Available online at www.sciencedirect.com
SCIENCE DIRECT®
Theoretical Computer Science 339 (2005) 257–271
Theoretical Computer Science
www.elsevier.com/locate/tcs

The super connectivity of the pancake graphs and the super laceability of the star graphs

Cheng-Kuan Lin[a], Hua-Min Huang[a], Lih-Hsing Hsu[b,*]

[a] *Department of Mathematics, National Central University, Chung-Li, Taiwan 32054, ROC*
[b] *Information Engineering Department, Ta Hwa Institute of Technology, Hsinchu, Taiwan 307, ROC*

Received 8 April 2004; received in revised form 31 January 2005; accepted 21 February 2005

Communicated by D-Z. Du

Abstract

A *k-container* $C(u, v)$ of a graph G is a set of k-disjoint paths joining u to v. A k-container $C(u, v)$ of G is a k^**-container* if it contains all the vertices of G. A graph G is k^**-connected* if there exists a k^*-container between any two distinct vertices. Let $\kappa(G)$ be the connectivity of G. A graph G is *super connected* if G is i^*-connected for all $1 \leqslant i \leqslant \kappa(G)$. A bipartite graph G is k^**-laceable* if there exists a k^*-container between any two vertices from different parts of G. A bipartite graph G is *super laceable* if G is i^*-laceable for all $1 \leqslant i \leqslant \kappa(G)$. In this paper, we prove that the n-dimensional pancake graph P_n is super connected if and only if $n \neq 3$ and the n-dimensional star graph S_n is super laceable if and only if $n \neq 3$.

Keywords: Hamiltonian; Hamiltonian connected; Hamiltonian laceable; Connectivity

1. Introduction

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer

* Corresponding author. Department of Computer and Information Science, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan, ROC. Tel.: +886 3 5720659; fax: +886 3 5721490.
E-mail address: lhhsu@cis.nctu.edu.tw (L.-H. Hsu).

0304-3975/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2005.02.007

networks. The n-cube is one of the most popular topologies [18]. The n-dimensional star network S_n was proposed in [1] as "an attractive alternative to the n-cube" topology for interconnecting processors in parallel computers. Since its introduction, the network has received considerable attention. Akers et al. [1] showed that the star graphs are vertex transitive and edge transitive. The diameter and fault diameters were computed in [1,17,22,23]. The hamiltonian and hamiltonian laceability of star graphs are studied in [12,15,19]. In particular, Fragopoulou and Akl [7,8] studied the embedding of $(n-1)$ directed edge-disjoint spanning trees on the star network S_n. These spanning trees are used in communication algorithms for star networks.

Akers et al. [1] also proposed another family of interesting graphs, the n-dimensional pancake graph P_n. They also showed that the pancake graphs are vertex transitive. Hung et al. [14] studied the hamiltonian connectivity on the faulty pancake graphs. The embedding of cycles and trees into the pancake graphs where discussed in [6,14,16]. Gates and Papadimitriou [10] studied the diameter of the pancake graphs. Until now, we do not know the exact value of the diameter of the pancake graphs [11].

For the graph definition and notation, we follow [3]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. A *path* of length k from x to y is a sequence of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k \rangle$, where $x = v_0$, $y = v_k$, and $(v_{i-1}, v_i) \in E$ for all $1 \leqslant i \leqslant k$. We also write the path $\langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle$, where Q is a path from v_i to v_j. Note that we allow Q to be a path of length zero. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ as $\langle v_0, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_t, \ldots, v_k \rangle$, where Q_1 is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_t \rangle$. We use $d(u, v)$ to denote the *distance* between u and v, i.e., the length of the shortest path joining u and v.

A path of graph G from u to v is a *hamiltonian path* if it contains all vertices of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct vertices. A *cycle* is a path (except that the first vertex is the same as the last vertex) containing at least three vertices. A cycle of G is a *hamiltonian cycle* if it contains all vertices. A graph is *hamiltonian* if it has a hamiltonian cycle.

The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's Theorem [20] that there are k *internal vertex-disjoint* (abbreviated as *disjoint*) *paths* joining any two distinct vertices u and v for any $k \leqslant \kappa(G)$. A k-container $C(u, v)$ of G is a set of k disjoint paths joining u to v. In this paper, we discuss another type of container. A k-container $C(u, v)$ is a *k^*-container* if it contains all vertices of G. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices. In particular, a graph G is 1^*-connected if and only if it is hamiltonian connected, and a graph G is 2^*-connected if and only if it is hamiltonian. All 1^*-connected graphs except that K_1 and K_2 are 2^*-connected. The study of k^*-connected graphs is motivated by the globally 3^*-connected graphs proposed by Albert et al. [2]. A graph G is *super connected* if it is i^*-connected for all $1 \leqslant i \leqslant \kappa(G)$. In this paper, we will prove that the pancake graph P_n is super connected if and only if $n \neq 3$.

A graph G is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge joins vertices between V_1 and V_2. Let G be a k-connected bipartite graph with bipartition V_1 and V_2 such that $|V_1| \geqslant |V_2|$. Suppose that there exists a k^*-container $C(u, v) = \{P_1, P_2, \ldots, P_k\}$ in a bipartite graph joining u to v with $u, v \in V_1$. Obviously, the number of vertices in P_i is $2k_i + 1$ for some integer k_i. There are $k_i - 1$ vertices of P_i in V_1 other than u and v, and k_i vertices of P_i in V_2. As a consequence, $|V_1| = \sum_{i=1}^{k}(k_i - 1) + 2$ and $|V_2| = \sum_{i=1}^{k} k_i$. Therefore, any bipartite graph G with $\kappa(G) \geqslant 3$ is not k^*-connected for any $3 \leqslant k \leqslant \kappa(G)$.

For this reason, a bipartite graph is *k^*-laceable* if there exists a k^*-container between any two vertices from different partite sets. Obviously, any bipartite k^*-laceable graph with $k \geqslant 2$ has the equal size of bipartition. A 1^*-laceable graph is also known as *hamiltonian laceable graph*. Moreover, a graph G is 2^*-laceable if and only if it is hamiltonian. All 1^*-laceable graphs except that K_1 and K_2 are 2^*-laceable. A bipartite graph G is *super laceable* if G is i^*-laceable for all $1 \leqslant i \leqslant \kappa(G)$. In this paper, we will prove that the star graph S_n is super laceable if and only if $n \neq 3$.

In the following section, we give the definition of the pancake graphs and discuss some of their properties. In Section 3, we prove that the pancake graph P_n is super connected if and only if $n \neq 3$. The definition of the star graphs and some of their properties are presented in Section 4. In Section 5, we prove that the star graph S_n is super laceable if and only if $n \neq 3$. In the final section, we discuss further research.

2. The pancake graphs

Let n be a positive integer. We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$. The *n-dimensional pancake graph*, denoted by P_n, is a graph with the vertex set $V(P_n) = \{u_1u_2 \ldots u_n \mid u_i \in \langle n \rangle$ and $u_i \neq u_j$ for $i \neq j\}$. The adjacency is defined as follows: $u_1u_2 \ldots u_i \ldots u_n$ is adjacent to $v_1v_2 \ldots v_i \ldots v_n$ through an edge of dimension i with $2 \leqslant i \leqslant n$ if $v_j = u_{i-j+1}$ for all $1 \leqslant j \leqslant i$ and $v_j = u_j$ for all $i < j \leqslant n$. We will use bold face to denote a vertex of P_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denote a sequence of vertices in P_n. In particular, $\mathbf{e}$ denotes the vertex $12 \ldots n$. By definition, P_n is an $(n-1)$-regular graph with $n!$ vertices.

Let $\mathbf{u} = u_1u_2 \ldots u_n$ be any vertex of P_n. We use $(\mathbf{u})_i$ to denote the ith component u_i of $\mathbf{u}$, and use $P_n^{\{i\}}$ to denote the ith subgraph of P_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Obviously, P_n can be decomposed into n vertex disjoint subgraphs $P_n^{\{i\}}$ for every $i \in \langle n \rangle$ such that each $P_n^{\{i\}}$ is isomorphic to P_{n-1}. Thus, the pancake graph can be constructed recursively. Let $H \subseteq \langle n \rangle$, we use P_n^H to denote the subgraph of P_n induced by $\cup_{i \in H} V(P_n^{\{i\}})$. By definition, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \leqslant i \leqslant n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in P_n^{\{(\mathbf{u})_1\}}$. For $1 \leqslant i, j \leqslant n$ and $i \neq j$, we use $E^{i,j}$ to denote the set of edges between $P_n^{\{i\}}$ and $P_n^{\{j\}}$. The pancake graphs P_2, P_3, and P_4 are shown in Fig. 1 for illustration.

The following theorem is proved by Hung et al. [14].

Theorem 1 (*Hung et al. [14]*). *P_n is 1^*-connected if $n \neq 3$, and P_n is 2^*-connected if $n \geqslant 3$.*

Lemma 1. *Assume that $n \geqslant 3$. $|E^{i,j}| = (n-2)!$ for any $1 \leqslant i, j \leqslant n$ with $i \neq j$.*

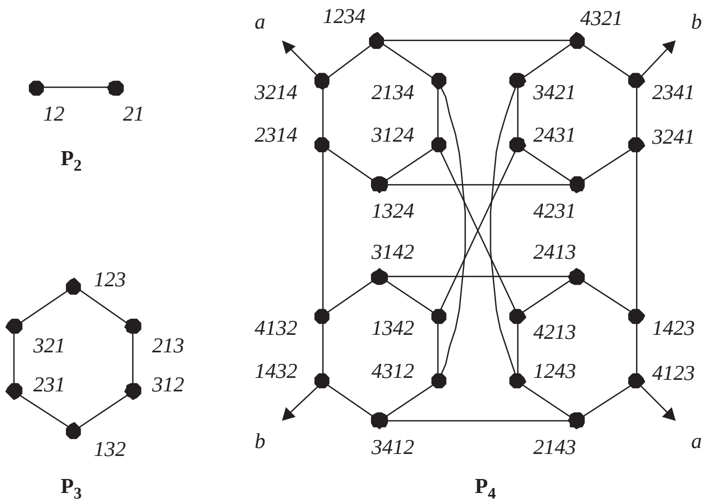

Fig. 1. The pancake graphs P_2, P_3, and P_4.

Lemma 2. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two distinct vertices of* P_n *with* $d(\mathbf{u}, \mathbf{v}) \leqslant 2$. *Then* $(\mathbf{u})_1 \neq (\mathbf{v})_1$. *Moreover,* $\{((\mathbf{u})^i)_1 \mid 2 \leqslant i \leqslant n-1\} = \langle n \rangle - \{(\mathbf{u})_1, (\mathbf{u})_n\}$ *if* $n \geqslant 3$.

Lemma 3. *Let* $n \geqslant 5$ *and* $H = \{i_1, i_2, \ldots, i_m\}$ *be any nonempty subset of* $\langle n \rangle$. *There is a hamiltonian path of* P_n^H *joining any vertex* $\mathbf{u} \in P_n^{\{i_1\}}$ *to any other vertex* $\mathbf{v} \in P_n^{\{i_m\}}$.

Proof. Note that $P_n^{\{i_j\}}$ is isomorphic to P_{n-1} for every $1 \leqslant j \leqslant m$. We set $\mathbf{x_1} = \mathbf{u}$ and $\mathbf{y_m} = \mathbf{v}$. By Theorem 1, this theorem holds for $m = 1$. Assume that $m \geqslant 2$. By Lemma 1, we choose $(\mathbf{y_j}, \mathbf{x_{j+1}}) \in E^{i_j, i_{j+1}}$ with $\mathbf{y_j} \neq \mathbf{x_j}$ and $\mathbf{y_m} \neq \mathbf{x_m}$ for every $1 \leqslant j \leqslant m-1$. By Theorem 1, there is a hamiltonian path Q_j of $P_n^{\{i_j\}}$ joining $\mathbf{x_j}$ to $\mathbf{y_j}$ for every $1 \leqslant j \leqslant m$. The path $\langle \mathbf{x_1}, Q_1, \mathbf{y_1}, \mathbf{x_2}, Q_2, \mathbf{y_2}, \ldots, \mathbf{x_m}, Q_m, \mathbf{y_m} \rangle$ forms a desired path. □

3. The super connectivity of the pancake graphs

Lemma 4. *Let* $n \geqslant 5$. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two distinct vertices in* $P_n^{\{t\}}$ *for some* $t \in \langle n \rangle$. *If* P_{n-1} *is* k^**-connected, then there is a* $(k+1)^*$*-container of* P_n *between* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. Since $P_n^{\{t\}}$ is isomorphic to P_{n-1}, there is a k^*-container $\{Q_1, Q_2, \ldots, Q_k\}$ of $P_n^{\{t\}}$ joining $\mathbf{u}$ to $\mathbf{v}$. We need to find a $(k+1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. We set $p = (\mathbf{u})_1$ and $q = (\mathbf{v})_1$.

Case 1: $p = q$. Thus, $(\mathbf{u})^n$ and $(\mathbf{v})^n$ are in $P_n^{\{p\}}$. By Lemma 3, there is a hamiltonian path Q of $P_n^{\{p\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We write Q as $\langle (\mathbf{u})^n, Q', \mathbf{y}, \mathbf{z}, (\mathbf{v})^n \rangle$. By Lemma 2, $(\mathbf{y})_1 \neq (\mathbf{z})_1$, $(\mathbf{y})_1 \neq t$, and $(\mathbf{z})_1 \neq t$. By Lemma 3, there is a hamiltonian path R of $P_n^{\langle n \rangle - \{t,p\}}$ joining $(\mathbf{y})^n$ to $(\mathbf{z})^n$. We set Q_{k+1} as $\langle \mathbf{u}, (\mathbf{u})^n, Q', \mathbf{y}, (\mathbf{y})^n, R, (\mathbf{z})^n, \mathbf{z}, (\mathbf{v})^n, \mathbf{v} \rangle$.

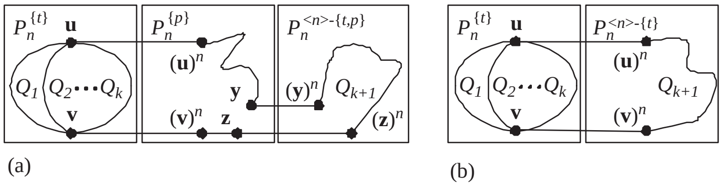

Fig. 2. Illustration for Lemma 4.

Then $\{Q_1, Q_2, \ldots, Q_{k+1}\}$ forms a $(k+1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. See Fig. 2a for illustration.

Case 2: $p \neq q$. Thus, $(\mathbf{u})^n$ and $(\mathbf{v})^n$ are in different subgraphs $P_n^{\{p\}}$ and $P_n^{\{q\}}$. By Lemma 3, there is a hamiltonian path Q of $P_n^{\langle n\rangle-\{t\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We set Q_{k+1} as $\langle \mathbf{u}, (\mathbf{u})^n, Q, (\mathbf{v})^n, \mathbf{v}\rangle$. Then $\{Q_1, Q_2, \ldots, Q_{k+1}\}$ forms a $(k+1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. See Fig. 2b for illustration.

Thus, the theorem is proved. □

Lemma 5. *Let $n \geqslant 5$ and k be any positive integer with $3 \leqslant k \leqslant n-1$. Let $\mathbf{u}$ be any vertex in $P_n^{\{s\}}$ and $\mathbf{v}$ be any vertex in $P_n^{\{t\}}$ such that $s \neq t$. Suppose that P_{n-1} is k^*-connected. Then there is a k^*-container of P_n between $\mathbf{u}$ and $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$ if $(\mathbf{u}, \mathbf{v}) \in E(P_n)$.*

Proof. Since $|E^{s,t}| = (n-2)! \geqslant 6$, we can choose a vertex $\mathbf{y}$ in $P_n^{\{s\}} - \{\mathbf{u}\}$ and a vertex $\mathbf{z}$ in $P_n^{\{t\}} - \{\mathbf{v}\}$ with $(\mathbf{y}, \mathbf{z}) \in E^{s,t}$. Note that $P_n^{\{s\}}$ and $P_n^{\{t\}}$ are both isomorphic to P_{n-1}. Let $\{R_1, R_2, \ldots, R_k\}$ be a k^*-container of $P_n^{\{s\}}$ joining $\mathbf{u}$ to $\mathbf{y}$, and $\{H_1, H_2, \ldots, H_k\}$ be a k^*-container of $P_n^{\{t\}}$ joining $\mathbf{z}$ to $\mathbf{v}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, \mathbf{y}\rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H_i', \mathbf{v}\rangle$. (Note that $\mathbf{y_i} = \mathbf{u}$ if the length of R_i' is zero and $\mathbf{z_i} = \mathbf{v}$ if the length of H_i' is zero.) Let $I = \{\mathbf{y_i} \mid 1 \leqslant i \leqslant k\}$ and $J = \{\mathbf{z_i} \mid 1 \leqslant i \leqslant k\}$. Note that $(\mathbf{y_i})_1 = (\mathbf{y})_j$ for some $j \in \{2, 3, \ldots, n-1\}$, and $(\mathbf{y})_l \neq (\mathbf{y})_m$ if $l \neq m$. By Lemma 2, $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant k\} \cap \{s, t\} = \emptyset$. Similarly, $\{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant k\} \cap \{s, t\} = \emptyset$. Let $A = \{\mathbf{y_i} \mid \mathbf{y_i} \in I$ and there exists an element $\mathbf{z_j} \in J$ such that $(\mathbf{y_i})_1 = (\mathbf{z_j})_1\}$. Then we relabel the indices of I and J such that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for $1 \leqslant i \leqslant |A|$. We set X as $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant k-2\} \cup \{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant k-2\} \cup \{s, t\}$. By Lemma 3, there is a hamiltonian path T_i of $P_n^{\{(\mathbf{y_i})_1, (\mathbf{z_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $1 \leqslant i \leqslant k-2$, and there is a hamiltonian path T_{k-1} of $P_n^{\langle n\rangle - X}$ joining $(\mathbf{y_{k-1}})^n$ to $(\mathbf{z_k})^n$. (Note that $\{(\mathbf{y_i})_1, (\mathbf{z_i})_1\} = \{(\mathbf{y_i})_1\}$ if $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$.) We set

$$Q_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant k-2,$$

$$Q_{k-1} = \langle \mathbf{u}, R_{k-1}', \mathbf{y_{k-1}}, (\mathbf{y_{k-1}})^n, T_{k-1}, (\mathbf{z_k})^n, \mathbf{z_k}, H_k', \mathbf{v}\rangle, \text{ and}$$

$$Q_k = \langle \mathbf{u}, R_k', \mathbf{y_k}, \mathbf{y}, \mathbf{z}, \mathbf{z_{k-1}}, H_{k-1}', \mathbf{v}\rangle.$$

It is easy to check that $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$ if $(\mathbf{u}, \mathbf{v}) \in E(P_n)$. See Fig. 3 for illustration. □

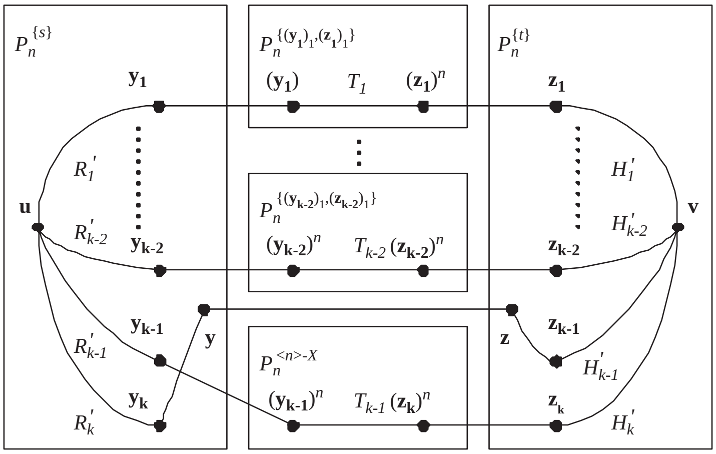

Fig. 3. Illustration for Lemma 5.

Theorem 2. *P_n is $(n-1)^*$-connected if $n \geqslant 2$.*

Proof. It is easy to see that P_2 is 1^*-connected and P_3 is 2^*-connected. Since the P_4 is vertex transitive, we claim that P_4 is 3^*-connected by listing all 3^*-containers from 1234 to any vertex as follows:

⟨(1234), (2134), (4312)⟩
⟨(1234), (3214), (4123), (2143), (3412), (4312)⟩
⟨(1234), (4321), (2341), (1432), (4132), (2314), (1324), (3124), (4213), (1243), (3421), (2431), (4231), (3241), (1423), (2413), (3142), (1342), (4312)⟩

⟨(1234), (2134), (3124), (1324), (4231), (2431), (1342)⟩
⟨(1234), (3214), (2314), (4132), (1432), (2341), (3241), (1423), (4123), (2143), (3412), (4312), (1342)⟩
⟨(1234), (4321), (3421), (1243), (4213), (2413), (3142), (1342)⟩

⟨(1234), (2134), (3124), (4213), (2413), (1423), (3241), (2341), (1432), (3412), (4312), (1342), (3142), (4132), (2314), (1324), (4231), (2431), (3421)⟩
⟨(1234), (3214), (4123), (2143), (1243), (3421)⟩
⟨(1234), (4321), (3421)⟩

⟨(1234), (2134), (3124), (1324), (4231), (2431), (3421), (4321)⟩
⟨(1234), (3214), (2314), (4132), (1432), (3412), (4312), (1342), (3142), (2413), (4213), (1243), (2143), (4123), (1423), (3241), (2341), (4321)⟩
⟨(1234), (4321)⟩

⟨(1234), (2134), (3124), (1324), (4231), (2431)⟩
⟨(1234), (3214), (2314), (4132), (3142), (2413), (4213), (1243), (2143), (4123), (1423), (3241), (2341), (1432), (3412), (4312), (1342), (2431)⟩
⟨(1234), (4321), (3421), (2431)⟩

⟨(1234), (2134), (3124), (4213), (1243), (2143), (3412), (4312), (1342), (3142), (2413), (1423)⟩
⟨(1234), (3214), (4123), (1423)⟩
⟨(1234), (4321), (3421), (2431), (4231), (1324), (2314), (4132), (1432), (2341), (3241), (1423)⟩

⟨(1234), (2134), (3124), (4213), (2413), (3142), (1342), (4312), (3412), (1432), (4132), (2314), (1324), (4231), (2431), (3421), (1243), (2143), (4123)⟩
⟨(1234), (3214), (4123)⟩
⟨(1234), (4321), (2341), (3241), (1423), (4123)⟩

⟨(1234), (2134), (3124), (4213), (2413), (1423), (3241), (4231)⟩
⟨(1234), (3214), (4123), (2143), (1243), (3421), (2431), (4231)⟩
⟨(1234), (4321), (2341), (1432), (3412), (4312), (1342), (3142), (4132), (2314), (1324), (4231)⟩

⟨(1234), (2134), (3124), (1324), (2314), (4132), (3142), (1342), (4312), (3412), (1432), (2341), (3241)⟩
⟨(1234), (3214), (4123), (2143), (1243), (4213), (2413), (1423), (3241)⟩
⟨(1234), (4321), (3421), (2431), (4231), (3241)⟩

⟨(1234), (2134), (4312), (1342), (2431), (3421), (1243), (4213), (3124), (1324), (4231), (3241), (2341)⟩ ⟨(1234), (3214), (2314), (4132), (3142), (2413), (1423), (4123), (2143), (3412), (1432), (2341)⟩ ⟨(1234), (4321), (2341)⟩
⟨(1234), (2134), (3124), (4213), (2413)⟩ ⟨(1234), (3214), (4123), (2143), (1243), (3421), (2431), (4231), (1324), (2314), (4132), (1432), (3412), (4312), (1342), (3142), (2413)⟩ ⟨(1234), (4321), (2341), (3241), (1423), (2413)⟩
⟨(1234), (2134), (3124), (1324), (4231), (2431), (3421), (1243)⟩ ⟨(1234), (3214), (2314), (4132), (1432), (3412), (4312), (1342), (3142), (2413), (4213), (1243)⟩ ⟨(1234), (4321), (2341), (3241), (1423), (4123), (2143), (1243)⟩
⟨(1234), (2134), (3124), (1324), (2314), (3214)⟩ ⟨(1234), (3214)⟩ ⟨(1234), (4321), (3421), (2431), (4231), (3241), (2341), (1432), (4132), (3142), (1342), (4312), (3412), (2143), (1243), (4213), (2413), (1423), (4123), (3214)⟩
⟨(1234), (2134), (3124), (4213), (2413), (3142), (4132), (2314)⟩ ⟨(1234), (3214), (2314)⟩ ⟨(1234), (4321), (3421), (1243), (2143), (4123), (1423), (3241), (2341), (1432), (3412), (4312), (1342), (2431), (4231), (1324), (2314)⟩
⟨(1234), (2134), (4312), (3412), (1432), (4132), (3142), (1342), (2431), (3421), (1243), (2143), (4123), (1423), (2413), (4213), (3124), (1324)⟩ ⟨(1234), (3214), (2314), (1324)⟩ ⟨(1234), (4321), (2341), (3241), (4231), (1324)⟩
⟨(1234), (2134), (3124)⟩ ⟨(1234), (3214), (4123), (1423), (3241), (4231), (2431), (3421), (1243), (2143), (3412), (4312), (1342), (3142), (2413), (4213), (3124)⟩ ⟨(1234), (4321), (2341), (1432), (4132), (2314), (1324), (3124)⟩
⟨(1234), (2134)⟩ ⟨(1234), (3214), (2314), (1324), (3124), (2134)⟩ ⟨(1234), (4321), (2341), (3241), (4231), (2431), (3421), (1243), (4213), (2413), (1423), (4123), (2143), (3412), (1432), (4132), (3142), (1342), (4312), (2134)⟩
⟨(1234), (2134), (4312), (1342), (3142)⟩ ⟨(1234), (3214), (4123), (1423), (2413), (3142)⟩ ⟨(1234), (4321), (3421), (2431), (4231), (3241), (2341), (1432), (3412), (2143), (1243), (4213), (3124), (1324), (2314), (4132), (3142)⟩
⟨(1234), (2134), (3124), (1324), (2314), (4132)⟩ ⟨(1234), (3214), (4123), (1423), (2413), (4213), (1243), (2143), (3412), (4312), (1342), (3142), (4132)⟩ ⟨(1234), (4321), (3421), (2431), (4231), (3241), (2341), (1432), (4132)⟩
⟨(1234), (2134), (4312), (1342), (3142), (2413), (1423), (3241), (4231), (2431), (3421), (1243), (4213), (3124), (1324), (2314), (4132), (1432)⟩ ⟨(1234), (3214), (4123), (2143), (3412), (1432)⟩ ⟨(1234), (4321), (2341), (1432)⟩
⟨(1234), (2134), (3124), (1324), (2314), (4132), (3142), (1342), (4312), (3412)⟩ ⟨(1234), (3214), (4123), (1423), (2413), (4213), (1243), (2143), (3412)⟩ ⟨(1234), (4321), (3412), (2431), (4231), (3241), (2341), (1432), (3412)⟩
⟨(1234), (2134), (4312), (1342), (3142), (4132), (2314), (1324), (3124), (4213)⟩ ⟨(1234), (3214), (4123), (1423), (2413), (4213)⟩ ⟨(1234), (4321), (3421), (2431), (4231), (3241), (2341), (1432), (3412), (2143), (1243), (4213)⟩
⟨(1234), (2134), (4312), (1342), (3142), (4132), (1432), (3412), (2143)⟩ ⟨(1234), (3214), (2314), (1324), (3124), (4213), (2413), (1423), (4123), (2143)⟩ ⟨(1234), (4321), (2341), (3241), (4231), (2431), (3421), (1243), (2143)⟩

Assume that P_k is $(k-1)^*$-connected for every $4 \leqslant k \leqslant n-1$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of P_n with $\mathbf{u} \in P_n^{\{s\}}$ and $\mathbf{v} \in P_n^{\{t\}}$. We need to find an $(n-1)^*$-container between $\mathbf{u}$ and $\mathbf{v}$ of P_n. Suppose that $s = t$. By Lemma 4, there is an $(n-1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. Thus, we assume that $s \neq t$. We set $p = (\mathbf{u})_1$ and $q = (\mathbf{v})_1$.

Case 1: $p = t$ and $q = s$. Thus, $(\mathbf{u})^n \in P_n^{\{t\}}$ and $(\mathbf{v})^n \in P_n^{\{s\}}$.

Subcase 1.1: $\mathbf{u} = (\mathbf{v})^n$. Thus, $(\mathbf{u}, \mathbf{v}) \in E(P_n)$. By Lemma 5, there is an $(n-2)^*$-container $\{Q_1, Q_2, \ldots, Q_{n-2}\}$ of P_n joining $\mathbf{u}$ to $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$. We set Q_{n-1} as $\langle \mathbf{u}, \mathbf{v} \rangle$. Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n-1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$.

Subcase 1.2: $\mathbf{u} \neq (\mathbf{v})^n$. We set $\mathbf{y} = (\mathbf{v})^n$ and $\mathbf{z} = (\mathbf{u})^n$. Let $\{R_1, R_2, \ldots, R_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{s\}}$ joining $\mathbf{u}$ to $\mathbf{y}$, and let $\{H_1, H_2, \ldots, H_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{t\}}$ joining $\mathbf{z}$ to $\mathbf{v}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, \mathbf{y} \rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H_i', \mathbf{v} \rangle$. We set $I = \{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant n-2\}$ and $J = \{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant n-2\}$. Note that $(\mathbf{y_i})_1 = (\mathbf{y})_j$ for some $j \in \{2, 3, \ldots, n-1\}$, and $(\mathbf{y})_k \neq (\mathbf{y})_l$ if $k \neq l$. By Lemma 2, $I = \{(\mathbf{y})_i \mid 2 \leqslant i \leqslant n-1\} =$

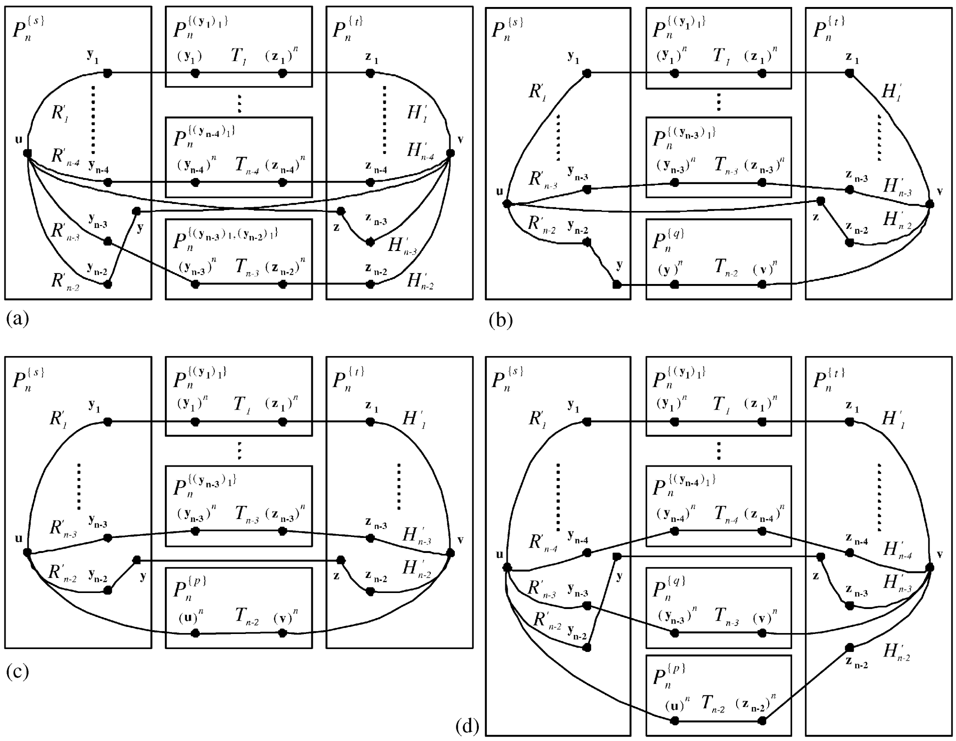

Fig. 4. Illustration for Theorem 2.

$\langle n\rangle - \{s, t\}$. Similarly, $J = \langle n\rangle - \{s, t\}$. We have $I = J$. Without loss of generality, we assume that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for every $1 \leqslant i \leqslant n-2$. By Lemma 3, there is a hamiltonian path T_i of $P_n^{\{(\mathbf{y_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $1 \leqslant i \leqslant n-4$, and there is a hamiltonian path T_{n-3} of $P_n^{\{(\mathbf{y_{n-3}})_1, (\mathbf{y_{n-2}})_1\}}$ joining $(\mathbf{y_{n-3}})^n$ to $(\mathbf{z_{n-2}})^n$. We set

$$
\begin{aligned}
Q_i &= \langle \mathbf{u}, R'_i, \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H'_i, \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-4,\\
Q_{n-3} &= \langle \mathbf{u}, R'_{n-3}, \mathbf{y_{n-3}}, (\mathbf{y_{n-3}})^n, T_{n-3}, (\mathbf{z_{n-2}})^n, \mathbf{z_{n-2}}, H'_{n-2}, \mathbf{v}\rangle,\\
Q_{n-2} &= \langle \mathbf{u}, \mathbf{z}, \mathbf{z_{n-3}}, H'_{n-3}, \mathbf{v}\rangle, \text{ and}\\
Q_{n-1} &= \langle \mathbf{u}, R'_{n-2}, \mathbf{y_{n-2}}, \mathbf{y}, \mathbf{v}\rangle.
\end{aligned}
$$

Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n-1)^*$-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. See Fig. 4a for illustration.

Case 2: $p = t$ and $q \in \langle n\rangle - \{s, t\}$. Since $|E^{s,q}| = (n-2)! \geqslant 6$, we can choose a vertex $\mathbf{y}$ in $P_n^{\{s\}} - \{\mathbf{u}\}$ with $(\mathbf{y})^n \in P_n^{\{q\}}$. We set $\mathbf{z} = (\mathbf{u})^n \in P_n^{\{t\}}$. Let $\{R_1, R_2, \ldots, R_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{s\}}$ joining $\mathbf{u}$ to $\mathbf{y}$, and $\{H_1, H_2, \ldots, H_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{t\}}$ joining $\mathbf{z}$ to $\mathbf{v}$. We write $R_i = \langle \mathbf{u}, R'_i, \mathbf{y_i}, \mathbf{y}\rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H'_i, \mathbf{v}\rangle$. We have $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant n-2\} = \{(\mathbf{y})_i \mid 2 \leqslant i \leqslant n-1\}$. By Lemma 2, $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant n-2\} = \langle n\rangle - \{s, q\}$. Similarly, $\{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant n-2\} = \langle n\rangle - \{s, t\}$. Without loss of generality, we assume that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for every $1 \leqslant i \leqslant n-3$, $(\mathbf{y_{n-2}})_1 = t$, and $(\mathbf{z_{n-2}})_1 = q$. By Lemma 3, there is a hamiltonian path T_i of $P_n^{\{(\mathbf{y_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $1 \leqslant i \leqslant n-3$, and there is a hamiltonian path T_{n-2} of $P_n^{\{q\}}$ joining $(\mathbf{y})^n$ to $(\mathbf{v})^n$. We set

$$\begin{aligned} Q_i &= \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-3, \\ Q_{n-2} &= \langle \mathbf{u}, R_{n-2}', \mathbf{y_{n-2}}, \mathbf{y}, (\mathbf{y})^n, T_{n-2}, (\mathbf{v})^n, \mathbf{v}\rangle, \text{ and} \\ Q_{n-1} &= \langle \mathbf{u}, \mathbf{z}, \mathbf{z_{n-2}}, H_{n-2}', \mathbf{v}\rangle. \end{aligned}$$

Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n-1)^*$-container of P_n joining **u** to **v**. See Fig. 4b for illustration.

Case 3: $p, q \in \langle n\rangle - \{s, t\}$. Since $|E^{s,t}| = (n-2)! \geqslant 6$, there exists an edge $(\mathbf{y}, \mathbf{z})$ in $E^{s,t}$ with $\mathbf{y} \in P_n^{\{s\}} - \{\mathbf{u}\}$ and $\mathbf{z} \in P_n^{\{t\}} - \{\mathbf{v}\}$. Let $\{R_1, R_2, \ldots, R_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{s\}}$ joining **u** to **y**, and let $\{H_1, H_2, \ldots, H_{n-2}\}$ be an $(n-2)^*$-container of $P_n^{\{t\}}$ joining **z** to **v**. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, \mathbf{y}\rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H_i', \mathbf{v}\rangle$. We set $I = \{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant n-2\}$ and $J = \{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant n-2\}$. We have $I = \{(\mathbf{y})_i \mid 2 \leqslant i \leqslant n-1\}$. By Lemma 2, $I = \langle n\rangle - \{s, t\}$. Similarly, $J = \langle n\rangle - \{s, t\}$. We have $I = J$. Without loss of generality, we assume that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for every $1 \leqslant i \leqslant n-2$ with $(\mathbf{y_{n-2}})_1 = p$.

Subcase 3.1: $p = q$. By Lemma 3, there is a hamiltonian path T_i of $P_n^{\{(\mathbf{y_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $i \in \langle n-3\rangle$, and there is a hamiltonian path T_{n-2} in $P_n^{\{p\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We set

$$\begin{aligned} Q_i &= \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-3, \\ Q_{n-2} &= \langle \mathbf{u}, R_{n-2}', \mathbf{y_{n-2}}, \mathbf{y}, \mathbf{z}, \mathbf{z_{n-2}}, H_{n-2}', \mathbf{v}\rangle, \text{ and} \\ Q_{n-1} &= \langle \mathbf{u}, (\mathbf{u})^n, T_{n-2}, (\mathbf{v})^n, \mathbf{v}\rangle. \end{aligned}$$

Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n-1)^*$-container of P_n joining **u** and **v**. See Fig. 4c for illustration.

Subcase 3.2: $p \neq q$. Without loss of generality, we assume that $(\mathbf{y_{n-3}})_1 = q$. By Theorem 1, there is a hamiltonian path T_i of $P_n^{\{(\mathbf{y_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $1 \leqslant i \leqslant n-4$, there is a hamiltonian path T_{n-3} of $P_n^{\{q\}}$ joining $(\mathbf{y_{n-3}})^n$ to $(\mathbf{v})^n$, and there is a hamiltonian path T_{n-2} of $P_n^{\{p\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{z_{n-2}})^n$. We set

$$\begin{aligned} Q_i &= \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-4, \\ Q_{n-3} &= \langle \mathbf{u}, R_{n-3}', \mathbf{y_{n-3}}, (\mathbf{y_{n-3}})^n, T_{n-3}, (\mathbf{v})^n, \mathbf{v}\rangle, \\ Q_{n-2} &= \langle \mathbf{u}, (\mathbf{u})^n, T_{n-2}, (\mathbf{z_{n-2}})^n, \mathbf{z_{n-2}}, H_{n-2}', \mathbf{v}\rangle, \text{ and} \\ Q_{n-1} &= \langle \mathbf{u}, R_{n-2}', \mathbf{y_{n-2}}, \mathbf{y}, \mathbf{z}, \mathbf{z_{n-3}}, H_{n-3}', \mathbf{v}\rangle. \end{aligned}$$

It is easy to check that $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ is an $(n-1)^*$-container of P_n from **u** to **v**. See Fig. 4d for illustration.

Thus, the theorem is proved. □

Theorem 3. *P_n is super connected if and only if $n \neq 3$.*

Proof. We prove this theorem by induction. Obviously, this theorem is true for P_1 and P_2. Since P_3 is isomorphic to a cycle with six vertices, P_3 is not 1^*-connected. Thus, P_3 is not super connected. By Theorems 1 and 2, this theorem holds on P_4. Assume that P_k is super connected for every $4 \leqslant k \leqslant n-1$. By Theorems 1 and 2, P_n is k^*-connected for any $k \in \{1, 2, n-1\}$. Thus, we still need to construct a k^*-container of P_n between any two distinct vertices $\mathbf{u} \in P_n^{\{s\}}$ and $\mathbf{v} \in P_n^{\{t\}}$ for every $3 \leqslant k \leqslant n-2$.

Suppose that $s = t$. By induction, P_{n-1} is $(k-1)^*$-connected. By Lemma 4, there is a k^*-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$. Suppose that $s \neq t$. By induction, P_{n-1} is k^*-connected. By Lemma 5, there is a k^*-container of P_n joining $\mathbf{u}$ to $\mathbf{v}$.

Hence, the theorem is proved. □

4. The star graphs

The *n-dimensional star graph*, denoted by S_n, is a graph with the vertex set $V(S_n) = \{u_1u_2\ldots u_n \mid u_i \in \langle n\rangle \text{ and } u_i \neq u_j \text{ for } i \neq j\}$. The adjacency is defined as follows: $u_1u_2\ldots u_i\ldots u_n$ is adjacent to $v_1v_2\ldots v_i\ldots v_n$ through an edge of dimension i with $2 \leqslant i \leqslant n$ if $v_j = u_j$ for $j \notin \{1, i\}$, $v_1 = u_i$, and $v_i = u_1$. Again, we use bold face to denote a vertex of S_n. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denote a sequence of vertices of S_n. In particular, $\mathbf{e}$ denotes the vertex $12\ldots n$. By definition, S_n is an $(n-1)$-regular graph with $n!$ vertices.

It is known that S_n is a bipartite graph with one partite set containing all odd permutations and the other partite set containing all even permutations. For convenience, we refer an even permutation as a white vertex, and refer an odd permutation as a black vertex. Let $\mathbf{u} = u_1u_2\ldots u_n$ be any vertex of S_n. We use $(\mathbf{u})_i$ to denote the ith component u_i of $\mathbf{u}$ and $S_n^{\{i\}}$ to denote the ith subgraph of S_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Obviously, S_n can be decomposed into n vertex disjoint subgraphs $S_n^{\{i\}}$ for $1 \leqslant i \leqslant n$, such that each $S_n^{\{i\}}$ is isomorphic to S_{n-1}. Thus, the star graph can be constructed recursively. Let $H \subseteq \langle n\rangle$. We use S_n^H to denote the subgraph of S_n induced by $\cup_{i\in H} V(S_n^{\{i\}})$. By the definition of S_n, there is exactly one neighbor $\mathbf{v}$ of $\mathbf{u}$ such that $\mathbf{u}$ and $\mathbf{v}$ are adjacent through an i-dimensional edge with $2 \leqslant i \leqslant n$. For this reason, we use $(\mathbf{u})^i$ to denote the unique i-neighbor of $\mathbf{u}$. We have $((\mathbf{u})^i)^i = \mathbf{u}$ and $(\mathbf{u})^n \in S_n^{\{(\mathbf{u})_1\}}$. For $1 \leqslant i, j \leqslant n$ and $i \neq j$, we use $E^{i,j}$ to denote the set of edges between $S_n^{\{i\}}$ and $S_n^{\{j\}}$. The star graphs S_2, S_3, and S_4 are shown in Fig. 5 for illustration.

The following theorem is proved by Hsieh et al. [12].

Theorem 4 (*Hsieh et al. [12]*). *S_n is 1^*-laceable if $n \neq 3$, and S_n is 2^*-connected if $n \geqslant 3$.*

Lemma 6. *Assume that $n \geqslant 3$. $|E^{i,j}| = (n-2)!$ for any $1 \leqslant i \neq j \leqslant n$. Moreover, there are $\frac{(n-2)!}{2}$ edges joining black vertices of $S_n^{\{i\}}$ to white vertices of $S_n^{\{j\}}$.*

Lemma 7. *Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of S_n with $d(\mathbf{u}, \mathbf{v}) \leqslant 2$. Then $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Moreover, $\{((\mathbf{u})^i)_1 \mid 2 \leqslant i \leqslant n-1\} = \langle n\rangle - \{(\mathbf{u})_1, (\mathbf{u})_n\}$ if $n \geqslant 3$.*

Lemma 8. *Let $n \geqslant 5$ and $H = \{i_1, i_2, \ldots, i_m\}$ be any nonempty subset of $\langle n\rangle$. There is a hamiltonian path of S_n^H joining any white vertex $\mathbf{u} \in S_n^{\{i_1\}}$ to any black vertex $\mathbf{v} \in S_n^{\{i_m\}}$.*

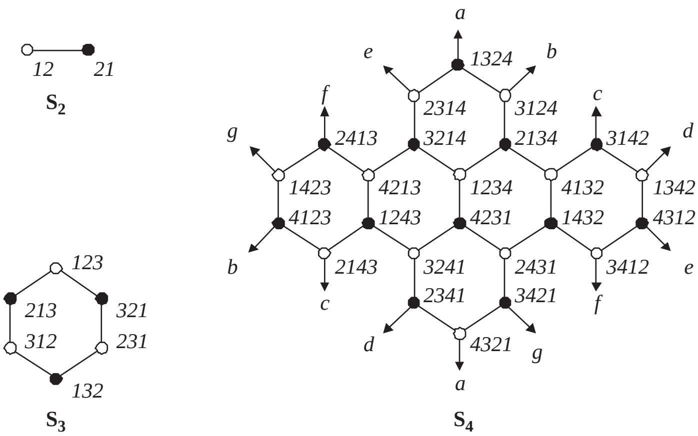

Fig. 5. The star graphs S_2, S_3, and S_4.

Proof. Note that $S_n^{\{i_j\}}$ is isomorphic to S_{n-1} for every $1 \leqslant j \leqslant m$. We set $\mathbf{x_1} = \mathbf{u}$ and $\mathbf{y_m} = \mathbf{v}$. By Theorem 4, this theorem holds for $m = 1$. Assume that $m \geqslant 2$. By Lemma 6, we choose $(\mathbf{y_j}, \mathbf{x_{j+1}}) \in E^{i_j, i_{j+1}}$ with $\mathbf{y_j}$ is a black vertex of $S_n^{\{j\}}$ and $\mathbf{x_{j+1}}$ is a white vertex of $S_n^{\{j+1\}}$ for every $1 \leqslant j \leqslant m-1$. By Theorem 4, there is a hamiltonian path Q_j of $S_n^{\{i_j\}}$ joining $\mathbf{x_j}$ to $\mathbf{y_j}$. The path $\langle \mathbf{x_1}, Q_1, \mathbf{y_1}, \mathbf{x_2}, Q_2, \mathbf{y_2}, \ldots, \mathbf{x_m}, Q_m, \mathbf{y_m} \rangle$ forms a desired path. □

5. The super laceability of the star graphs

In this section, we are going to prove that S_n is super laceable if and only if $n \neq 3$. As you will observe, the proof is very similar to the proof that P_n is super connected if and only if $n \neq 3$.

Lemma 9. *Let $n \geqslant 5$ and k be any positive integer with $3 \leqslant k \leqslant n-1$. Let $\mathbf{u}$ be any white vertex and $\mathbf{v}$ be any black vertex of S_n. Suppose that S_{n-1} is k^*-laceable. Then there is a k^*-container of S_n between $\mathbf{u}$ and $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$ if $(\mathbf{u}, \mathbf{v}) \in E(S_n)$.*

Proof. Since S_n is edge transitive, we may assume that $\mathbf{u} \in S_n^{\{n\}}$ and $\mathbf{v} \in S_n^{\{n-1\}}$. By Lemma 6, there are $\frac{(n-2)!}{2} \geqslant 3$ edges joining black vertices of $S_n^{\{n\}}$ to white vertices of $S_n^{\{n-1\}}$. We can choose an edge $(\mathbf{y}, \mathbf{z}) \in E^{n-1,n}$ where $\mathbf{y}$ is a black vertex in $S_n^{\{n\}}$ and $\mathbf{z}$ is a white vertex in $S_n^{\{n-1\}}$. By induction, there is a k^*-container $\{R_1, R_2, \ldots, R_k\}$ of $S_n^{\{n\}}$ joining $\mathbf{u}$ to $\mathbf{y}$, and there is a k^*-container $\{H_1, H_2, \ldots, H_k\}$ of $S_n^{\{n-1\}}$ joining $\mathbf{z}$ to $\mathbf{v}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, \mathbf{y} \rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H_i', \mathbf{v} \rangle$. Note that $\mathbf{y_i}$ is a white vertex and $\mathbf{z_i}$ is a black vertex for every $1 \leqslant i \leqslant k$. Let $I = \{\mathbf{y_i} \mid (\mathbf{y_i}, \mathbf{y}) \in E(R_i)$ and $1 \leqslant i \leqslant k\}$, and $J = \{\mathbf{z_i} \mid (\mathbf{z_i}, \mathbf{x}) \in E(H_i)$ and $1 \leqslant i \leqslant k\}$. Note that $(\mathbf{y_i})_1 = (\mathbf{y})_j$ for some $j \in \{2, 3, \ldots, n-1\}$, and $(\mathbf{y})_l \neq (\mathbf{y})_m$ if $l \neq m$. By Lemma 7, $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant k\} \cap \{n-1, n\} = \emptyset$. Similarly, $\{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant k\} \cap \{n-1, n\} = \emptyset$. Let $A = \{\mathbf{y_i} \mid \mathbf{y_i} \in I$ and there exists an element $\mathbf{z_j} \in J$

such that $(\mathbf{y_i})_1 = (\mathbf{z_j})_1\}$. Then we relabel the indices of I and J such that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for $1 \leqslant i \leqslant |A|$. We set X as $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant k-2\} \cup \{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant k-2\} \cup \{n-1, n\}$. By Lemma 8, there is a hamiltonian path T_i of $S_n^{\{(\mathbf{y_i})_1, (\mathbf{z_i})_1\}}$ joining the black vertex $(\mathbf{y_i})^n$ to the white vertex $(\mathbf{z_i})^n$ for every $1 \leqslant i \leqslant k-2$, and there is a hamiltonian path T_{k-1} of $S_n^{\langle n \rangle - X}$ joining the black vertex $(\mathbf{y_{k-1}})^n$ to the white vertex $(\mathbf{z_k})^n$. (Note that $\{(\mathbf{y_i})_1, (\mathbf{z_i})_1\} = \{(\mathbf{y_i})_1\}$ if $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$.) We set

$$Q_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v} \rangle \text{ for } 1 \leqslant i \leqslant k-2,$$
$$Q_{k-1} = \langle \mathbf{u}, R_{k-1}', \mathbf{y_{k-1}}, (\mathbf{y_{k-1}})^n, T_{k-1}, (\mathbf{z_k})^n, \mathbf{z_k}, H_k', \mathbf{v} \rangle, \text{ and}$$
$$Q_k = \langle \mathbf{u}, R_k', \mathbf{y_k}, \mathbf{y}, \mathbf{z}, \mathbf{z_{k-1}}, H_{k-1}', \mathbf{v} \rangle.$$

It is easy to check that $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of S_n joining $\mathbf{u}$ to $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$ if $(\mathbf{u}, \mathbf{v}) \in E(S_n)$. □

Theorem 5. *S_n is $(n-1)^*$-laceable if $n \geqslant 2$.*

Proof. It is easy to see that S_2 is 1^*-laceable and S_3 is 2^*-laceable. Since the S_4 is vertex transitive, we claim that S_4 is 3^*-laceable by listing all 3^*-containers from the white vertex 1234 to any black vertex as follows:

⟨(1234), (2134)⟩
⟨(1234), (3214), (2314), (4312), (1342), (2341), (4321), (1324), (3124), (2134)⟩
⟨(1234), (4231), (3241), (1243), (4213), (2413), (3412), (1432), (2431), (3421), (1423), (4123), (2143), (3142), (4132), (2134)⟩

⟨(1234), (3214)⟩
⟨(1234), (4231), (3241), (2341), (1342), (3142), (2143), (1243), (4213), (3214)⟩
⟨(1234), (2134), (4132), (1432), (2431), (3421), (4321), (1324), (3124), (4123), (1423), (2413), (3412), (4312), (2314), (3214)⟩

⟨(1234), (4231)⟩
⟨(1234), (2134), (4132), (3142), (1342), (4312), (3412), (1432), (2431), (4231)⟩
⟨(1234), (3214), (2314), (1324), (3124), (4123), (2143), (1243), (4213), (2413), (1423), (3421), (4321), (2341), (3241), (4231)⟩

⟨(1234), (2134), (3124), (1324), (2314), (4312), (1342), (3142), (4132), (1432), (3412), (2413), (1423), (4123), (2143), (1243)⟩
⟨(1234), (3214), (4213), (1243)⟩
⟨(1234), (4231), (2431), (3421), (4321), (2341), (3241), (1243)⟩

⟨(1234), (2134), (4132), (1432)⟩
⟨(1234), (3214), (2314), (1324), (3124), (4123), (1423), (2413), (4213), (1243), (2143), (3142), (1342), (4312), (3412), (1432)⟩
⟨(1234), (4231), (3241), (2341), (4321), (3421), (2431), (1432)⟩

⟨(1234), (2134), (4132), (3142), (1342), (4312), (3412), (1432), (2431), (3421), (1423), (2413), (4213), (1243), (2143), (4123), (3124), (1324)⟩
⟨(1234), (3214), (2314), (1324)⟩
⟨(1234), (4231), (3241), (2341), (4321), (1324)⟩

⟨(1234), (2134), (3124), (1324), (2314), (4312), (3412), (1432), (4132), (3142), (1342), (2341)⟩
⟨(1234), (3214), (4213), (1423), (2413), (4123), (2143), (1243), (3241), (2341)⟩
⟨(1234), (4231), (2431), (3421), (4321), (2341)⟩

⟨(1234), (2134), (4132), (3142), (1342), (4312), (3412), (1432), (2431), (3421)⟩
⟨(1234), (3214), (2314), (1324), (3124), (4123), (2143), (1243), (4213), (2413), (1423), (3421)⟩
⟨(1234), (4231), (3241), (2341), (4321), (3421)⟩

⟨(1234), (2134), (3124), (1324), (2314), (4312)⟩
⟨(1234), (3214), (4213), (1243), (2143), (4123), (1423), (2413), (3412), (4312)⟩
⟨(1234), (4231), (3241), (2341), (4321), (3421), (2431), (1432), (4132), (3142), (1342), (4312)⟩

⟨(1234), (2134), (4132), (1432), (3412), (4312), (1342), (3142), (2143), (4123)⟩
⟨(1234), (3214), (2314), (1324), (3124), (4123)⟩
⟨(1234), (4231), (2431), (3421), (4321), (2341), (3241), (1243), (4213), (2413), (1423), (4123)⟩

⟨(1234), (2134), (4132), (3142)⟩
⟨(1234), (3214), (2314), (1324), (3124), (4123), (1423), (2413), (4213), (1243), (2143), (3142)⟩
⟨(1234), (4231), (3241), (2341), (4321), (3421), (2431), (1432), (3412), (4312), (1342), (3142)⟩

⟨(1234), (2134), (3124), (1324), (2314), (4312), (1342), (3142), (4132), (1432), (3412), (2413)⟩
⟨(1234), (3214), (4213), (2413)⟩
⟨(1234), (4231), (2431), (3421), (4321), (2341), (3241), (1243), (2143), (4123), (1423), (2413)⟩

Assume that S_k is $(k-1)^*$-laceable for every $4 \leqslant k \leqslant n-1$. We need to construct an $(n-1)^*$-container of S_n between any white vertex $\mathbf{u}$ to any black vertex $\mathbf{v}$.

Case 1: $d(\mathbf{u},\mathbf{v}) = 1$. We have $(\mathbf{u},\mathbf{v}) \in E(S_n)$. By induction, S_{n-1} is $(n-2)^*$-laceable. By Lemma 9, there exists a $(n-2)^*$-container $\{Q_1, Q_2, \dots, Q_{n-2}\}$ of S_n joining $\mathbf{u}$ to $\mathbf{v}$ not using the edge $(\mathbf{u},\mathbf{v})$. We set Q_{n-1} as $\langle \mathbf{u}, \mathbf{v}\rangle$. Then $\{Q_1, Q_2, \dots, Q_{n-1}\}$ forms an $(n-1)^*$-container of S_n joining $\mathbf{u}$ to $\mathbf{v}$.

Case 2: $d(\mathbf{u},\mathbf{v}) \geqslant 3$. We have star graph is edge transitive. Without loss of generality, we may assume that $\mathbf{u} \in S_n^{\{n\}}$ and $\mathbf{v} \in S_n^{\{n-1\}}$ with $(\mathbf{u})_1 \neq n-1$ and $(\mathbf{v})_1 \neq n$. By Lemma 6, there are $\frac{(n-2)!}{2} \geqslant 3$ edges joining black vertices of $S_n^{\{n\}}$ to white vertices of $S_n^{\{n-1\}}$. We can choose an edge $(\mathbf{y},\mathbf{z}) \in E^{n-1,n}$ where $\mathbf{y}$ is a black vertex in $S_n^{\{n\}}$ and $\mathbf{z}$ is a white vertex in $S_n^{\{n-1\}}$. Let $\{R_1, R_2, \dots, R_{n-2}\}$ be an $(n-2)^*$-container of $S_n^{\{n\}}$ joining $\mathbf{u}$ to $\mathbf{y}$, and let $\{H_1, H_2, \dots, H_{n-2}\}$ be an $(n-2)^*$-container of $S_n^{\{n-1\}}$ joining $\mathbf{z}$ to $\mathbf{v}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{y_i}, \mathbf{y}\rangle$ and $H_i = \langle \mathbf{z}, \mathbf{z_i}, H_i', \mathbf{v}\rangle$. Note that $\mathbf{y_i}$ is a white vertex and $\mathbf{z_i}$ is a black vertex for every $1 \leqslant i \leqslant n-2$. We have $\{(\mathbf{y_i})_1 \mid 1 \leqslant i \leqslant n-2\} = \{(\mathbf{z_i})_1 \mid 1 \leqslant i \leqslant n-2\} = \langle n-2\rangle$. Without loss of generality, we assume that $(\mathbf{y_i})_1 = (\mathbf{z_i})_1$ for every $1 \leqslant i \leqslant n-2$ with $(\mathbf{y_{n-2}})_1 = (\mathbf{u})_1$.

Subcase 2.1: $(\mathbf{u})_1 = (\mathbf{v})_1$. By Theorem 4, there is a hamiltonian path T_i of $S_n^{\{(\mathbf{y_i})_1\}}$ joining the black vertex $(\mathbf{y_i})^n$ to the white vertex $(\mathbf{z_i})^n$ for every $i \in \langle n-3\rangle$, and there is a hamiltonian path H of $S_n^{\{(\mathbf{y_{n-2}})_1\}}$ joining the black vertex $(\mathbf{u})^n$ to the white vertex $(\mathbf{v})^n$. We set

$$
\begin{aligned}
Q_i &= \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-3,\\
Q_{n-1} &= \langle \mathbf{u}, R_{n-2}', \mathbf{y_{n-2}}, \mathbf{y}, \mathbf{z}, \mathbf{z_{n-2}}, H_{n-2}', \mathbf{v}\rangle, \text{ and}\\
Q_{n-2} &= \langle \mathbf{u}, (\mathbf{u})^n, H, (\mathbf{v})^n, \mathbf{v}, \rangle.
\end{aligned}
$$

Then $\{Q_1, Q_2, \dots, Q_{n-1}\}$ forms an $(n-1)^*$-container of S_n joining $\mathbf{u}$ and $\mathbf{v}$.

Subcase 2.2: $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Without loss of generality, we assume that $(\mathbf{y_{n-3}})_1 = (\mathbf{v})_1$. By Theorem 4, there is a hamiltonian path T_i of $S_n^{\{(\mathbf{y_i})_1\}}$ joining $(\mathbf{y_i})^n$ to $(\mathbf{z_i})^n$ for every $i \in \langle n-4\rangle$, there is a hamiltonian path H of $S_n^{\{(\mathbf{y_{n-3}})_1\}}$ joining the black vertex $(\mathbf{y_{n-3}})^n$ to the white vertex $(\mathbf{v})^n$, and there is a hamiltonian path P of $S_n^{\{(\mathbf{y_{n-2}})_1\}}$ joining the black vertex $(\mathbf{u})^n$ to the white vertex $(\mathbf{z_{n-2}})^n$. We set

$$
\begin{aligned}
Q_i &= \langle \mathbf{u}, R_i', \mathbf{y_i}, (\mathbf{y_i})^n, T_i, (\mathbf{z_i})^n, \mathbf{z_i}, H_i', \mathbf{v}\rangle \text{ for } 1 \leqslant i \leqslant n-4,\\
Q_{n-3} &= \langle \mathbf{u}, R_{n-3}', \mathbf{y_{n-3}}, (\mathbf{y_{n-3}})^n, H, (\mathbf{v})^n, \mathbf{v}, \rangle,\\
Q_{n-2} &= \langle \mathbf{u}, (\mathbf{u})^n, P, (\mathbf{z_{n-2}})^n, \mathbf{z_{n-2}}, H_{n-2}', \mathbf{v}, \rangle, \text{ and}\\
Q_{n-1} &= \langle \mathbf{u}, R_{n-2}', \mathbf{y_{n-2}}, \mathbf{y}, \mathbf{z}, \mathbf{z_{n-3}}, H_{n-3}', \mathbf{v}\rangle.
\end{aligned}
$$

It is easy to check that $\{Q_1, Q_2, \dots, Q_{n-1}\}$ is an $(n-1)^*$-container of S_n joining $\mathbf{u}$ to $\mathbf{v}$.

Thus, this theorem is proved. □

Theorem 6. *S_n is super laceable if and only if $n \neq 3$.*

Proof. It is easy to see that this theorem is true for S_1 and S_2. Since S_3 is isomorphic to a cycle with six vertices, S_3 is not 1^*-laceable. Thus, S_3 is not super laceable. By Theorems 4 and 5, this theorem holds on S_4. Assume that S_k is super laceable for every $4 \leqslant k \leqslant n-1$. By Theorems 4 and 5, S_n is k^*-laceable for any $k \in \{1, 2, n-1\}$. Thus, we still need to construct a k^*-container of S_n between any white vertex **u** and any black vertex **v** for every $3 \leqslant k \leqslant n-2$. By induction, S_{n-1} is k^*-laceable. By Lemma 9, there is a k^*-container of S_n joining **u** to **v**. □

6. Further study

In this paper, we prove that the pancake graph P_n is super connected for $n \neq 3$ and the star graphs S_n is super laceable for $n \neq 3$. We believe that there are other super connected and super laceable graphs. It would be very interesting to classify such graphs.

We may also study the fault tolerant k^*-connectivity for any super connected graph. For example, let $F \subset V(P_n) \cup E(P_n)$ with $|F| = f \leqslant n-3$. Obviously, $P_n - F$ is $(n-1-f)$ connected. However, we believe that $P_n - F$ is $(n-1-f)^*$-connected. Similarly, we can study the fault tolerant k^*-laceability for any super laceable graph. For example, let $F \subset E(S_n)$ with $|F| = f \leqslant n-3$. Obviously, $S_n - F$ is $(n-1-f)$ connected. However, we believe that $S_n - F$ is $(n-1-f)^*$-connected.

Assume that G is k^*-connected. We may also define the *k^*-connected distance* between any two vertices u and v, denoted by $d_k^s(u,v)$, which is the minimum length among all k^*-containers between u and v. The k^*-diameter of G, denote by $D_k^s(G)$, is $\max\{d_k^s(u,v) \mid u$ and v are two different vertices of $G\}$. In particular, we are intrigued in $D_{\kappa(G)}^s(G)$ and $D_2^s(G)$. Similarly, we define the *k^{s_L}-laceable distance* on bipartite graph between any two vertices u and v from different partite sets, denoted by $d_k^{s_L}(u,v)$, which is the minimum length among all k^*-containers between u and v. The k^{s_L}-diameter of G, denoted by $D_k^{s_L}(G)$, is $\max\{d_k^{s_L}(u,v) \mid u$ and v are vertices from different partite sets$\}$. Again, we are intrigued in $D_{\kappa(G)}^{s_L}(G)$ and $D_2^{s_L}(G)$.

References

[1] S.B. Akers, B. Krishnameurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (1989) 555–566.
[2] M. Albert, R.E.L. Aldred, D. Holton, On 3*-connected graphs, Australasian J. Combin. 24 (2001) 193–208.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North Holland, New York, 1980.
[4] A. Bouabdallah, M.C. Heydemann, J. Opatmy, D. Sotteau, Embedding complete binary trees into star and pancake graphs, Theory of Comput. Syst. 31 (1998) 279–305.
[5] K. Day, A. Tripathi, A comparative study of topological properties, IEEE Trans. Parallel and Distributed Syst. 5 (1994) 31–38.
[6] W.C. Fang, C.C. Hsu, On the fault-tolerant embedding of complete binary tree in the pancake graph interconnection network, Inform. Sci. 126 (2000) 191–204.
[7] P. Fragopoulou, S.G. Akl, Optimal communication algorithms on the star graphs using spanning tree constructions, J. Parallel and Distributed Comput. 23 (1995) 55–71.
[8] P. Fragopoulou, S.G. Akl, Edge-disjoint spanning trees on the star networks with applications to fault tolerance, IEEE Trans. Comput. 45 (1996) 174–185.
[9] L. Gargano, U. Vaccaro, A. Vozella, Fault tolerant routing in the star and pancake interconnection networks, Inform. Process. Lett. 45 (1993) 315–320.
[10] W.H. Gates, C.H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Math. 27 (1979) 47–57.
[11] M.H. Heydari, I.H. Sudborough, On the diameter of the pancake network, J. Algorithms 25 (1997) 67–94.
[12] S.Y. Hsieh, G.H. Chen, C.W. Ho, Hamiltonian-laceability of star graphs, Networks 36 (2000) 225–232.

[13] D.F. Hsu, On container width and length in graphs, groups, and networks, IEICE Trans. Fundamentals of Electron. E 77A (1994) 668–680.
[14] C.N. Hung, H.C. Hsu, K.Y. Liang, L.H. Hsu, Ring embedding in faulty pancake graphs, Inform. Process. Lett. 86 (2003) 271–275.
[15] J.S. Jwo, S. Lakshmivarahan, S.K. Dhall, Embedding of cycles and grids in star graphs, J. Circuits, Systems, and Comput. 1 (1991) 43–74.
[16] A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs, Parallel Comput. 21 (1995) 923–936.
[17] S. Latifi, On the fault-diameter of the star graph, Inform. Process. Lett. 46 (1993) 143–150.
[18] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays. Trees. Hypercubes, Morgan Kaufmann, San Mateo, CA, 1992.
[19] T.K. Li, J.M. Tan, L.H. Hsu, Hyper Hamiltonian Laceability on the Edge Fault Star Graph, Inform. Sci. 165 (2004) 59–71.
[20] K. Menger, Zur allgemeinen Kurventheorie, Fundamenta Math. 10 (1927) 95–115.
[21] K. Qiu, H. Meijer, S.G. Akl, Decomposing a star graph into disjoint cycles, Inform. Process. Lett. 39 (1991) 125–129.
[22] Y. Rouskov, S. Latifi, P.K. Srimani, Conditional fault diameter of star graph networks, J. Parallel and Distributed Comput. 33 (1996) 91–97.
[23] Y. Rouskov, P.K. Srimani, Fault diameter of star graph networks, Inform. Process. Lett. 48 (1993) 243–251.