q(0) = 1$, $\alpha(0) = 0, \dot{\alpha}(0) = 1$, $\sigma_+(0) = 0$, and $\dot{\sigma}_+(0) = -1$. In Fig. 1(b) the thick red line is the same as the red line contour shown in Fig. 1(a). Blue and thin green lines are the contours with $\dot{\sigma}_+(0) = 0$ and $\dot{\sigma}_+(0) = 1$, respectively. The rest of the chosen constants and initial

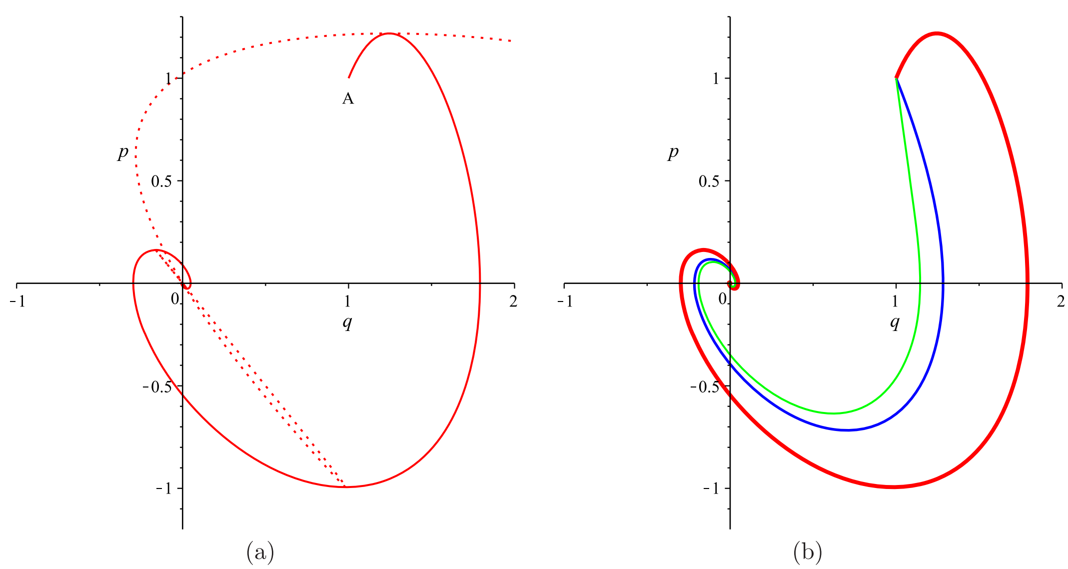

FIG. 1 (color online). (a) The dotted curve L is described by the equation $\dot{\alpha}_e p + \kappa q = 0$. The solid curves are the contour evolution of the numerical solutions to Eq. (8.65) with the constants and initial conditions chosen as $\kappa = 1$, $\Lambda_e = 3$, $p(0) = q(0) = 1$, $\alpha(0) = 0, \dot{\alpha}(0) = 1$, $\sigma_+(0) = 0$, and $\dot{\sigma}_+(0) = -1$. (b) Thick red line is the same as the red line contour shown in (a). Blue and thin green lines are the contours with $\dot{\sigma}_+(0) = 0$ and $\dot{\sigma}_+(0) = 1$, respectively. The rest of the chosen constants and initial conditions are the same as the constants and initial conditions of the red line shown in (a).

conditions are the same as the constants and initial conditions of the red line shown in Fig. 1(a).

For convenience, we will say "A is left to the curve L" if A is on the left to the straight line $\overline{Oa}$ connecting the origin and the point a. Otherwise, we will say "the point A is right to the curve L." If A is left to the curve L, $Q < 0$ and vice versa. Note that $\dot{p} = -Q > 0$; therefore $\dot{p} > 0$, if the contour point A is left to the curve L. Hence p will increase as long as the contour point is still left to the curve L. Similarly, p will start to decrease when the contour point is right to the curve L.

Consider that the contour is on the upper half-plane with $p > 0$ for the moment. As a result, points on L will act as attractor points that will attract the contour C on its left. On the other hand, the points on the curve L act as repeller points that will repel all the contour points on its right. The attracting and repelling action of the curve L will, however, interchange if the contour is on the lower half-plane with $p < 0$. The total effect as shown in Fig. 1(a) is that L and C tend to spiral with each other toward the origin.

Note that the q-direction motion is determined by the sign on p. If the contour point is in quadrant I and II where $p > 0$, the motion is heading toward the right direction. Similarly, if the contour point is in quadrant III and IV where $p < 0$, the motion is leftward.

If the contour starts at point A in the first quadrant as shown in Fig. 1(a), p will start to increase since A is left to the curve L. Once the contour touches briefly with the curve L, p will start to decrease until the contour touches again with the curve L in quadrant IV. p starts to increase again thereafter until the contour eventually touches the curve L for the third time in the second quadrant. As a result, the contour tends to spiral to the origin and drives $S_1 \to 0$ when $t \to \infty$. Note that all the touch points of C and L are the points satisfying $Q = 0$. In other words, the tangent lines at these points are flat lines with vanishing slope.

In addition, Fig. 1(a) shows that $\dot{\alpha}_e$ is negative in quadrant I and III, and positive in quadrant II and IV. Note that Eq. (8.59) shows that

$$\ddot{\alpha}_e = \Lambda_e - 3\dot{\alpha}\dot{\alpha}_e + \frac{1}{6\epsilon'}\exp[-3\alpha + \sigma_+](5p^2 - 2q^2). \quad (8.66)$$

Therefore, $\ddot{\alpha}_e > 0$ if $\Lambda_e \gg 1$ as compared to the other terms on the right-hand side of Eq. (8.66). Therefore, $\dot{\alpha}_e$ tends to remain positive unless $\dot{\alpha}_e$ is initially large enough such that the second negative term dominates.

In particular, $\dot{\alpha}_e$ tends to increase rapidly if the initial values of $\dot{\alpha}_e(o)$ are negative. This implies that the curve L will tends to bend toward quadrant II as long as $\ddot{\alpha}_e$ remains positive. Since we are interested in the model with a large cosmological constant, curve L will eventually tend to remain confined in the neighborhood of the origin contouring between quadrants II and IV.

In addition, at some large time $t \gg t_0$, the strong expansion will tend to keep $\dot{\alpha}^2 \to \Lambda_e/3$ a good approximation. Therefore, $\dot{\sigma}_+ \to 0$ when $t \to \infty$. As a result, $\dot{\alpha}_e \to \dot{\alpha}$ when $t \to \infty$. This feature is shown clearly in Fig. 1(a) near the end of the contour evolution when $t \to \infty$. This is also the reason why L tends to settle in quadrants II and IV with a close-to-constant slope when $t \to \infty$.

3. Alternative approach

There is another way to look at the evolution of the S_1 equation. Writing $S_1 = k_s \exp[s]$ with $k_s = \pm 1$, we can show that the S_1 equation can be written as

$$\frac{\dot{\mathcal{C}}}{\mathcal{C}} = \dot{s} - \kappa^2 \frac{1}{\dot{s}} \quad (8.67)$$

with $\mathcal{C} \equiv S_1 \dot{S}_1 \exp[\alpha + 4\sigma_+]$. This equation implies that $\mathcal{C} \to 0$ if $\dot{s} \to 0^+$ when $t \to \infty$ as shown in Fig. 1(a) with the curve L contouring with close-to-constant slope.

In addition, the S_1 equation can also be written as

$$\ddot{s} + \dot{s}^2 + \kappa^2 + \dot{\alpha}\,\dot{s} + \frac{4}{3\epsilon'}\exp[-3\alpha](\mathcal{C} + 3k_+)\dot{s} = 0, \quad (8.68)$$

with the solution of σ_+ solved in Eq. (8.58) included. Note again that Eq. (8.68) is independent of the $\pm$ sign of k_s associated with the definition $S_1 = k_s \exp$. This then implies, assuming $\dot{s} \neq 0$,

$$\frac{\ddot{s}}{\dot{s}} + 2\kappa \leq \frac{\ddot{s}}{\dot{s}} + \dot{s} + \kappa^2 \frac{1}{\dot{s}} + \dot{\alpha} + \frac{4}{3\epsilon'}\exp[-3\alpha]\mathcal{C} \to 0 \quad (8.69)$$

when $t \to \infty$ for all expanding solutions as long as $\dot{s} > 0$ (and hence $\mathcal{C} \geq 0$). Therefore, we find that

$$\frac{\ddot{s}}{\dot{s}} \leq -2\kappa \quad (8.70)$$

if $\dot{s} > 0$ when $t \to \infty$. This result is derived from the fact that $\dot{s} + \kappa^2/\dot{s} \geq 2\kappa$ for all positive $\dot{s}$. Hence we reach the conclusion that $\dot{s} \to 0^+$ and $\mathcal{C} \to 0^+$ when $t \to \infty$ for all expanding solutions if $\dot{s} > 0$. Therefore, Eq. (8.62) implies that $\dot{\sigma}_+ \to 0$ when $t \to \infty$.

On the other hand, $\dot{s} < 0$ implies that $S_1 \dot{S}_1 < 0$. This implies immediately that $|S_1|$ is monotonically decreasing. Once this happens, $\dot{S}_1$ will also tend to zero when $t \to \infty$. As a result, $\dot{s} \to 0^-$ when $t \to \infty$ if $\dot{s} < 0$. Therefore, Eq. (8.62) also implies that $\dot{\sigma}_+ \to 0$ when $t \to \infty$. The conclusion is that whether $\dot{s}$ is positive or not when $t \to \infty$, they will both tend to oscillate along the $\dot{s} = 0$ line and tend to force $\dot{\sigma}_+ \to 0$ when $t \to \infty$. As a result, the contribution of the Weyl vector boson appears to be negligible in the cosmological evolution of the BI expanding Universe.

4. Isotropic limit

It was just shown that the presence of the S_μ field tends to drive the evolution of the Universe toward an isotropic limit. It can be shown, however, that there is no isotropic solution unless $S_1 = 0$.

Indeed, Eq. (8.56) implies that $\dot{S}_1 = \pm\kappa S_1$. Together with Eq. (8.65), the solution is

$$\dot{\alpha} = \mp 2\kappa. \tag{8.71}$$

This solution does not agree with Eq. (8.61) unless $S_1 = 0$. Therefore the isotropic limit of the field equation does not exist unless S_1 is turned off identically.

E. Solution for the case (iv) $S_1 = S_2 = S_3 = 0$

In this section, we will review the result for an effective theory with $S_\mu = 0$ in BI metric space. Consequently, the effective theory obeys the effective Einstein equation (2.12) of the following form:

$$\frac{1}{2}Rg_{\mu\nu} - R_{\mu\nu} = \Lambda_e g_{\mu\nu}. \tag{8.72}$$

Here the effective cosmological constant is given by $\Lambda_e = -m_g^2\mathcal{L}_M/2 = \frac{\lambda}{4\epsilon}\phi_0^2\mathcal{L}_M$. This equation can also be written as

$$R_{\mu\nu} = \Lambda_e g_{\mu\nu}. \tag{8.73}$$

The analytic solutions can be solved by a standard method [62]. We will present a brief version of this procedure for heuristic reasons.

As a result, the metric field equations can be shown to be $A_0 = \Sigma_0 = 0$ by setting $S_1 = 0$ in Eqs. (8.60)–(8.61). Hence the equations of motion are

$$\ddot{\alpha} + 3\dot{\alpha}^2 = \Lambda_e, \tag{8.74}$$

$$(\partial_t + 3\dot{\alpha})\dot{\sigma}_+ = 0, \tag{8.75}$$

and Eq. (8.75) can be solved to give

$$\dot{\sigma}_+ = k_+ \exp[-3\alpha] \tag{8.76}$$

with k_+ some integration constant. In addition, Eq. (8.74) can be solved by defining the volume factor $V = a_1a_2a_3 = \exp[3\alpha]$. Indeed, we can write Eq. (8.74) as

$$\ddot{V} = 3\Lambda_e V. \tag{8.77}$$

This is an equation linear in V. Therefore, it can be solved to give

$$V = a\exp[\sqrt{3\Lambda_e}t] + b\exp[-\sqrt{3\Lambda_e}t] \tag{8.78}$$

as a linear combination of the exponential solutions $\exp[\pm\sqrt{3\Lambda_e t}]$. Here a and b are constant coefficients to be determined by the boundary conditions. Therefore $\exp[3\alpha]$ becomes

$$V = \exp[3\alpha] = \exp[3\alpha_0]\left[\cosh\sqrt{3\Lambda_e}t + \frac{\dot{\alpha}_0}{\sqrt{\Lambda_e/3}}\sinh\sqrt{3\Lambda_e t}\right] \tag{8.79}$$

with $\alpha_0 = \alpha(t=0)$ and $\dot{\alpha}_0 = \dot{\alpha}(t=0)$ as appropriate initial values.

Moreover, we can also show that $\dot{\alpha}^2$ can be added with $-\Lambda_e/3$ as

$$\dot{\alpha}^2 - \frac{\Lambda_e}{3} = \left(\dot{\alpha}_0^2 - \frac{\Lambda_e}{3}\right)\exp[6\alpha_0]\exp[-6\alpha]. \tag{8.80}$$

As a result, the Friedmann equation implies

$$G^0{}_0 = 3(\dot{\alpha}^2 - \dot{\sigma}_+^2 - \dot{\sigma}_-^2) = \Lambda_e. \tag{8.81}$$

We can further show that the solution

$$\dot{\sigma}_- = \pm\left[\dot{\alpha}_0^2 - \frac{\Lambda_e}{3} - k_+^2\exp[-6\alpha_0]\right]^{1/2} \times\left[\cosh\sqrt{3\Lambda_e}t + \frac{3\dot{\alpha}_0}{\sqrt{3\Lambda_e}}\sinh\sqrt{3\Lambda_e}t\right]^{-1} \tag{8.82}$$

exists only when

$$\dot{\alpha}_0^2 \geq \frac{\Lambda_e}{3} + k_+^2\exp[-6\alpha_0]. \tag{8.83}$$

In summary, we have found a set of analytic solutions of the following form:

$$\dot{\sigma}_+ = k_+\exp[-3\alpha], \tag{8.84}$$

$$\dot{\sigma}_- = k_-\exp[-3\alpha], \tag{8.85}$$

$$\dot{\alpha}^2 - \frac{\Lambda_e}{3} = k_\alpha^2\exp[-6\alpha] \tag{8.86}$$

with

$$k_- = \pm\left[\dot{\alpha}_0^2 - \frac{\Lambda_e}{3} - k_+^2\exp[-6\alpha_0]\right]^{1/2}\exp[3\alpha_0], \tag{8.87}$$

$$k_\alpha^2 = \left(\dot{\alpha}_0^2 - \frac{\Lambda_e}{3}\right)\exp[6\alpha_0]. \tag{8.88}$$

Therefore the constraint (8.83) becomes

$$\dot{\alpha}_0^2 = \frac{\Lambda_e}{3} + (k_+^2 + k_-^2)\exp[-6\alpha_0] \tag{8.89}$$

with $k_\alpha^2 = k_+^2 + k_-^2$. In addition, Eq. (8.84) can be integrated directly to give the result

$$\sigma_+ = \sigma_1 + \left[\frac{k_+ \exp[-3\alpha_0]}{3\sqrt{\dot{\alpha}_0^2 - \Lambda_e/3}}\right]\left\{\left[\ln\frac{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}}\exp[\sqrt{3\Lambda_e}t] - \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}} - \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}\right] + \ln\left[\frac{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}} + \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}}\exp[\sqrt{3\Lambda_e}t] + \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}\right]\right\} \tag{8.90}$$

with $\sigma_1 = \sigma_+(0)$. Similarly, σ_- can be written as

$$\sigma_- = \sigma_2 + \left[\frac{k_- \exp[-3\alpha_0]}{3\sqrt{\dot{\alpha}_0^2 - \Lambda_e/3}}\right]\left\{\left[\ln\frac{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}}\exp[\sqrt{3\Lambda_e}t] - \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}} - \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}\right] + \ln\left[\frac{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}} + \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}{\sqrt{\dot{\alpha}_0 + (\Lambda_e/3)^{1/2}}\exp[\sqrt{3\Lambda_e}t] + \sqrt{\dot{\alpha}_0 - (\Lambda_e/3)^{1/2}}}\right]\right\} \tag{8.91}$$

with $\sigma_2 = \sigma_-(0)$. Note that both $\sigma_\pm$ approach constants when $t \to \infty$. Therefore anisotropy will decrease as the Universe expands. In particular, we can also show explicitly that the expanding solutions are stable under perturbations of the following form: $(\delta\alpha, \delta\sigma_+, \delta\sigma_-) = (k_1, k_2, k_3)\exp[\nu t]$ [62]. Note also that the isotropic solution can be obtained by setting $k_\pm = 0$.

IX. CONCLUSIONS

The solution of this model in the Schwarzschild metric space has been shown explicitly in Ref. [63]. Similar result shows that the S_μ field also tends to vanish in the presence of a black hole solution. Evidence shown in this paper indicates that the massive gauge field does not seem to provide much physical impact in BI metric space.

In summary, we have studied the cosmological implications of a Weyl-invariant generalization of the dRGT theory. In particular, we showed that the massive terms serve as an effective cosmological constant for all fiducial metric spaces and physical metric spaces. We also show that the Weyl vector meson decouples effectively from the Weyl-invariant model in the BI metric space when $t \to \infty$. This is done by showing that $S_\mu \to 0$ in the unitary gauge $\phi = \phi_0$. Specific solutions are solved as examples for this model in the BI metric space. The fiducial metric is treated as an auxiliary field contributing only through the trace components of $\mathcal{K} = \delta - M$ with $M^2 = g^{-1}f$. In addition, we also present some general conservation properties of the bimetric theory. A specific example is also given by assuming that $\mathcal{K}$, g, and f commute with each other and can be diagonalized by the same similarity transformation matrix S.

Hopefully, the result shown in this paper may shed light on the research on the generalization and applications of the massive gravity theories. In addition, the results shown in this paper also indicate that the structure and properties associated with the dRGT theory deserve more attention.

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APPENDIX: THE RECURRENCE RELATION OF THE MASSIVE LAGRANGIAN

We will briefly review the algebra in the determinant of $M = \delta - \mathcal{K}$ in this section. The determinant of an $n \times n$ matrix M_{ab} is defined as

$$|M| \equiv \det M = \frac{1}{n!} e^{a_1,a_2,\cdots,a_n} e^{b_1,b_2,\cdots,b_n} M_{a_1b_1} M_{a_2b_2} \cdots M_{a_nb_n} \tag{A1}$$

with the help of the flat totally skew-symmetric Levi-Civita tensor $e^{a_1,a_2,\cdots,a_n}$. In addition, the inverse matrix M^{-1} can be shown to be

$$M^{ab} = \frac{\tilde{M}_{ba}}{|M|} \tag{A2}$$

with $\tilde{M}_{ba}$ the cofactor of M_{ab} defined by

$$\tilde{M}_{ab} = \frac{s(ab)}{(n-1)!} e^{a,a_2,\cdots,a_n} e^{b,b_2,\cdots,b_n} \langle M_{ab} \rangle M_{a_2 b_2} M_{a_3 b_3} \cdots M_{a_n b_n}. \tag{A3}$$

Here $s(ab)$ stands for the sign derived from the permutation of a and b with respect to the indices a_i and b_i from their original ordered position. To be more specific, the sign occurs when the ordered series is permuted from $(a_2, a_3, \cdots, a, \cdots, a_n; b_2, b_3, \cdots, b, \cdots, b_n)$ to $(a, a_2, a_3, \cdots, a_n; b, b_2, b_3, \cdots, b_n)$. In addition, $\langle M_{ab} \rangle$ denotes the omission of the matrix element M_{ab} from the original definition of the cofactor $\tilde{M}_{ab}$ in Eq. (A3). As a result, we can prove that $M^{ab} M_{bc} = \delta^a{}_c$ with the inverse matrix given above.

In the paper, we will set $n = 4$ for the four-dimensional space. As a result,

$$|M| = \sum_{i=0}^{4} (-1)^i \mathcal{L}_i \tag{A4}$$

with $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = [\mathcal{K}]$ and with $\mathcal{L}_i$ defined by Eqs. (2.9)–(2.11) for $i = 2, 3, 4$. Therefore, the massive Lagrangian is equivalent to the polynomial components of $2|M|$. Moreover, the variation of $|M|$ with respect to $\mathcal{K}^a{}_b$, can be shown to be

$$\frac{\delta |M|}{\delta \mathcal{K}^a{}_b} \mathcal{K}^a{}_c = |M| \delta^b{}_c - \tilde{M}_c{}^b = |M| \delta^b{}_c - \frac{\delta |M|}{\delta \mathcal{K}^c{}_b}. \tag{A5}$$

Note that we have resumed the upper and lower indices in order to reflect the tensor properties of these indices. This equation can be further expanded as a set of recurrence relations in order of $\mathcal{K}^n$. The result is

$$\frac{\delta \mathcal{L}_n}{\delta \mathcal{K}^b{}_a} = \mathcal{L}_{n-1} \delta^a{}_b - \frac{\delta \mathcal{L}_{n-1}}{\delta \mathcal{K}^c{}_a} \mathcal{K}^c{}_b. \tag{A6}$$

Note that this recurrence relation can also be verified directly from the f equation. In particular, the $n = 5$ recurrence relation gives

$$\mathcal{L}_4 \delta^a{}_b - \frac{\delta \mathcal{L}_4}{\delta \mathcal{K}^c{}_a} \mathcal{K}^c{}_b = 0 \tag{A7}$$

because the expansion of $|M|$ in Eq. (A4) terminates when $n \geq 5$. To be more specific, $\mathcal{L}_5 = 0$.

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