High energy scattering amplitudes of superstring theory

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Abstract

We use three different methods to calculate the proportionality constants among high-energy scattering amplitudes of different string states with polarizations on the scattering plane. These are the decoupling of high-energy zero-norm states (HZNS), the Virasoro constraints and the saddle-point calculation. These calculations are performed at arbitrary but fixed mass level for the NS sector of 10D open superstring. All three methods give the consistent results, which generalize the previous works on the high-energy 26D open bosonic string theory. In addition, we discover new leading order high-energy scattering amplitudes, which are still proportional to the previous ones, with polarizations *orthogonal* to the scattering plane. These scattering amplitudes are of subleading order in energy for the case of 26D open bosonic string theory. The existence of these new high-energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions and is, presumably, related to the high-energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory.

1. Introduction

It has long been believed that string theory consists of a huge hidden symmetry. This is strongly suggested by the ultraviolet finiteness of quantum string theory, which contains no free parameter and an infinite number of states. To probe the structure and the origin of the symmetry has been one of the fundamental issue ever since the discovery of string theory.

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The first key idea to uncover the hidden stringy symmetry was to study the high-energy behavior of the theory, as suggested by Gross in 1988 [1]. This was based on the saddle-point calculations of high-energy fixed-angle string scattering amplitudes in [2,3]. There are two main conjectures of Gross's pioneer work on this subject. The first one is the existence of an infinite number of linear relations among the scattering amplitudes of different string states that are valid order by order in perturbation theory at high energies. The second is that this symmetry is so powerful as to determine the scattering amplitudes of all the infinite number of string states in terms of, say, the tachyon scattering amplitudes (for the bosonic open string case). However, the symmetry charges of his proposed stringy symmetries were not understood and the proportionality constants among high-energy scattering amplitudes of different string states were not calculated.

The second key idea to uncover the fundamental symmetry of string theory was the identification of symmetry charges from an infinite number of stringy zero-norm states with arbitrarily high spins in the old covariant first quantized (OCFQ) string spectrum [4]. The importance of zero-norm states and their implication on stringy symmetries were first pointed out [4] in the context of massive σ-model approach [5,6] of string theory. Some implications of the corresponding stringy Ward identities on the scattering amplitudes were discussed in [7,8]. In addition to the continuous symmetries, the discrete T-duality symmetry was shown to be related to the existence of compactified closed string soliton zero-norm states [9]. The enhanced gauge symmetry of N coincident D-branes can also be shown to be related to the existence of compactified open string zero-norm states at some discrete values of compactified radii [10]. On the other hand, zero-norm states were also shown [11] to carry the spacetime ω_∞ symmetry [12] charges of 2D string theory [13]. This is in parallel with the work of [14] where the ground ring structure of ghost number zero operators was identified in the BRST quantization. All the above interesting results of 26D and 2D string theories strongly suggest that a clearer understanding of zero-norm states holds promise to uncover the fundamental symmetry of string theory. Incidentally, it was also shown [15,16] that off-shell gauge transformations of Witten string field theory [17], after imposing the no-ghost condition, are identical to the on-shell stringy gauge symmetries generated by two types of zero-norm states in the massive σ-model approach of string theory [4]. Other approaches of stringy symmetries can be found in [18–25].

Recently high-energy Ward identities derived from the decoupling of zero-norm states, which combines the previous two key ideas of probing stringy symmetry, were used to explicitly prove Gross's two conjectures [26,27]. An infinite number of linear relations among high-energy scattering amplitudes of different string states were derived. Moreover, these linear relations can be used to fix the proportionality constants among high-energy scattering amplitudes of different string states algebraically up to mass level $M^2 = 6$. Thus there is only one independent component of high-energy scattering amplitude at each fixed mass level. It is important to discover that the result of saddle-point calculation in [1–3] was inconsistent with high-energy stringy Ward identities of zero-norm state calculation in [26–28]. A corrected saddle-point calculation was given in [28], where the missing terms of the calculation in [1–3] were identified to recover the stringy Ward identities. Soon after, the calculations of the proportionality constants among high-energy scattering amplitudes of different string states were generalized to arbitrary but fixed mass level [29–31]. Based on the general formula for the independent component of high-energy scattering amplitude at each fixed mass level calculated previously in [26,28], one can then derive the general formula of high-energy scattering amplitude for four arbitrary string states, and express them in terms of those of tachyons. This completes the general proofs of Gross's two conjectures on high-energy symmetry of string theory stated above.

In this paper, we consider the high-energy scattering amplitudes for the NS sector of 10D open superstring theory. Based on the calculations of 26D bosonic open string [29–31], all the three independent calculations of bosonic string, namely the decoupling of high-energy zero-norm states (HZNS), the Virasoro constraints and the saddle-point calculation, can be generalized to scattering amplitudes of string states with polarizations on the scattering plane of superstring. All three methods give the consistent results. In addition, we discover new leading order high-energy scattering amplitudes, which are still proportional to the previous ones, with polarizations *orthogonal* to the scattering plane. These scattering amplitudes are of subleading order in energy for the case of 26D open bosonic string theory. The existence of these new high-energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions and is, presumably, related to the high-energy massive fermionic scattering amplitudes in the R-sector of the theory. We thus conjecture that the validity of Gross's two conjectures on high-energy stringy symmetry persists for superstring theory. This paper is organized as follows. In Section 2, after a brief review of previous calculations of the decoupling of HZNS for the bosonic string, we show that the calculations can be generalized to the superstring. The ratios among the scattering amplitudes of different string states *with polarizations on the scattering plane* can be determined by using two types of HZNS in the NS sector. In Section 3, we use the "dual method", the Virasoro constraints, to calculate the ratios among the scattering amplitudes of different string states. The results are consistent with those obtained in Section 2. In Section 4, a set of scattering amplitudes are calculated by the saddle-point method to justify our results in Sections 2 and 3. In Section 5, we present some new high-energy scattering amplitudes of string states with polarizations *orthogonal* to the scattering plane. Finally a brief conclusion is given in Section 6.

2. Decoupling of HZNS

We will consider four-point correlation functions in this paper. We begin with a brief review of the high-energy calculation of 26D bosonic open string theory. At a fixed mass level $M^2 = 2(n-1)$, it was shown that [29–31] in the high-energy limit, only states of the following form

$$|n, 2m, q\rangle \equiv \left(\alpha_{-1}^T\right)^{n-2m-2q} \left(\alpha_{-1}^L\right)^{2m} \left(\alpha_{-2}^L\right)^q |0, k\rangle, \quad \text{where } n - 2m - 2q, m, q \geqslant 0, \qquad (1)$$

are relevant for four-point functions (we use the notation of [32]). The state in Eq. (1) is arbitrarily chosen to be the second vertex of the four-point function. The other three points can be any string states. We have defined the normalized polarization vectors of the second string state to be [26,27]

$$e^P = \frac{1}{M_2}(E_2, \mathrm{k}_2, 0) = \frac{k_2}{M_2}, \qquad (2)$$

$$e^L = \frac{1}{M_2}(\mathrm{k}_2, E_2, 0), \qquad (3)$$

$$e^T = (0, 0, 1), \qquad (4)$$

in the CM frame contained in the plane of scattering. In the OCFQ spectrum of open bosonic string theory, the solutions of physical states conditions include positive-norm propagating states and two types of zero-norm states. The latter are [32]

Type I: $\quad L_{-1}|x\rangle, \quad$ where $L_1|x\rangle = L_2|x\rangle = 0, \; L_0|x\rangle = 0;$ $\qquad (5)$

Type II: $\quad \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\tilde{x}\rangle, \quad$ where $L_1|\tilde{x}\rangle = L_2|\tilde{x}\rangle = 0, \; (L_0 + 1)|\tilde{x}\rangle = 0.$ $\qquad (6)$

While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at D = 26. The decoupling of the following type I HZNS

$$L_{-1}|n-1, 2m-1, q\rangle = M|n, 2m, q\rangle + (2m-1)|n, 2m-2, q+1\rangle \tag{7}$$

gives the first high-energy Ward identities

$$\mathcal{T}^{(n,2m,q)} = \left(-\frac{2m-1}{M}\right)\cdots\left(-\frac{3}{M}\right)\left(-\frac{1}{M}\right)\mathcal{T}^{(n,0,q+m)}, \tag{8}$$

where $\mathcal{T}^{(n,2m,q)}$ represents the four-point functions with the second particle at level n. Similarly, the decoupling of the following type II HZNS

$$L_{-2}|n-2, 0, q\rangle = \frac{1}{2}|n, 0, q\rangle + M|n, 0, q+1\rangle \tag{9}$$

gives the second high-energy Ward identities

$$\mathcal{T}^{(n,0,q)} = \left(-\frac{1}{2M}\right)^q \mathcal{T}^{(n,0,0)}. \tag{10}$$

Combining Eqs. (8) and (10) gives the master formula [29–31]

$$\mathcal{T}^{(n,2m,q)} = \left(-\frac{1}{M}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!\mathcal{T}^{(n,0,0)}, \tag{11}$$

which shows that there is only one independent high-energy scattering amplitudes at each fixed mass level.

We now consider the superstring case. We will first consider high-energy scattering amplitudes of string states with polarizations on the scattering plane. Those with polariations orthogonal to the scattering plane will be discussed in Section 5. It can be argued that there are four types of high-energy scattering amplitudes for states in the NS sector with even GSO parity

$$|n, 2m, q\rangle \otimes \left|b^T_{-\frac{1}{2}}\right\rangle \equiv (\alpha^T_{-1})^{n-2m-2q}(\alpha^L_{-1})^{2m}(\alpha^L_{-2})^q(b^T_{-\frac{1}{2}})|0, k\rangle, \tag{12}$$

$$|n, 2m+1, q\rangle \otimes \left|b^L_{-\frac{1}{2}}\right\rangle \equiv (\alpha^T_{-1})^{n-2m-2q-1}(\alpha^L_{-1})^{2m+1}(\alpha^L_{-2})^q(b^L_{-\frac{1}{2}})|0, k\rangle, \tag{13}$$

$$|n, 2m, q\rangle \otimes \left|b^L_{-\frac{3}{2}}\right\rangle \equiv (\alpha^T_{-1})^{n-2m-2q}(\alpha^L_{-1})^{2m}(\alpha^L_{-2})^q(b^L_{-\frac{3}{2}})|0, k\rangle, \tag{14}$$

$$|n, 2m, q\rangle \otimes \left|b^T_{-\frac{1}{2}}b^L_{-\frac{1}{2}}b^L_{-\frac{3}{2}}\right\rangle \equiv (\alpha^T_{-1})^{n-2m-2q}(\alpha^L_{-1})^{2m}(\alpha^L_{-2})^q(b^T_{-\frac{1}{2}})(b^L_{-\frac{1}{2}})(b^L_{-\frac{3}{2}})|0, k\rangle. \tag{15}$$

Note that the number of α^L_{-1} operator in Eq. (13) is odd. In the OCFQ spectrum of open superstring, the solutions of physical states conditions include positive-norm propagating states and two types of zero-norm states. In the NS sector, the latter are [32]

Type I:　$G_{-\frac{1}{2}}|x\rangle$,　　where $G_{\frac{1}{2}}|x\rangle = G_{\frac{3}{2}}|x\rangle = 0$, $L_0|x\rangle = 0$; $\tag{16}$

Type II:　$(G_{-\frac{3}{2}} + 2G_{-\frac{1}{2}}L_{-1})|\tilde{x}\rangle$,　　where $G_{\frac{1}{2}}|\tilde{x}\rangle = G_{\frac{3}{2}}|\tilde{x}\rangle = 0$, $(L_0+1)|\tilde{x}\rangle = 0$. $\tag{17}$

While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at D = 10. We will show that, for each fixed mass level, all high-energy scattering amplitudes corresponding to states in Eqs. (12)–(15) are proportional to each other, and the proportionality constants can be determined from the decoupling of two types of zero-norm states, Eqs. (16)

and (17) in the high-energy limit. For simplicity, based on the result of Eq. (11), one needs only calculate the proportionality constants among the scattering amplitudes of the following four lower mass level states

$$|2, 0, 0\rangle \otimes |b^T_{-\frac{1}{2}}\rangle \equiv (\alpha^T_{-1})^2 (b^T_{-\frac{1}{2}}) |0, k\rangle, \tag{18}$$

$$|2, 1, 0\rangle \otimes |b^L_{-\frac{1}{2}}\rangle \equiv (\alpha^T_{-1})(\alpha^L_{-1})(b^L_{-\frac{1}{2}}) |0, k\rangle, \tag{19}$$

$$|1, 0, 0\rangle \otimes |b^L_{-\frac{3}{2}}\rangle \equiv (\alpha^T_{-1})(b^L_{-\frac{3}{2}}) |0, k\rangle, \tag{20}$$

$$|0, 0, 0\rangle \otimes |b^T_{-\frac{1}{2}} b^L_{-\frac{1}{2}} b^L_{-\frac{3}{2}}\rangle \equiv (b^T_{-\frac{1}{2}})(b^L_{-\frac{1}{2}})(b^L_{-\frac{3}{2}}) |0, k\rangle. \tag{21}$$

Other proportionality constants for higher mass level can be obtained through Eqs. (11) and (18)–(21). To calculate the ratio among the high-energy scattering amplitudes corresponding to states in Eqs. (19) and (20), we use the decoupling of the type I HZNS at mass level $M^2 = 2$

$$G_{-\frac{1}{2}}(\alpha^L_{-1})|0, k\rangle = \left[M(\alpha^L_{-1})(b^L_{-\frac{1}{2}}) + (b^L_{-\frac{3}{2}}) \right] |0, k\rangle. \tag{22}$$

Eq. (22) gives the ratio for states at mass level $M^2 = 4$

$$(\alpha^T_{-1})(b^L_{-\frac{3}{2}})|0, k\rangle : (\alpha^T_{-1})(\alpha^L_{-1})(b^L_{-\frac{1}{2}})|0, k\rangle = M : -1. \tag{23}$$

We have used an abbreviated notation for the scattering amplitudes on the l.h.s. of Eq. (23). The HZNS in Eq. (22) is the high-energy limit of the vector zero-norm state at mass level $M^2 = 2$

$$G_{-\frac{1}{2}}|x\rangle = \left[k_{(\mu}\theta_{\nu)}\alpha^\mu_{-1}b^\nu_{-\frac{1}{2}} + \theta \cdot b_{-\frac{3}{2}} \right] |0, k\rangle, \tag{24}$$

where

$$|x\rangle = \left[\theta \cdot \alpha_{-1} + \frac{1}{2} k \cdot b_{-\frac{1}{2}} \theta \cdot b_{-\frac{1}{2}} \right] |0, k\rangle, \qquad k \cdot \theta = 0. \tag{25}$$

In fact, in the high-energy limit, $\theta = e^L$, so $|x\rangle \to (\alpha^L_{-1})|0, k\rangle$ and Eq. (24) reduces to Eq. (22). To calculate the ratio among the high-energy scattering amplitudes corresponding to states in Eqs. (18) and (20), we use the decoupling of the type II HZNS at mass level $M^2 = 4$

$$G_{-\frac{3}{2}}(\alpha^T_{-1})|0, k\rangle = \left[M(\alpha^T_{-1})(b^L_{-\frac{3}{2}}) + (\alpha^T_{-1})^2 (b^T_{-\frac{1}{2}}) \right] |0, k\rangle. \tag{26}$$

Eq. (26) gives the ratio

$$(\alpha^T_{-1})(b^L_{-\frac{3}{2}})|0, k\rangle : (\alpha^T_{-1})^2 (b^T_{-\frac{1}{2}})|0, k\rangle = 1 : -M. \tag{27}$$

Finally, to calculate the ratio among the high-energy scattering amplitudes corresponding to states in Eqs. (18) and (21), we use the decoupling of the type II HZNS at mass level $M^2 = 4$

$$G_{-\frac{3}{2}}(b^T_{-\frac{1}{2}})(b^L_{-\frac{1}{2}})|0, k\rangle \equiv \left[M(b^T_{-\frac{1}{2}})(b^L_{-\frac{1}{2}})(b^L_{-\frac{3}{2}}) + (\alpha^L_{-2})(b^T_{-\frac{1}{2}}) \right] |0, k\rangle. \tag{28}$$

Eq. (28) gives the ratio

$$(b^T_{-\frac{1}{2}})(b^L_{-\frac{1}{2}})(b^L_{-\frac{3}{2}})|0, k\rangle : (\alpha^L_{-2})(b^T_{-\frac{1}{2}})|0, k\rangle = 1 : -M. \tag{29}$$

On the other hand, Eq. (10) gives

$$(\alpha^L_{-2})(b^T_{-\frac{1}{2}})|0, k\rangle : (\alpha^T_{-1})^2 (b^T_{-\frac{1}{2}})|0, k\rangle = 1 : -2M. \tag{30}$$

We conclude that

$$\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle : \left(\alpha^T_{-1}\right)^2\left(b^T_{-\frac{1}{2}}\right)|0,k\rangle = 1 : 2M^2. \tag{31}$$

Eqs. (23), (27) and (31) give the proportionality constants among high-energy scattering amplitudes corresponding to states in Eqs. (18)–(21). Finally, by using Eq. (11), one can then easily calculate the proportionality constants among high-energy scattering amplitudes corresponding to states in Eqs. (12)–(15).

3. Virasoro constraints

In this section, we will use the method of Virasoro constrains to derive the ratios between the physical states in the NS sector. In the superstring theory, the physical state $|\phi\rangle$ in the NS sector should satisfy the following conditions:

$$\left(L_0 - \frac{1}{2}\right)|\phi\rangle = 0, \tag{32}$$

$$L_m|\phi\rangle = 0, \quad m = 1, 2, 3, \ldots, \tag{33}$$

$$G_r|\phi\rangle = 0, \quad r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \tag{34}$$

where the L_m and G_r are super Virasoro operators in the NS sector,

$$L_m = \frac{1}{2}\sum_n :\alpha_{m-n}\cdot\alpha_n: + \frac{1}{4}\sum_r (2r-m):\psi_{m-r}\cdot\psi_r:, \tag{35}$$

$$G_r = \sum_n \alpha_n \cdot \psi_{r-n}. \tag{36}$$

These super Virasoro operators satisfy the following superconformal algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{8}D(m^3 - m)\delta_{m+n},$$

$$[L_m, G_r] = \left(\frac{1}{2}m - r\right)G_{m+r},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2}D\left(r^2 - \frac{1}{4}\right)\delta_{r+s}. \tag{37}$$

Using the above superconformal algebra, the Virasoro conditions (33) and (34) reduce to the following simple form,

$$G_{1/2}|\phi\rangle = 0, \tag{38}$$

$$G_{3/2}|\phi\rangle = 0. \tag{39}$$

In the following, we will use the reduced Virasoro conditions (38) and (39) to determine the ratios between the physical states in the NS sector in the high-energy limit.

To warm up, let us consider the mass level at $M^2 = 2$ first. The most general state in the NS sector at this mass level can be written as

$$|2\rangle = \left\{ \boxed{\mu}\,\psi^\mu_{-\frac{3}{2}} + \boxed{\mu}\otimes\boxed{\nu}\,\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}} + \boxed{\begin{array}{c}\mu\\\nu\\\sigma\end{array}}\,\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}\psi^\sigma_{-\frac{1}{2}} \right\} |0\rangle_{\text{NS}}, \tag{40}$$

where we use the Young tableaux to represent the coefficients of different tensors. The properties of symmetry and anti-symmetry can be easily and clearly described in this representation.

We then apply the reduced Virasoro conditions (38) and (39) to the state (40). It is easy to obtain

$$
G_{1/2}|2\rangle = \alpha_{-1}^{\mu}\left\{\boxed{\mu} + k^{\nu}\boxed{\mu}\otimes\boxed{\nu}\right\} + \psi_{-\frac{1}{2}}^{\mu}\psi_{-\frac{1}{2}}^{\nu}\left\{\boxed{\mu}\otimes\boxed{\nu} - \boxed{\nu}\otimes\boxed{\mu} + 3k^{\sigma}\boxed{\begin{array}{c}\mu\\\nu\\\sigma\end{array}}\right\}, \tag{41a}
$$

$$
G_{3/2}|2\rangle = \boxed{\mu}k^{\mu} + \boxed{\mu}\otimes\boxed{\nu}\eta^{\mu\nu}, \tag{41b}
$$

which leads to the following equations

$$
\boxed{\mu} + k^{\nu}\boxed{\mu}\otimes\boxed{\nu} = 0, \tag{42a}
$$

$$
\boxed{\mu}\otimes\boxed{\nu} - \boxed{\nu}\otimes\boxed{\mu} + 3k^{\sigma}\boxed{\begin{array}{c}\mu\\\nu\\\sigma\end{array}} = 0, \tag{42b}
$$

$$
\boxed{\mu}k^{\mu} + \boxed{\mu}\otimes\boxed{\nu}\eta^{\mu\nu} = 0. \tag{42c}
$$

To solve the above equation, we first take the high-energy limit by letting $\mu \to (L, T)$ and

$$
k^{\mu} \to M\left(e^{L}\right)^{\mu}, \qquad \eta^{\mu\nu} \to \left(e^{T}\right)^{\mu}\left(e^{T}\right)^{\nu}. \tag{43}
$$

The above equations reduce to

$$
\boxed{\mu} + M\boxed{\mu}\otimes\boxed{L} = 0, \tag{44}
$$

$$
\boxed{\mu}\otimes\boxed{\nu} - \boxed{\nu}\otimes\boxed{\mu} = 0, \tag{45}
$$

$$
M\boxed{L} + \boxed{T}\otimes\boxed{T} = 0. \tag{46}
$$

At this mass level, the terms with odd number of T's will be sub-leading in the high-energy limit and be ignored, the resulting equations contain only terms with even number of T's as following,

$$
\boxed{L} + M\boxed{L}\otimes\boxed{L} = 0, \tag{47}
$$

$$
M\boxed{L} + \boxed{T}\otimes\boxed{T} = 0. \tag{48}
$$

The ratio of the coefficients then can be obtained as

ε_{TT}	$M^2\ (=2)$
ε_{LL}	1
ε_L	$-M\ (=-\sqrt{2})$

(49)

In the following, we will consider the general mass level at $M^2 = (2n - 1)$. At this mass level, the most general state can be written as

$$|n\rangle = \left\{ \sum_{m_j, m_r} \bigotimes_{j=1}^{n} \frac{1}{j^{m_j} m_j!} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r=1/2}^{n-1/2} \frac{1}{m_r!} \boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T \psi_{-r}^{v_1^r \cdots v_{m_r}^r} \right\} |0, k\rangle,$$

(50)

where

$$\boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T = \boxed{\begin{array}{c} v_1^r \\ \vdots \\ v_{m_r}^r \end{array}},$$

(51)

and we have defined the abbreviation

$$\alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \equiv \alpha_{-j}^{\mu_1^j} \cdots \alpha_{-j}^{\mu_{m_j}^j} \quad \text{and} \quad \psi_{-r}^{v_1^r \cdots v_{m_r}^r} \equiv \psi_{-r}^{v_1^r} \cdots \psi_{-r}^{v_{m_r}^r},$$

(52)

with m_j (m_r) is the number of the operator α_{-j}^{μ} (ψ_{-r}^{v}) for $j \in Z$ and $r \in Z + 1/2$. The summation runs over all possible m_j (m_r) with the constrain

$$\sum_{j=1}^{n} j m_j + \sum_{r=1/2}^{n-1/2} r m_r = n - \frac{1}{2} \quad \text{with } m_j, m_r \geqslant 0,$$

(53)

so that the total mass square is $2(n - 1)$.

Next, we will apply the reduced Virasoro conditions (38) and (39) to the state (50),

$$G_{1/2}|n\rangle = \sum_{m_j} \left[k^{v_1^{1/2}} \boxed{v_1^{1/2}} \cdots \boxed{v_{m_{1/2}}^{1/2}}^T \bigotimes_{j=1}^{k} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \bigotimes_{r \neq 1/2}^{s} \boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T \right.$$

$$+ \sum_{l \geqslant 1} \sum_{i=1}^{m_l} l \boxed{\mu_1^l} \cdots \boxed{\hat{\mu}_i^l} \cdots \boxed{\mu_{m_l}^l} \otimes \boxed{\mu_i^l \, v_1^{l+1/2}} \cdots \boxed{v_{m_{l+1/2}}^{l+1/2}}^T$$

$$\otimes \boxed{v_2^{1/2}} \cdots \boxed{v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^{k} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \bigotimes_{r \neq 1/2, l+1/2}^{s} \boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T$$

$$+ \sum_{i=2}^{m_{1/2}} \boxed{v_i^{1/2} \, \mu_1^1} \cdots \boxed{\mu_{m_1}^1} \otimes (-1)^{i+1} \boxed{v_2^{1/2}} \cdots \boxed{\hat{v}_i^{1/2}} \cdots \boxed{v_{m_{1/2}}^{1/2}}^T$$

$$\bigotimes_{j \neq l}^{k} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \bigotimes_{r \neq 1/2}^{s} \boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T$$

$$+ \sum_{l \geqslant 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \boxed{v_i^{l-1/2} \, \mu_1^l} \cdots \boxed{\mu_{m_l}^l} \otimes \boxed{v_1^{l-1/2}} \cdots \boxed{\hat{v}_i^{l-1/2}} \cdots \boxed{v_{m_{l-1/2}}^{l-1/2}}^T$$

$$\left. \otimes \boxed{v_2^{1/2}} \cdots \boxed{v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^{k} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \bigotimes_{r \neq 1/2, l-1/2}^{s} \boxed{v_1^r} \cdots \boxed{v_{m_r}^r}^T \right]$$

$$\times \frac{1}{(m_{1/2}-1)!}\psi_{-1/2}^{v_2^{1/2}\cdots v_{m_{1/2}}^{1/2}}\prod_{j=1}^{k}\frac{1}{j^{m_j}m_j!}\alpha_{-j}^{\mu_1^j\cdots\mu_{m_j}^j}\prod_{r\neq 1/2}^{s}\frac{1}{m_r!}\psi_{-r}^{v_1^r\cdots v_{m_r}^r}, \tag{54a}$$

and

$$G_{3/2}|n\rangle=\sum_{m_j}\left[\boxed{v_1^{3/2}\cdots v_{m_{3/2}}^{3/2}}^{T}k^{v_1^{3/2}}\bigotimes_{j=1}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^{T}\right.$$

$$+\eta^{\mu v}\boxed{\mu\;\mu_1^1\cdots\mu_{m_1}^1}\otimes\boxed{v\;v_1^{1/2}\cdots v_{m_{1/2}}^{1/2}}^{T}\otimes\boxed{v_2^{3/2}\cdots v_{m_{3/2}}^{3/2}}^{T}$$

$$\bigotimes_{j\neq 1}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 1/2,3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^{T}$$

$$+\sum_{l\geqslant 1}\sum_{i=1}^{m_l}l\boxed{\mu_1^l\cdots\hat{\mu}_i^l\cdots\mu_{m_l}^l}\otimes\boxed{\mu_i^l\;v_1^{l+3/2}\cdots v_{m_{l+3/2}}^{l+3/2}}^{T}$$

$$\otimes\boxed{v_2^{3/2}\cdots v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 3/2,l+3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^{T}$$

$$+\sum_{i=2}^{m_{3/2}}3\boxed{v_i^{3/2}\;\mu_1^3\cdots\mu_{m_3}^3}\otimes(-1)^{i+1}\boxed{v_2^{3/2}\cdots\hat{v}_i^{3/2}\cdots v_{m_{3/2}}^{3/2}}^{T}$$

$$\bigotimes_{j\neq 3}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^{T}$$

$$+\sum_{l\geqslant 2,l\neq 3}\sum_{i=1}^{m_{l-3/2}}\boxed{v_i^{l-3/2}\;\mu_1^l\cdots\mu_{m_l}^l}\otimes\boxed{v_1^{l-3/2}\cdots\hat{v}_i^{l-3/2}\cdots v_{m_{l-3/2}}^{l-3/2}}^{T}$$

$$\left.\otimes\boxed{v_2^{3/2}\cdots v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 3/2,l-3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^{T}\right]$$

$$\times\frac{1}{(m_{3/2}-1)!}\psi_{-3/2}^{v_2^{3/2}\cdots v_{m_{3/2}}^{3/2}}\prod_{j=1}^{k}\frac{1}{j^{m_j}m_j!}\alpha_{-j}^{\mu_1^j\cdots\mu_{m_j}^j}\prod_{r\neq 3/2}^{s}\frac{1}{m_r!}\psi_{-r}^{v_1^r\cdots v_{m_r}^r}, \tag{54b}$$

where we have used the identities of the Young tableaux,

$$\boxed{1\cdots p}=\frac{1}{p}[1+\sigma_{(21)}+\sigma_{(321)}+\cdots+\sigma_{(p\cdots 1)}]\boxed{1}\otimes\boxed{2\cdots p}$$

$$=\frac{1}{p}\sum_{i=1}^{p}\sigma_{(i1)}\boxed{1}\otimes\boxed{2\cdots p}, \tag{55}$$

$$\boxed{1\cdots p}^{T}=\frac{1}{p}\big[1-\sigma_{(21)}+\sigma_{(321)}-\cdots+(-1)^{p+1}\sigma_{(p\cdots 1)}\big]\boxed{1}\otimes\boxed{2\cdots p}^{T}$$

$$=\frac{1}{p}\sum_{i=1}^{p}(-1)^{i+1}\sigma_{(i\cdots 1)}\boxed{1}\otimes\boxed{2\cdots p}^{T}. \tag{56}$$

We then obtain the constraint equations

$$
0 = k^{v_1^{1/2}}\boxed{v_1^{1/2}\,\cdots\,v_{m_{1/2}}^{1/2}}^{T}\bigotimes_{j=1}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 1/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{l\geqslant 1}\sum_{i=1}^{m_l} l\,\boxed{\mu_1^{l}\,\cdots\,\hat{\mu}_i^{l}\,\cdots\,\mu_{m_l}^{l}}\otimes\boxed{\mu_i^{l}\,v_1^{l+1/2}\,\cdots\,v_{m_{l+1/2}}^{l+1/2}}^{T}
$$

$$
\otimes\boxed{v_2^{1/2}\,\cdots\,v_{m_{1/2}}^{1/2}}^{T}\bigotimes_{j\neq l}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 1/2,l+1/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{i=2}^{m_{1/2}}\boxed{v_i^{1/2}\,\mu_1^{1}\,\cdots\,\mu_{m_1}^{1}}\otimes(-1)^{i+1}\boxed{v_2^{1/2}\,\cdots\,\hat{v}_i^{1/2}\,\cdots\,v_{m_{1/2}}^{1/2}}^{T}
$$

$$
\bigotimes_{j\neq l}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 1/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{l\geqslant 2}\sum_{i=1}^{m_{l-1/2}}(-1)^{i+1}\boxed{v_i^{l-1/2}\,\mu_1^{l}\,\cdots\,\mu_{m_l}^{l}}\otimes\boxed{v_1^{l-1/2}\,\cdots\,\hat{v}_i^{l-1/2}\,\cdots\,v_{m_{l-1/2}}^{l-1/2}}^{T}
$$

$$
\otimes\boxed{v_2^{1/2}\,\cdots\,v_{m_{1/2}}^{1/2}}^{T}\bigotimes_{j\neq l}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 1/2,l-1/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}, \tag{57a}
$$

$$
0 = \boxed{v_1^{3/2}\,\cdots\,v_{m_{3/2}}^{3/2}}^{T} k^{v_1^{3/2}}\bigotimes_{j=1}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\eta^{\mu\nu}\boxed{\mu\,\mu_1^{1}\,\cdots\,\mu_{m_1}^{1}}\otimes\boxed{v\,v_1^{1/2}\,\cdots\,v_{m_{1/2}}^{1/2}}^{T}\otimes\boxed{v_2^{3/2}\,\cdots\,v_{m_{3/2}}^{3/2}}^{T}
$$

$$
\bigotimes_{j\neq 1}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 1/2,3/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{l\geqslant 1}\sum_{i=1}^{m_l} l\,\boxed{\mu_1^{l}\,\cdots\,\hat{\mu}_i^{l}\,\cdots\,\mu_{m_l}^{l}}\otimes\boxed{\mu_i^{l}\,v_1^{l+3/2}\,\cdots\,v_{m_{l+3/2}}^{l+3/2}}^{T}
$$

$$
\otimes\boxed{v_2^{3/2}\,\cdots\,v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 3/2,l+3/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{i=2}^{m_{3/2}} 3\,\boxed{v_i^{3/2}\,\mu_1^{3}\,\cdots\,\mu_{m_3}^{3}}\otimes(-1)^{i+1}\boxed{v_2^{3/2}\,\cdots\,\hat{v}_i^{3/2}\,\cdots\,v_{m_{3/2}}^{3/2}}^{T}
$$

$$
\bigotimes_{j\neq 3}^{k}\boxed{\mu_1^{j}\,\cdots\,\mu_{m_j}^{j}}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^{r}\,\cdots\,v_{m_r}^{r}}^{T}
$$

$$
+\sum_{l\geqslant 2,\,l\neq 3}\sum_{i=1}^{m_{l-3/2}}\boxed{v_i^{l-3/2}\,\mu_1^{l}\,\cdots\,\mu_{m_l}^{l}}\otimes(-1)^{i+1}\boxed{v_1^{l-3/2}\,\cdots\,\hat{v}_i^{l-3/2}\,\cdots\,v_{m_{l-3/2}}^{l-3/2}}^{T}
$$

$$\otimes \boxed{v_2^{3/2}\cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k}\boxed{\mu_1^j\cdots\mu_{m_j}^j}\bigotimes_{r\neq 3/2, l-3/2}^{s}\boxed{v_1^r\cdots v_{m_r}^r}^T. \tag{57b}$$

Taking the high-energy limit in the above equations by letting $(\mu_i, v_i)\to (L, T)$, and

$$k^{\mu_i}\to M\big(e^L\big)^{\mu_i},\qquad \eta^{\mu_1\mu_2}\to \big(e^T\big)^{\mu_1}\big(e^T\big)^{\mu_2}, \tag{58}$$

and using the following lemma.

Lemma.

$$\boxed{T\cdots T}\underbrace{\boxed{L\cdots L}}_{l_1}\otimes\underbrace{\boxed{T\cdots T}\boxed{L\cdots L}}_{m_2-l_2}\otimes\underbrace{\boxed{v_1^{1/2}\,v_2^{1/2}}}_{m_{1/2}}\otimes\boxed{v_2^{3/2}}\otimes\boxed{\cdots}\equiv 0, \tag{59}$$

except for (i) $m_{j\geqslant 3}=m_{r\geqslant 3/2}=0$, $l_2=m_2$, $l_{3/2}=m_{3/2}=1$ *and* (ii) $l_1+l_{1/2}=2k$.

We solve the equations in Appendix A, the ratios between the physical states in the NS sector in the high-energy limit are given as

$$\underbrace{\boxed{T\cdots T}}_{n-2m_2-2k}\underbrace{\boxed{L\cdots L}}_{2k}\otimes\underbrace{\boxed{L\cdots L}}_{m_2}\otimes\boxed{0}\otimes\boxed{L}$$

$$=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^k}\underbrace{\boxed{T\cdots T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{60}$$

$$\underbrace{\boxed{T\cdots T}}_{n-2m_2-2k}\underbrace{\boxed{L\cdots L}}_{2k+1}\otimes\underbrace{\boxed{L\cdots L}}_{m_2}\otimes\boxed{L}\otimes\boxed{0}$$

$$=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k+1)!!}{(-M)^{k+1}}\underbrace{\boxed{T\cdots T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{61}$$

$$\underbrace{\boxed{T\cdots T}}_{n-2m_2-2k+1}\underbrace{\boxed{L\cdots L}}_{2k}\otimes\underbrace{\boxed{L\cdots L}}_{m_2}\otimes\boxed{T}\otimes\boxed{0}$$

$$=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^{k-1}}\underbrace{\boxed{T\cdots T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{62}$$

$$\underbrace{\boxed{T\cdots T}}_{n-2m_2-2k+1}\underbrace{\boxed{L\cdots L}}_{2k}\otimes\underbrace{\boxed{L\cdots L}}_{m_2-1}\otimes\boxed{T\,L}^T\otimes\boxed{L}$$

$$=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^{k}}\underbrace{\boxed{T\cdots T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{63}$$

which are exactly consistent with the results obtained by using the decoupling of HZNS in Section 2 and the saddle-point calculation in the following section.

4. Saddle-point approximation

In this section, we shall calculate the high-energy limits of various scattering amplitudes based on saddle-point approximation. Since the decoupling of zero-norm states holds true for arbitrary physical processes, in order to check the ratios among scattering amplitudes at the same mass level, it is helpful to choose low-lying states to simplify calculations. For instance, in the case of 4-point amplitudes, we fix the first vertex to be a $M^2 = 0$ photon with polarization vector ϵ^μ (in the -1 ghost picture, and ϕ is the bosonized ghost operator),

$$V_1 \equiv \epsilon^\mu \psi_\mu e^{-\phi} e^{ik_1 X_1}, \qquad \epsilon \cdot k_1 = k_1^2 = 0; \tag{64}$$

and the third and fourth vertices to be $M^2 = -1$ tachyon (in the 0 ghost picture),

$$V_{3,4} \equiv k_{3,4}^\mu \psi_\mu e^{ik_{3,4} X_{3,4}}, \qquad k_{3,4}^2 = -1. \tag{65}$$

We shall vary the second vertex at the same level and compare the scattering amplitudes to obtain the proportional constants.

4.1. $M^2 = 2$

The second vertex operators at mass level $M^2 = 2$, are given by (in the -1 ghost picture),

$$(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0, k\rangle \Rightarrow \psi^T \partial X^T e^{-\phi} e^{ikX}, \tag{66}$$

$$(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)|0, k\rangle \Rightarrow \psi^L \partial X^L e^{-\phi} e^{ikX}, \tag{67}$$

$$(b_{-\frac{3}{2}}^L)|0, k\rangle \Rightarrow \partial \psi^L e^{-\phi} e^{ikX}. \tag{68}$$

Here we have used the polarization basis to specify the particle spins, e.g., $\psi^T \equiv e_\mu^T \cdot \psi^\mu$.

To illustrate the procedure, we take the first state, Eq. (66), as an example to calculate the scattering amplitude among one massive tensor ($M^2 = 2$) with one photon (V_1) and two tachyons (V_3, V_4). As in the case of open bosonic string theory, we list the contributions of s–t channel only. The 4-point function is given by

$$\int_0^1 dx_2 \left\langle \left(\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}\right)\left(\psi_2^{T_2} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2}\right)\left(k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}\right)\left(k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}\right)\right\rangle, \tag{69}$$

where we have suppressed the $SL(2, R)$ gauge-fixed worldsheet coordinates $x_1 = 0$, $x_3 = 1$, $x_4 = \infty$. Notice that in both the first and second vertices, it is possible to allow fermion operators ψ^μ to have polarization in transverse direction T_i out of the scattering plane. As we shall see in next section that this leads to a new feature of supersymmetric stringy amplitudes in the high-energy limit. At this moment, we only choose the polarization vector to be in the P, L, T directions for a comparison with results obtained by the previous two methods.

A direct application of Wick contraction among fermions ψ, ghosts ϕ, and bosons X leads to the following result

$$\int_0^1 dx \left[\frac{(3,4)(e^{T_1} \cdot e^{T_2})}{x} - (e^{T_1} \cdot k_3)(e^{T_2} \cdot k_4) + \frac{(e^{T_2} \cdot k_3)(e^{T_1} \cdot k_4)}{1-x} \right]$$

$$\times \frac{1}{x} \left[\frac{e^{T_2} \cdot k_3}{1-x} \right] x^{(1,2)} (1-x)^{(2,3)}, \tag{70}$$

where we have used the short-hand notation, $(3,4) \equiv k_3 \cdot k_4$. Based on the kinematic variables and the master formula for saddle-point approximation,

$$\int dx \, u(x) e^{-Kf(x)}$$

$$= u_0 e^{-Kf_0} \sqrt{\frac{2\pi}{Kf_0''}}$$

$$\times \left\{ 1 + \left[\frac{u_0''}{2u_0 f_0''} - \frac{u_0' f_0^{(3)}}{2u_0 (f_0'')^2} - \frac{f_0^{(4)}}{8(f_0'')^2} + \frac{5[f_0^{(3)}]^2}{24(f_0'')^3} \right] \frac{1}{K} + O\left(\frac{1}{K^2} \right) \right\}, \tag{71}$$

where u_0, f_0, u_0', f_0'', etc., stand for the values of functions and their derivatives evaluated at the saddle point $f'(x_0) = 0$. In order to apply this master formula to calculate stringy amplitudes, we need the following substitutions ($\alpha' = 1/2$)

$$K \equiv 2E^2, \tag{72}$$

$$f(x) \equiv \ln(x) - \tau \ln(1-x), \tag{73}$$

$$\tau \equiv -\frac{(2,3)}{(1,2)} \to \sin^2 \frac{\theta}{2}, \tag{74}$$

where θ is the scattering angle in center of momentum frame and the saddle point for the integration of moduli is $x_0 = \frac{1}{1-\tau}$. In the first scattering amplitude corresponding to Eq. (66), we can identify the $u(x)$ function as

$$u_I(x) \equiv \left[\frac{(3,4)(e^{T_1} \cdot e^{T_2})}{x} - (e^{T_1} \cdot k_3)(e^{T_2} \cdot k_4) + \frac{(e^{T_2} \cdot k_3)(e^{T_1} \cdot k_4)}{1-x} \right] \frac{1}{x} \left[\frac{e^{T_2} \cdot k_3}{1-x} \right]. \tag{75}$$

Equipped with this, we obtain the high-energy limit of the first amplitude,

$$2E^2(1-\tau)(e^T \cdot k_3) x_0^{(1,2)-1} (1-x_0)^{(2,3)-1} \sqrt{\frac{\pi \tau}{E^2(1-\tau)^3}}$$

$$= 4\sqrt{\pi} E^2(1-\tau)^2 x_0^{(1,2)} (1-x_0)^{(2,3)}. \tag{76}$$

Next, we replace the second vertex operator in Eq. (69) by Eq. (67), and the 4-point function is given by

$$\int_0^1 dx \frac{1}{M^2} \left[(e^T \cdot k_3)(2,4) - \frac{(e^T \cdot k_4)(2,3)}{1-x} \right] \frac{1}{x} \left[\frac{(1,2)}{x} - \frac{(2,3)}{1-x} \right] x^{(1,2)} (1-x)^{(2,3)}. \tag{77}$$

Here we can identify the $u(x)$ function for saddle-point master formula, Eq. (71)

$$u_{II}(x) \equiv \frac{(e^T \cdot k_3)(1,2)}{M^2 x} \left[(2,4) + \frac{(2,3)}{1-x} \right] f'(x). \tag{78}$$

One can check that $u_{II}(x_0) = u'_{II}(x_0) = 0$, and

$$u''_{II}(x_0) = \frac{2(1,2)(2,3)(e^T \cdot k_3)}{M^2 x(1-x)^2} f''(x_0). \tag{79}$$

Thus, the amplitude associated with the massive state, Eq. (67), is given by

$$-\frac{2}{M^2} E^2 \tau (e^T \cdot k_3) x_0^{(1,2)-1} (1-x_0)^{(2,3)-2} \sqrt{\frac{\pi \tau}{E^2 (1-\tau)^3}}$$
$$= \frac{4}{M^2} \sqrt{\pi} E^2 (1-\tau)^2 x_0^{(1,2)} (1-x_0)^{(2,3)}. \tag{80}$$

In the third case, after replacing the second vertex operator in Eq. (69) by Eq. (68), we get the Wick contraction

$$\int_0^1 dx \frac{1}{M} \left[-\frac{(e^T \cdot k_4)(2,3)}{(1-x)^2} \right] \frac{1}{x} x^{(1,2)} (1-x)^{(2,3)}. \tag{81}$$

The high-energy limit of this amplitude, after applying the master formula of saddle-point approximation, is

$$\frac{2}{M} E^2 \tau (e^T \cdot k_3) x^{(1,2)-1} (1-x)^{(2,3)-2} \sqrt{\frac{\pi \tau}{E^2 (1-\tau)^3}}$$
$$= -\frac{4}{M} \sqrt{\pi} E^2 (1-\tau)^2 x_0^{(1,2)} (1-x_0)^{(2,3)}. \tag{82}$$

In conclusion, from these results, Eqs. (76), (80), (82), we find the ratios between the 4-point amplitudes associated with $(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0,k\rangle$, $(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)|0,k\rangle$, and $(b_{-\frac{3}{2}}^L)|0,k\rangle$ to be $1 : \frac{1}{M^2} : -\frac{1}{M}$, in perfect agreement with Eqs. (23), (27) and Eq. (49).

4.2. $M^2 = 4$

Our previous examples only involve one fermion operator $b_{-\frac{1}{2}}^T$, $b_{-\frac{1}{2}}^L$, $b_{-\frac{3}{2}}^L$. Since in the 4-point functions with the fixed states $V_1 \rightarrow$ photon, $V_{3,4} \rightarrow$ tachyons, the maximum fermion number of the second vertex is three, it is of interest to see the pattern of stringy amplitudes associated with the next massive vertices at $M^2 = 4$.

At this mass level, the relevant states and the vertex operators are (in the -1 ghost picture)

$$(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L)|0,k\rangle \Rightarrow \psi^T \psi^L \partial \psi^L e^{-\phi} e^{ikX}, \tag{83}$$

$$(\alpha_{-1}^T)(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0,k\rangle \Rightarrow \psi^T \partial X^T \partial X^T e^{-\phi} e^{ikX}. \tag{84}$$

To calculate 4-point functions, we can fix the first vertex (V_1) to be a photon state in the -1 ghost picture, Eq. (64), and the third and the fourth vertices to be tachyon state in the 0 ghost picture, Eq. (65).

Since the applications of saddle-point approximation is essentially identical to previous cases, we simply list the results of our calculations

$$\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle$$

$$\Rightarrow \int_0^1 dx_2 \left\langle \left(\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}\right)\left(\psi_2^{T_2}\psi_2^{L_2}\partial\psi_2^{L_2} e^{-\phi_2} e^{ik_2 X_2}\right)\left(k_{3\lambda}\psi_3^{\lambda} e^{ik_3 X_3}\right)\left(k_{4\sigma}\psi_4^{\sigma} e^{ik_4 X_4}\right)\right\rangle$$

$$= \frac{4\sqrt{\pi}}{M^2} E^3 \tau^{-\frac{1}{2}}(1-\tau)^{\frac{7}{2}} x_0^{(1,2)}(1-x_0)^{(2,3)}, \tag{85}$$

$$\left(\alpha^T_{-1}\right)\left(\alpha^T_{-1}\right)\left(b^T_{-\frac{1}{2}}\right)|0,k\rangle$$

$$\Rightarrow \int_0^1 dx_2 \left\langle \left(\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}\right)\left(\psi_2^{T_2} X_2^{L_2} X_2^{L_2} e^{-\phi_2} e^{ik_2 X_2}\right)\left(k_{3\lambda}\psi_3^{\lambda} e^{ik_3 X_3}\right)\left(k_{4\sigma}\psi_4^{\sigma} e^{ik_4 X_4}\right)\right\rangle$$

$$= 8\sqrt{\pi} E^3 \tau^{-\frac{1}{2}}(1-\tau)^{\frac{7}{2}} x_0^{(1,2)}(1-x_0)^{(2,3)}. \tag{86}$$

Combining these results, we conclude that the ratio between the $M^2 = 4$ vertices is given by

$$\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle : \left(\alpha^T_{-1}\right)\left(\alpha^T_{-1}\right)\left(b^L_{-\frac{1}{2}}\right)|0,k\rangle = \frac{1}{M^2} : 2 = 1 : 8. \tag{87}$$

4.3. GSO odd vertices at $M^2 = 5$

In addition to the stringy amplitudes associated with GSO even vertices we have calculated in the previous subsections, we can also apply the same method to those associated with the GSO odd vertices. While it is a common practice to project out the GSO odd states in order to maintain spacetime supersymmetry, it turns out that we do find linear relation among these amplitudes. This seems to suggest a hidden structure of superstring theory in the high-energy limit.

To see this, we examine the vertices of odd GSO parity, at the mass level $M^2 = 5$. Based on the power-counting rule as in the bosonic string case, we can identify the relevant vertices and the associated vertex operators as follows

$$\left(\alpha^T_{-1}\right)\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle \Rightarrow \psi^T \partial\psi^L \partial X^T e^{-\phi} e^{ikX}, \tag{88}$$

$$\left(\alpha^L_{-1}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle \Rightarrow \psi^L \partial\psi^L \partial X^L e^{-\phi} e^{ikX}. \tag{89}$$

To calculate 4-point functions, we can fix the first vertex (V_1) to be a tachyon state in the -1 ghost picture,

$$V_1 = e^{-\phi_1} e^{ik_1 \cdot X_1}, \tag{90}$$

and the third and the fourth vertices to be tachyon state in the 0 ghost picture, as Eq. (65).

Since the applications of saddle-point approximation is essentially identical to previous cases, we simply list the results of our calculations

$$\left(\alpha^T_{-1}\right)\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle$$

$$\Rightarrow \int_0^1 dx_2 \left\langle \left(e^{-\phi_1}e^{ik_1X_1}\right)\left(\psi_2^{T_2}\partial\psi_2^{L_2}\partial X_2^{T_2}e^{-\phi_2}e^{ik_2X_2}\right)\left(k_{3\lambda}\psi_3^{\lambda}e^{ik_3X_3}\right)\left(k_{4\sigma}\psi_4^{\sigma}e^{ik_4X_4}\right)\right\rangle$$

$$= -\frac{8\sqrt{\pi}}{M}E^3\tau^{-\frac{1}{2}}(1-\tau)^{\frac{7}{2}}x_0^{(1,2)}(1-x_0)^{(2,3)}, \tag{91}$$

$$\left(\alpha^L_{-1}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle$$

$$\Rightarrow \int_0^1 dx_2 \left\langle \left(e^{-\phi_1}e^{ik_1X_1}\right)\left(\psi_2^{L_2}\partial\psi_2^{L_2}\partial X_2^{L_2}e^{-\phi_2}e^{ik_2X_2}\right)\left(k_{3\lambda}\psi_3^{\lambda}e^{ik_3X_3}\right)\left(k_{4\sigma}\psi_4^{\sigma}e^{ik_4X_4}\right)\right\rangle$$

$$= -\frac{4\sqrt{\pi}}{M^3}E^3\tau^{-\frac{1}{2}}(1-\tau)^{\frac{7}{2}}x_0^{(1,2)}(1-x_0)^{(2,3)}. \tag{92}$$

It is worth noting that in the second calculations, we need to include both $u''(x_0)$ and $u'(x_0)$ terms of the first order corrections in saddle-point approximation, Eq. (71), to get the correct answer.

Combining these results, we conclude that the ratio between the $M^2 = 5$ vertices is given by

$$\left(\alpha^T_{-1}\right)\left(b^T_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle : \left(\alpha^L_{-1}\right)\left(b^L_{-\frac{1}{2}}\right)\left(b^L_{-\frac{3}{2}}\right)|0,k\rangle = 2M^2 : 1 = 10 : 1. \tag{93}$$

Notice that here we also find an interesting connection between GSO even $M^2 = 4$ amplitudes and those of GSO odd parity at $M^2 = 5$. The high-energy limits of the four amplitudes, Eqs. (85), (86), (91), (92), are proportional to each other. and their ratios are $\sqrt{5} : 8\sqrt{5} : (-8) : -\frac{4}{5}$.

5. Polarizations orthogonal to the scattering plane

In this section we consider high-energy scattering amplitudes of string states with polarizations e_{T^i}, $i = 3, 4, \ldots, 25$, orthogonal to the scattering plane. We will present some examples with saddle-point calculations and compare them with those calculated in Section 4. We will find that they are all proportional to the previous ones considered before. These scattering amplitudes are of subleading order in energy for the case of 26D open bosonic string theory. The existence of these new high-energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions as we will see in the following examples. Our first example is to consider Eq. (69) and replace $\psi_1^{T_1}$ and $\psi_2^{T_2}$ by $\psi_1^{T_1^i}$ and $\psi_2^{T_2^i}$ respectively

$$\int_0^1 dx_2 \left\langle \left(\psi_1^{T_1^i}e^{-\phi_1}e^{ik_1X_1}\right)\left(\psi_2^{T_2^i}\partial X_2^{T_2}e^{-\phi_2}e^{ik_2X_2}\right)\left(k_{3\lambda}\psi_3^{\lambda}e^{ik_3X_3}\right)\left(k_{4\sigma}\psi_4^{\sigma}e^{ik_4X_4}\right)\right\rangle. \tag{94}$$

The calculation of Eq. (94) is similar to that of Eq. (69) except that, for this new case, one ends up with only the first term in Eq. (70), and the second and the third terms vanish. Remarkably, the final answer is

$$-2E^2(1-\tau)\left(e^T\cdot k_3\right)x_0^{(1,2)-1}(1-x_0)^{(2,3)-1}\sqrt{\frac{\pi\tau}{E^2(1-\tau)^3}}$$

$$= -4\sqrt{\pi}E^2(1-\tau)^2 x_0^{(1,2)}(1-x_0)^{(2,3)}, \tag{95}$$

which is proportional to Eq. (76). Our second example is again to replace $\psi_1^{T_1}$ and $\psi_2^{T_2}$ in Eq. (85) by $\psi_1^{T_1^i}$ and $\psi_2^{T_2^i}$, respectively. One gets exactly the same answer as Eq. (85). The two examples above seem to suggest that high-energy scattering of string states with polarizations e_{T^i} are the same as that of polarization e_T up to a sign. Let us consider the third example to justify this point. It is straightforward to show the following

$$
\int_0^1 dx_2 \left\langle \left(\psi_1^L \psi_1^{T_1} \psi_1^{T_1^i} e^{-\phi_1} e^{ik_1 X_1} \right) \left(\psi_2^L \psi_2^{T_2} \psi_2^{T_2^i} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2} \right) \left(k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3} \right) \right.
$$
$$
\left. \times \left(k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4} \right) \right\rangle
$$
$$
= N \left[4E^4 (1-\tau) - 4E^4 (1-\tau)^2 - 4E^4 \tau (1-\tau) \right] = 0. \tag{96}
$$

On the other hand, if we assume the symmetry for all transverse polarization vectors T, T^i in the scattering amplitudes, one can easily derive the same conclusion without detailed calculations. Since replacing T^i polarization vectors of both vertices in Eq. (96) by T will naturally leads to a null result due to anticommuting property of fermions.

It is clear from the above calculations that the existence of these new high-energy scattering amplitudes of string states with polarizations e_{T^i} orthogonal to the scattering plane is due to the worldsheet fermion exchange in the correlation functions. These fermion exchanges do not exist in the pure bosonic string correlation functions and is, presumably, related to the high-energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory. Physically, the high-energy scattering amplitudes of spacetime fermion will enjoy the symmetry of rotations among different polarizations in the spin space and our results here seem to justify this observation. If this conjecture turns out to be true, then the list of vertices we considered in Eqs. (18)–(21) for high-energy stringy amplitudes should be extended and includes the cases with $b_{-\frac{1}{2}}^{T}$ replaced by $b_{-\frac{1}{2}}^{T^i}$. Obviously, these new high-energy amplitudes create complications and textures for a full understanding of stringy symmetry. Nevertheless, the claim that there is only one independent high-energy scattering amplitude at each fixed mass level of the string spectrum persists in the case of superstring theory, at least, for the NS sector of the theory.

6. Conclusion

In this paper we have explicitly calculated all high-energy scattering amplitudes of string states with polarizations on the scattering plane of open superstring theory. In particular, the proportionality constants among high-energy scattering amplitudes of different string states at each fixed but arbitrary mass level are determined by using three different methods. These constants are shown to originate from zero-norm states in the spectrum as in the case of open bosonic string theory. In addition, we discover new high-energy scattering amplitudes, which are still proportional to the previous ones, with polarizations *orthogonal* to the scattering plane. We conjecture the existence of a symmetry among high-energy scattering amplitudes with polarizations e_{T^i} and e_T. These scattering amplitudes are subleading order in energy for the case of open bosonic string theory. The existence of these new high-energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions and is argued to be related to the high-energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory. Finally, our study also suggests that the nature of GSO projection in superstring theory might

be simplified in the high-energy limit. Hopefully, this is in connection with the conjecture that supersymmetry is realized in broken phase without GSO projection in the open string theory [33].

It would be of crucial importance to calculate high-energy massive fermion scattering amplitudes in the R-sector to complete the proof of Gross's two conjectures on high-energy symmetry of superstring theory. The construction of general *massive* spacetime fermion vertex, involving picture changing, will be the first step toward understanding of the high-energy behavior of superstring theory.

Acknowledgements

Appendix A. Solve the Virasoro conditions in the high-energy limit

Let repeat the Virasoro conditions on the general state at the mass level $M^2 = (2n - 1)$,

$$
G_{1/2}|n\rangle = \sum_{m_j} \left[k^{v_1^{1/2}} \boxed{v_1^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j=1}^k \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r \neq 1/2}^s \boxed{v_1^r \cdots v_{m_r}^r}^T \right.
$$

$$
+ \sum_{l \geq 1} \sum_{i=1}^{m_l} l \boxed{\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l} \otimes \boxed{\mu_i^l \, v_1^{l+1/2} \cdots v_{m_{l+1/2}}^{l+1/2}}^T
$$

$$
\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^k \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r \neq 1/2, l+1/2}^s \boxed{v_1^r \cdots v_{m_r}^r}^T
$$

$$
+ \sum_{i=2}^{m_{1/2}} \boxed{v_i^{1/2} \, \mu_1^1 \cdots \mu_{m_1}^1} \otimes (-1)^{i+1} \boxed{v_2^{1/2} \cdots \hat{v}_i^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T
$$

$$
\bigotimes_{j \neq l}^k \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r \neq 1/2}^s \boxed{v_1^r \cdots v_{m_r}^r}^T
$$

$$
+ \sum_{l \geq 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \boxed{v_i^{l-1/2} \, \mu_1^l \cdots \mu_{m_l}^l} \otimes \boxed{v_1^{l-1/2} \cdots \hat{v}_i^{l-1/2} \cdots v_{m_{l-1/2}}^{l-1/2}}^T
$$

$$
\left. \otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^k \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r \neq 1/2, l-1/2}^s \boxed{v_1^r \cdots v_{m_r}^r}^T \right]
$$

$$
\times \frac{1}{(m_{1/2}-1)!} \psi_{-1/2}^{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}} \prod_{j=1}^k \frac{1}{j^{m_j} m_j!} \alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \prod_{r \neq 1/2}^s \frac{1}{m_r!} \psi_{-r}^{v_1^r \cdots v_{m_r}^r}, \tag{A.1}
$$

and

$$
\begin{aligned}
G_{3/2}|n\rangle = \sum_{m_j} \Bigg[& \boxed{\nu_1^{3/2}\,\cdots\,\nu_{m_{3/2}}^{3/2}}^{\,T} k^{\nu_1^{3/2}} \bigotimes_{j=1}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \eta^{\mu\nu} \boxed{\mu\;\mu_1^1\,\cdots\,\mu_{m_1}^1} \otimes \boxed{\nu\;\nu_1^{1/2}\,\cdots\,\nu_{m_{1/2}}^{1/2}}^{\,T} \otimes \boxed{\nu_2^{3/2}\,\cdots\,\nu_{m_{3/2}}^{3/2}}^{\,T} \\
& \bigotimes_{j\neq 1}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 1/2,3/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \sum_{l\geqslant 1}\sum_{i=1}^{m_l} l \boxed{\mu_1^l\,\cdots\,\hat{\mu}_i^l\,\cdots\,\mu_{m_l}^l} \otimes \boxed{\mu_i^l\;\nu_1^{l+3/2}\,\cdots\,\nu_{m_{l+3/2}}^{l+3/2}}^{\,T} \\
& \otimes \boxed{\nu_2^{3/2}\,\cdots\,\nu_{m_{3/2}}^{3/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \sum_{i=2}^{m_{3/2}} 3 \boxed{\nu_i^{3/2}\;\mu_1^3\,\cdots\,\mu_{m_3}^3} \otimes (-1)^{i+1} \boxed{\nu_2^{3/2}\,\cdots\,\hat{\nu}_i^{3/2}\,\cdots\,\nu_{m_{3/2}}^{3/2}}^{\,T} \\
& \bigotimes_{j\neq 3}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \sum_{l\geqslant 2,l\neq 3}\sum_{i=1}^{m_{l-3/2}} \boxed{\nu_i^{l-3/2}\;\mu_1^l\,\cdots\,\mu_{m_l}^l} \otimes \boxed{\nu_1^{l-3/2}\,\cdots\,\hat{\nu}_i^{l-3/2}\,\cdots\,\nu_{m_{l-3/2}}^{l-3/2}}^{\,T} \\
& \otimes \boxed{\nu_2^{3/2}\,\cdots\,\nu_{m_{3/2}}^{3/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 3/2,l-3/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \Bigg] \\
& \times \frac{1}{(m_{3/2}-1)!}\psi_{-3/2}^{\nu_2^{3/2}\cdots\nu_{m_{3/2}}^{3/2}} \prod_{j=1}^{k} \frac{1}{j^{m_j} m_j!}\alpha_{-j}^{\mu_1^j\cdots\mu_{m_j}^j} \prod_{r\neq 3/2}^{s} \frac{1}{m_r!}\psi_{-r}^{\nu_1^r\cdots\nu_{m_r}^r}.
\end{aligned}
\tag{A.2}
$$

We then obtain the constrain equations

$$
\begin{aligned}
0 = & k^{\nu_1^{1/2}} \boxed{\nu_1^{1/2}\,\cdots\,\nu_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j=1}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 1/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \sum_{l\geqslant 1}\sum_{i=1}^{m_l} l \boxed{\mu_1^l\,\cdots\,\hat{\mu}_i^l\,\cdots\,\mu_{m_l}^l} \otimes \boxed{\mu_i^l\;\nu_1^{l+1/2}\,\cdots\,\nu_{m_{l+1/2}}^{l+1/2}}^{\,T} \\
& \otimes \boxed{\nu_2^{1/2}\,\cdots\,\nu_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\cdots\,\mu_{m_j}^j} \bigotimes_{r\neq 1/2,l+1/2}^{s} \boxed{\nu_1^r\,\cdots\,\nu_{m_r}^r}^{\,T} \\
& + \sum_{i=2}^{m_{1/2}} \boxed{\nu_i^{1/2}\;\mu_1^1\,\cdots\,\mu_{m_1}^1} \otimes (-1)^{i+1} \boxed{\nu_2^{1/2}\,\cdots\,\hat{\nu}_i^{1/2}\,\cdots\,\nu_{m_{1/2}}^{1/2}}^{\,T}
\end{aligned}
$$

$$\bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \sum_{l\geqslant 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \boxed{v_i^{l-1/2}\, \mu_1^l \cdots \mu_{m_l}^l} \otimes \boxed{v_1^{l-1/2} \cdots \hat{v}_i^{l-1/2} \cdots v_{m_{l-1/2}}^{l-1/2}}^{T}$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^{T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2,\,l-1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}, \tag{A.3a}$$

$$0 = \boxed{v_1^{3/2} \cdots v_{m_{3/2}}^{3/2}}^{T} k^{v_1^{3/2}} \bigotimes_{j=1}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \eta^{\mu v} \boxed{\mu\, \mu_1^1 \cdots \mu_{m_1}^1} \otimes \boxed{v\, v_1^{1/2} \cdots v_{m_{1/2}}^{1/2}}^{T} \otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^{T}$$

$$\bigotimes_{j\neq 1}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2,\,3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \sum_{l\geqslant 1} \sum_{i=1}^{m_l} l \boxed{\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l} \otimes \boxed{\mu_i^l\, v_1^{l+3/2} \cdots v_{m_{l+3/2}}^{l+3/2}}^{T}$$

$$\otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^{T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2,\,l+3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \sum_{i=2}^{m_{3/2}} 3 \boxed{v_i^{3/2}\, \mu_1^3 \cdots \mu_{m_3}^3} \otimes (-1)^{i+1} \boxed{v_2^{3/2} \cdots \hat{v}_i^{3/2} \cdots v_{m_{3/2}}^{3/2}}^{T}$$

$$\bigotimes_{j\neq 3}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \sum_{l\geqslant 2,\,l\neq 3} \sum_{i=1}^{m_{l-3/2}} \boxed{v_i^{l-3/2}\, \mu_1^l \cdots \mu_{m_l}^l} \otimes (-1)^{i+1} \boxed{v_1^{l-3/2} \cdots \hat{v}_i^{l-3/2} \cdots v_{m_{l-3/2}}^{l-3/2}}^{T}$$

$$\otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^{T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2,\,l-3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}. \tag{A.3b}$$

Taking the high-energy limit in the above equations by letting $(\mu_i, v_i) \to (L, T)$, and

$$k^{\mu_i} \to M e^L, \qquad \eta^{\mu_1 \mu_2} \to e^T e^T. \tag{A.4}$$

We get

$$0 = M \boxed{L\, v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^{T} \bigotimes_{j=1}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^{T}$$

$$+ \sum_{l\geqslant 1} \sum_{i=1}^{m_l} l \boxed{\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l} \otimes \boxed{\mu_i^l\, v_1^{l+1/2} \cdots v_{m_{l+1/2}}^{l+1/2}}^{T}$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2,l+1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{i=2}^{m_{1/2}} \boxed{v_i^{1/2} \mu_1^1 \cdots \mu_{m_1}^1} \otimes (-1)^{i+1} \boxed{v_2^{1/2} \cdots \hat{v}_i^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T$$

$$\bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \boxed{v_i^{l-1/2} \mu_1^l \cdots \mu_{m_l}^l} \otimes \boxed{v_1^{l-1/2} \cdots \hat{v}_i^{l-1/2} \cdots v_{m_{l-1/2}}^{l-1/2}}^T$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2,l-1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T , \qquad (A.5a)$$

$$0 = M \boxed{L \; v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j=1}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \boxed{T \; \mu_1^1 \cdots \mu_{m_1}^1} \otimes \boxed{T \; v_1^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T$$

$$\bigotimes_{j\neq 1}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 1/2,3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 1} \sum_{i=1}^{m_l} l \boxed{\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l} \otimes \boxed{\mu_i^l \; v_1^{l+3/2} \cdots v_{m_{l+3/2}}^{l+3/2}}^T$$

$$\otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{i=2}^{m_{3/2}} 3 \boxed{v_i^{3/2} \mu_1^3 \cdots \mu_{m_3}^3} \otimes (-1)^{i+1} \boxed{v_2^{3/2} \cdots \hat{v}_i^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T$$

$$\bigotimes_{j\neq 3}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 2,\, l\neq 3} \sum_{i=1}^{m_{l-3/2}} \boxed{v_i^{l-3/2} \mu_1^l \cdots \mu_{m_l}^l} \otimes (-1)^{i+1} \boxed{v_1^{l-3/2} \cdots \hat{v}_i^{l-3/2} \cdots v_{m_{l-3/2}}^{l-3/2}}^T$$

$$\otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r\neq 3/2,l-3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T . \qquad (A.5b)$$

The indices $\{\mu_i^j\}$ are symmetric and can be chosen to have l_j of $\{L\}$ and $\{T\}$, while $\{v_i^r\}$ are antisymmetric and we keep them as what they are at this moment. Thus

$$
0 = M \boxed{L\,\big|\,v_2^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j=1}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}
$$

$$
+ \sum_{l\geqslant 1} l\,\boxed{\underbrace{T\,\big|\cdots\big|\,T}_{m_l-1-l_l}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_l}} \otimes \boxed{\mu_1^l\,\big|\,v_1^{l+1/2}\,\big|\cdots\big|\,v_{m_{l+1/2}}^{l+1/2}}^{\,T}
$$

$$
\otimes \boxed{v_2^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2,\,l+1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}
$$

$$
+ \sum_{l\geqslant 1} l(m_l-1-l_l)\,\boxed{\mu_1^l\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_l-2-l_l}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_l}} \otimes \boxed{T\,\big|\,v_1^{l+1/2}\,\big|\cdots\big|\,v_{m_{l+1/2}}^{l+1/2}}^{\,T}
$$

$$
\otimes \boxed{v_2^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2,\,l+1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}
$$

$$
+ \sum_{l\geqslant 1} l\,l_l\,\boxed{\mu_1^l\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_l-1-l_l}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_l-1}} \otimes \boxed{L\,\big|\,v_1^{l+1/2}\,\big|\cdots\big|\,v_{m_{l+1/2}}^{l+1/2}}^{\,T}
$$

$$
\otimes \boxed{v_2^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2,\,l+1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}
$$

$$
+ \sum_{i=2}^{m_{1/2}} \boxed{v_i^{1/2}\,\big|\,\mu_1^1\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_1-1-l_1}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_1}} \otimes (-1)^{i+1}\boxed{v_2^{1/2}\,\big|\cdots\big|\,\hat{v}_i^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T}
$$

$$
\bigotimes_{j\neq 1}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}
$$

$$
+ \sum_{l\geqslant 2}\sum_{i=1}^{m_{l-1/2}} (-1)^{i+1}\boxed{v_i^{l-1/2}\,\big|\,\mu_1^l\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_l-1-l_l}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_l}}
$$

$$
\otimes \boxed{v_1^{l-1/2}\,\big|\cdots\big|\,\hat{v}_i^{l-1/2}\,\big|\cdots\big|\,v_{m_{l-1/2}}^{l-1/2}}^{\,T}
$$

$$
\otimes \boxed{v_2^{1/2}\,\big|\cdots\big|\,v_{m_{1/2}}^{1/2}}^{\,T} \bigotimes_{j\neq l}^{k} \boxed{\mu_1^j\,\big|\,\underbrace{T\,\big|\cdots\big|\,T}_{m_j-1-l_j}\,\big|\,\underbrace{L\,\big|\cdots\big|\,L}_{l_j}} \bigotimes_{r\neq 1/2,\,l-1/2}^{s} \boxed{v_1^r\,\big|\cdots\big|\,v_{m_r}^r}^{\,T}, \tag{A.6a}
$$

$$0 = M \boxed{L}\boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T \bigotimes_{j=1}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \mu_1^1 \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \boxed{T}\boxed{v_1^{1/2}}\cdots\boxed{v_{m_{1/2}}^{1/2}}^T \otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T$$

$$\bigotimes_{j\neq 1}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 1} l \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l} \otimes \mu_1^l \boxed{v_1^{l+3/2}}\cdots\boxed{v_{m_{l+3/2}}^{l+3/2}}^T$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 1} l(m_l - 1 - l_l) \mu_1^l \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-2-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l} \otimes \boxed{T}\boxed{v_1^{l+3/2}}\cdots\boxed{v_{m_{l+3/2}}^{l+3/2}}^T$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 1} l l_l \mu_1^l \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l-1} \otimes \boxed{L}\boxed{v_1^{l+3/2}}\cdots\boxed{v_{m_{l+3/2}}^{l+3/2}}^T$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T \bigotimes_{j\neq l}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \sum_{i=2}^{m_{3/2}} 3 \boxed{v_i^{3/2}}\mu_1^3 \underbrace{\boxed{T}\cdots\boxed{T}}_{m_3-1-l_3}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_3} \otimes (-1)^{i+1}\boxed{v_2^{3/2}}\cdots\boxed{\hat{v}_i^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^T$$

$$\bigotimes_{j\neq 3}^{k} \mu_1^j \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-1-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^T$$

$$+ \sum_{l\geqslant 2, l\neq 3} \sum_{i=1}^{m_{l-3/2}} \boxed{v_i^{l-3/2}}\mu_1^l \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}$$

$$\otimes (-1)^{i+1}\boxed{v_1^{l-3/2}}\cdots\boxed{\hat{v}_i^{l-3/2}}\cdots\boxed{v_{m_{l-3/2}}^{l-3/2}}^T$$

$$\otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j \neq l}^{k} \boxed{\mu_1^j \; \underbrace{\boxed{T \cdots T}}_{m_j - 1 - l_j} \underbrace{\boxed{L \cdots L}}_{l_j}} \bigotimes_{r \neq 3/2, l-3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T . \tag{A.6b}$$

There are some undetermined parameters, which can be L or T, in the above equations. However, it is easy to see that both choice lead to the same equations. Therefore, we will set all of them to be T in the following. Thus, the constrain equations become

$$0 = M \boxed{L \; v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j=1}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l \geqslant 1} l(m_l - l_l) \underbrace{\boxed{T \cdots T}}_{m_l - 1 - l_l} \underbrace{\boxed{L \cdots L}}_{l_l} \otimes \boxed{T \; v_1^{l+1/2} \cdots v_{m_{l+1/2}}^{l+1/2}}^T$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 1/2, l+1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l \geqslant 1} l l_l \underbrace{\boxed{T \cdots T}}_{m_l - l_l} \underbrace{\boxed{L \cdots L}}_{l_l - 1} \otimes \boxed{L \; v_1^{l+1/2} \cdots v_{m_{l+1/2}}^{l+1/2}}^T$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 1/2, l+1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{i=2}^{m_{1/2}} \boxed{v_i^{1/2}} \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes (-1)^{i+1} \boxed{v_2^{1/2} \cdots \hat{v}_i^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T$$

$$\bigotimes_{j \neq 1}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \sum_{l \geqslant 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \boxed{v_i^{l-1/2}} \underbrace{\boxed{T \cdots T}}_{m_l - l_l} \underbrace{\boxed{L \cdots L}}_{l_l} \otimes \boxed{v_1^{l-1/2} \cdots \hat{v}_i^{l-1/2} \cdots v_{m_{l-1/2}}^{l-1/2}}^T$$

$$\otimes \boxed{v_2^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \bigotimes_{j \neq l}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 1/2, l-1/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T , \tag{A.7a}$$

$$0 = M \boxed{L \; v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T \bigotimes_{j=1}^{k} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \bigotimes_{r \neq 3/2}^{s} \boxed{v_1^r \cdots v_{m_r}^r}^T$$

$$+ \underbrace{\boxed{T \cdots T}}_{m_1 + 1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \boxed{T \; v_1^{1/2} \cdots v_{m_{1/2}}^{1/2}}^T \otimes \boxed{v_2^{3/2} \cdots v_{m_{3/2}}^{3/2}}^T$$

$$\bigotimes_{j\neq 1}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,3/2}^{s} \boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^{T}$$

$$+\sum_{l\geqslant 1} l(m_l-l_l)\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}\otimes \boxed{T}\boxed{v_1^{l+3/2}}\cdots\boxed{v_{m_{l+3/2}}^{l+3/2}}^{T}$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 3/2,l+3/2}^{s}\boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^{T}$$

$$+\sum_{l\geqslant 1} ll_l\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l-1}\otimes \boxed{L}\boxed{v_1^{l+3/2}}\cdots\boxed{v_{m_{l+3/2}}^{l+3/2}}^{T}$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 3/2,l+3/2}^{s}\boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^{T}$$

$$+\sum_{i=2}^{m_{3/2}} 3\,\boxed{v_i^{3/2}}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_3-l_3}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_3}\otimes (-1)^{i+1}\boxed{v_2^{3/2}}\cdots\boxed{\hat{v}_i^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^{T}$$

$$\bigotimes_{j\neq 3}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^{T}$$

$$+\sum_{l\geqslant 2,l\neq 3}\sum_{i=1}^{m_{l-3/2}}\boxed{v_i^{l-3/2}}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}$$

$$\otimes (-1)^{i+1}\boxed{v_1^{l-3/2}}\cdots\boxed{\hat{v}_i^{l-3/2}}\cdots\boxed{v_{m_{l-3/2}}^{l-3/2}}^{T}$$

$$\otimes \boxed{v_2^{3/2}}\cdots\boxed{v_{m_{3/2}}^{3/2}}^{T}\bigotimes_{j\neq l}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 3/2,l-3/2}^{s}\boxed{v_1^r}\cdots\boxed{v_{m_r}^r}^{T}. \qquad (A.7b)$$

Next, we will deal with those antisymmetric indices $\{v_i^r\}$. In this case, there are much fewer possibilities which we can chosen, i.e.,

$$\boxed{v_1^r}\cdots\boxed{v_{m_r}^r}\equiv\boxed{v_1^r}\boxed{v_2^r}\quad\left(=\boxed{T}\boxed{L},\ \boxed{T},\ \boxed{L}\text{ or }\boxed{0}\right). \qquad (A.8)$$

Therefore

$$0=M\boxed{L}\boxed{v_2^{1/2}}^{T}\bigotimes_{j=1}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 1/2}^{s}\boxed{v_1^r}\boxed{v_2^r}^{T}$$

$$+\sum_{l\geqslant 1} l(m_l-l_l)\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}\otimes \boxed{T}\boxed{v_1^{l+1/2}}\boxed{v_2^{l+1/2}}^{T}$$

$$
\otimes \boxed{v_2^{1/2}} \bigotimes_{j\neq l}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,l+1/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ \sum_{l\geqslant 1} l l_l \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l-1}\otimes\boxed{L}\boxed{v_1^{l+1/2}}\boxed{v_2^{l+1/2}}^{T}
$$

$$
\otimes \boxed{v_2^{1/2}} \bigotimes_{j\neq l}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,l+1/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ (-1)\boxed{v_2^{1/2}}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1}\bigotimes_{j\neq 1}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 1/2}^{s}\boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ \sum_{l\geqslant 2}\boxed{v_1^{l-1/2}}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}\otimes\boxed{v_2^{l-1/2}}
$$

$$
\otimes \boxed{v_2^{1/2}} \bigotimes_{j\neq l}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,l-1/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ \sum_{l\geqslant 2}(-1)\boxed{v_2^{l-1/2}}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}\otimes\boxed{v_1^{l-1/2}}
$$

$$
\otimes \boxed{v_2^{1/2}} \bigotimes_{j\neq l}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,l-1/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}, \tag{A.9}
$$

$$
0 = M\boxed{L}\boxed{v_2^{3/2}}^{T}\bigotimes_{j=1}^{k}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j}\bigotimes_{r\neq 3/2}^{s}\boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1+1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1}\otimes\boxed{T}\boxed{v_1^{1/2}}\boxed{v_2^{1/2}}^{T}
$$

$$
\otimes \boxed{v_2^{3/2}} \bigotimes_{j\neq 1}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 1/2,3/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$
+ \sum_{l\geqslant 1} l(m_l-l_l) \underbrace{\boxed{T}\cdots\boxed{T}}_{m_l-1-l_l}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_l}\otimes\boxed{T}\boxed{v_1^{l+3/2}}\boxed{v_2^{l+3/2}}^{T}
$$

$$
\otimes \boxed{v_2^{3/2}} \bigotimes_{j\neq l}^{k} \underbrace{\boxed{T}\cdots\boxed{T}}_{m_j-l_j}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_j} \bigotimes_{r\neq 3/2,l+3/2}^{s} \boxed{v_1^r}\boxed{v_2^r}^{T}
$$

$$+ \sum_{l \geqslant 1} l l_l \underbrace{\boxed{T \cdots T}}_{m_l - l_l} \underbrace{\boxed{L \cdots L}}_{l_l - 1} \otimes \boxed{L \; v_1^{l+3/2} \; v_2^{l+3/2}}^T$$

$$\otimes \boxed{v_2^{3/2}} \overset{k}{\underset{j \neq l}{\bigotimes}} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \overset{s}{\underset{r \neq 3/2, l+3/2}{\bigotimes}} \boxed{v_1^r \; v_2^r}^T$$

$$+ 3(-1) \boxed{v_2^{3/2}} \underbrace{\boxed{T \cdots T}}_{m_3 - l_3} \underbrace{\boxed{L \cdots L}}_{l_3} \overset{k}{\underset{j \neq 3}{\bigotimes}} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \overset{s}{\underset{r \neq 3/2}{\bigotimes}} \boxed{v_1^r \; v_2^r}^T$$

$$+ \sum_{l \geqslant 2, l \neq 3} \boxed{v_1^{l-3/2}} \underbrace{\boxed{T \cdots T}}_{m_l - l_l} \underbrace{\boxed{L \cdots L}}_{l_l} \otimes \boxed{v_2^{l-3/2}}$$

$$\otimes \boxed{v_2^{3/2}} \overset{k}{\underset{j \neq l}{\bigotimes}} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \overset{s}{\underset{r \neq 3/2, l-3/2}{\bigotimes}} \boxed{v_1^r \; v_2^r}^T$$

$$+ \sum_{l \geqslant 2, l \neq 3} \boxed{v_2^{l-3/2}} \underbrace{\boxed{T \cdots T}}_{m_l - l_l} \underbrace{\boxed{L \cdots L}}_{l_l} \otimes (-1) \boxed{v_1^{l-3/2}}$$

$$\otimes \boxed{v_2^{3/2}} \overset{k}{\underset{j \neq l}{\bigotimes}} \underbrace{\boxed{T \cdots T}}_{m_j - l_j} \underbrace{\boxed{L \cdots L}}_{l_j} \overset{s}{\underset{r \neq 3/2, l-3/2}{\bigotimes}} \boxed{v_1^r \; v_2^r}^T. \tag{A.10}$$

Using the lemma (59), Eqs. (A.9) and (A.10) reduce to

$$0 = M \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L \; v_2^{1/2} \; v_3^{1/2}}^T \otimes \boxed{v_1^{3/2}}$$

$$+ l_1 \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1 - 1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2} \; v_3^{1/2}}^T \otimes \boxed{L \; v_1^{3/2}}^T$$

$$- \boxed{v_2^{1/2}} \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_3^{1/2}} \otimes \boxed{v_1^{3/2}}$$

$$+ \boxed{v_3^{1/2}} \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2}} \otimes \boxed{v_1^{3/2}}$$

$$+ \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \boxed{v_1^{3/2}} \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2} \; v_3^{1/2}}^T \otimes \boxed{0}$$

$$- \underbrace{\boxed{T \cdots T}}_{m_1 - l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2} \; v_3^{1/2}}^T \otimes \boxed{v_1^{3/2}}, \tag{A.11a}$$

$$
0 = M \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2}\, v_3^{1/2}}^{T} \otimes \boxed{L\ v_1^{3/2}}^{T}
$$

$$
+ \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T\ v_2^{1/2}\, v_3^{1/2}}^{T} \otimes \boxed{v_1^{3/2}}
$$

$$
- \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \boxed{v_2^{1/2}} \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_3^{1/2}} \otimes \boxed{v_1^{3/2}}
$$

$$
+ \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \boxed{v_3^{1/2}} \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{v_2^{1/2}} \otimes \boxed{v_1^{3/2}}. \tag{A.11b}
$$

From the first equation we have:

For $v_2^{1/2} = 0$, $v_3^{1/2} = 0$ and $v_1^{3/2} = 0$,

$$
0 = M \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L} \otimes \boxed{0}
$$

$$
+ l_1 \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1-1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L}. \tag{A.12}
$$

For $v_2^{1/2} = T$, $v_3^{1/2} = L$ and $v_1^{3/2} = 0$,

$$
0 = l_1 \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1-1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T\ L}^{T} \otimes \boxed{L}
$$

$$
- \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1+1-l_1 \quad l_1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L} \otimes \boxed{0}
$$

$$
+ \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1+1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T} \otimes \boxed{0}. \tag{A.13}
$$

For $v_2^{1/2} = T$, $v_3^{1/2} = 0$ and $v_1^{3/2} = L$,

$$
0 = -M \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T\ L}^{T} \otimes \boxed{L}
$$

$$
+ (-1) \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1+1-l_1 \quad l_1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L}
$$

$$
+ \underbrace{\boxed{T \cdots T}\boxed{L \cdots L}}_{\substack{m_1-l_1 \quad l_1}} \otimes \underbrace{\boxed{L \cdots L}}_{m_2+1} \otimes \boxed{T} \otimes \boxed{0}. \tag{A.14}
$$

For $\nu_2^{1/2} = L$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$0 = (-1)\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1+1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L}$$
$$+ \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2+1} \otimes \boxed{L} \otimes \boxed{0}. \tag{A.15}$$

From the second equation we have:
For $\nu_2^{1/2} = 0$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = 0$,

$$0 = M\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L}$$
$$+ \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{T} \otimes \boxed{0}. \tag{A.16}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = L$ and $\nu_1^{3/2} = 0 \Rightarrow$,

$$0 = M\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{T}\boxed{L}^T \otimes \boxed{L}$$
$$+ \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2+1} \otimes \boxed{T} \otimes \boxed{0}. \tag{A.17}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$0 = 0. \tag{A.18}$$

For $\nu_2^{1/2} = L$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$0 = \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{T}\boxed{L}^T \otimes \boxed{L}$$
$$- \underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2+1} \otimes \boxed{0} \otimes \boxed{L}. \tag{A.19}$$

Using Eqs. (A.12) and (A.15), we get

$$\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1-1} \otimes \underbrace{\boxed{L}\cdots\boxed{L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L}$$
$$= \frac{l_1!!(-M)^{m_2}}{(l_1+2m_2-2)!!}\underbrace{\boxed{T}\cdots\boxed{T}}_{m_1-l_1}\underbrace{\boxed{L}\cdots\boxed{L}}_{l_1-1+2m_2} \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \tag{A.20}$$

then using Eqs. (A.14), (A.16) and (A.19), we obtain

$$
\underbrace{\boxed{T}\cdots\boxed{T}}_{n-2m_2-2k}\underbrace{\boxed{L}\cdots\boxed{L}}_{2k}\otimes\underbrace{\boxed{L}\cdots\boxed{L}}_{m_2}\otimes\boxed{0}\otimes\boxed{L}
$$

$$
=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^{k}}\underbrace{\boxed{T}\cdots\boxed{T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{A.21}
$$

Eq. (A.12) leads to

$$
\underbrace{\boxed{T}\cdots\boxed{T}}_{n-2m_2-2k}\underbrace{\boxed{L}\cdots\boxed{L}}_{2k+1}\otimes\underbrace{\boxed{L}\cdots\boxed{L}}_{m_2}\otimes\boxed{L}\otimes\boxed{0}
$$

$$
=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k+1)!!}{(-M)^{k+1}}\underbrace{\boxed{T}\cdots\boxed{T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{A.22}
$$

Eq. (A.16) leads to

$$
\underbrace{\boxed{T}\cdots\boxed{T}}_{n-2m_2-2k+1}\underbrace{\boxed{L}\cdots\boxed{L}}_{2k}\otimes\underbrace{\boxed{L}\cdots\boxed{L}}_{m_2}\otimes\boxed{T}\otimes\boxed{0}
$$

$$
=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^{k-1}}\underbrace{\boxed{T}\cdots\boxed{T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}, \tag{A.23}
$$

Eq. (A.19) leads to

$$
\underbrace{\boxed{T}\cdots\boxed{T}}_{n-2m_2-2k+1}\underbrace{\boxed{L}\cdots\boxed{L}}_{2k}\otimes\underbrace{\boxed{L}\cdots\boxed{L}}_{m_2-1}\otimes\boxed{T}\,\boxed{L}^{T}\otimes\boxed{L}
$$

$$
=\left(-\frac{1}{2M}\right)^{m_2}\left(-\frac{1}{2M}\right)^{k}\frac{(2k-1)!!}{(-M)^{k}}\underbrace{\boxed{T}\cdots\boxed{T}}_{n}\otimes\boxed{0}\otimes\boxed{0}\otimes\boxed{L}. \tag{A.24}
$$

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