STABILITY EQUATION METHOD
PART I: THE GENERAL PRINCIPLE

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Abstract *This paper (PART I) presents the general principle of the stability equation method. A detail proof of a criterion for predicting the locations of roots of polynomials with real or complex coefficients is given and comparisons with other methods are considered.*

Following this paper, additional papers will be presented concerning the applications of the stability-equation method to varivus engineering problems.

INTRODUCTION

The root distribution of an algebraic equation has been studied by many authors [1-13]. Among them, Han and Thaler [1, 2, 12] have introduced a method, called stability equation method, to predict whether all roots of a polynomial with either real or complex coefficients are in LHP of the domain of the polynomial. Starting from the point of view of stability equation method, the authors of this paper will present a general criterion concerning the root distribution of a polynomial with complex coefficients.

The polynomial $F(S)$ is transformed to the λ-plane by the relation $S=j\lambda$, and a root distribution index $\delta_{\updownarrow}(A, \lambda)$, which indicates the root distribution of $A(\lambda)=0$ in UHP (up half plane), on R-axis (Real axis) and in DHP (down half plane) of λ-plane, is determined, then the root distribution of the polynomial $F(S)=0$ in LHP, on I-axis (Imaginary axis) and in RHP of S-plane is indicated by the root distribution index $\delta_{\leftrightarrow}(F, S)$. The main purpose of this transformation is that the mathematical operations and proofs for the criterion in λ-domain are easier and clearer than those in S-domain, and $\delta_{\leftrightarrow}(F, S)$ is evident once $\delta_{\updownarrow}(A, \lambda)$ is obtained.

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I. THE ROOT DISTRIBUTION AND THE ROOT LOCUS FO TESTING FUNCTION

In this section, the root distribution of a polynomial in UHP, DHP and real axis of λ-plane will be discussed. The rational function for testing the root distribution will be defined. For ease of presentation some definitions and notations are given along with the Lemmas and Theorems.

Definition 1. Roots distribution index $\delta_{\updownarrow}(A, \lambda)$ of an algebraic equation $A(\lambda)=0$ is an ordered set or vector (n_u, n_0, n_d), if there are n_u roots of $A(\lambda)=0$ in UHP of λ, no roots on the real axis, and n_d roots in DHP of λ.

Definition 2. If $A(\lambda)=\sum_{i=0}^{n} a_i\lambda^i$, where $a_i=\alpha_i+j\beta_i$, and assume $\alpha_n \neq 0$ [otherwise replace $A(\lambda)$ by $jA(\lambda)$], the testing function of $\delta_{\updownarrow}(A, \lambda)$ is defined by

$$T(\lambda)=\frac{A_I(\lambda)}{A_R(\lambda)} = \frac{\sum_{i=0}^{m} \beta_i\lambda^i}{\sum_{i=0}^{n} \alpha_i\lambda^i} \tag{1}$$

where $T(\lambda)$ is a real rational function. $A_I(\lambda)$ and $A_R(\lambda)$, which are the imaginary part and the real part of $A(\lambda)$ respectively, can be written in factored form:

$$A_I(\lambda)=\beta_m \prod_{i=1}^{m} (\lambda-z_i)=K_I A_I^*(\lambda) \tag{2}$$

$$A_R(\lambda)=\alpha_n \prod_{i=1}^{n} (\lambda-p_i)=K_R A_R^*(\lambda) \tag{3}$$

where $m \leq n$. $T(\lambda)$ can be written in factored form also:

$$T(\lambda)=K_T\frac{A_I^*(\lambda)}{A_R^*(\lambda)} \triangleq K_T\ T^*(\lambda) \tag{4}$$

where $K_T=\frac{K_I}{K_R}=\beta_m/\alpha_n$, and

$$T^*(\lambda) = \frac{\prod_{i=1}^{m} (\lambda-z_i)}{\prod_{i=1}^{n} (\lambda_i-p_i)} \tag{5}$$

Definition 3. The notation $RL(K, T^*(\lambda), \theta)|_{k1}^{k2}$ represents the conventional terminology, "root locus of $KT^*(\lambda)=e^{j\theta}$", except the range of K is bounded in $[k_1, k_2]$.

Definition 4. Branch distribution index $B_{\updownarrow}(K, T^*(\lambda), \theta)|_{k1}^{k2}$ of a root locus $RL(K, T^*(\lambda), \theta)|_{k1}^{k2}$ is an ordered set (m_u, m_0, m_d) if there are m_u branches of $RL(K, T^*(\lambda), \theta)|_{k1}^{k2}$ in UHP of λ, m_0 branches on the real axis and m_d branches in DHP of λ. If there is a branch starting from a point on the real axis and going into the UHP or DHP then, this branch must count to m_u or m_d, not m_0. If there is a root locus starting from a point in UHP (DHP), terminating in UHP (DHP) and crossing the real axis many times, then this root locus must be cut into many branches at the crossing points of the root locus and the real axis, and each branch in the UHP or DHP must count to m_u or m_d. If there is a pole p_i identical to a zero z_i then $p_i = z_i$ is a singular branch of root locus, otherwise the branch is non-singular.

From the "Branch distribution index", the general picture of root loci can be seen.

Definition 5. The root locus of a rational function is called separable if there is no branch starting from UHP (DHP), crossing the real axis and terminating in DHP (UHP).

Lemma 1. The root locus $RL(K, T^*(\lambda), \theta)|_{k1}^{k2}$ is separable, if and only if $m_0 + m_u + m_d = M$, where $M = \max\ (n, m)$ and $(m_u, m_0, m_d) = \beta_{\updownarrow}$ $(K, T^*(\lambda), \theta)\ |_{k1}^{k2}$. Lemma 1 can be proved directly from the definition of $B_{\updownarrow}\ (K,\ T^*(\lambda), \theta)|_{k1}^{k2}$

Lemma 2. If $T^*(\lambda)$ is defined as in Eq. (5), and let $A(\lambda) = A_R^*(\lambda) - K_T e^{-j\theta} A_I^*(\lambda)$ then $\delta_{\updownarrow}\ (A,\ \lambda) = B_{\updownarrow}\ (K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$, for $K_T \in [k_1,\ k_2]$.

Proof: $\beta_{\updownarrow}\ (K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$ is the distribution index for the root locus $RL(K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$ or $K_T T^*(\lambda) = e^{j\theta}$ for $K_T \in [k_1,\ k_2]$. By the definition of $T^*(\lambda) = \dfrac{A_I^*(\lambda)}{A_R^*(\lambda)}$, one gets

$$K_T \frac{A_I^*(\lambda)}{A_R^*(\lambda)} = e^{j\theta}, \qquad K_T \in [k_1,\ k_2]$$

or $\qquad A_R^*(\lambda) - K_T e^{-j\theta} A_I^*(\lambda) = 0,\ K_T \in [k_1,\ k_2]$

If $K_T \in [k_1,\ k_2]$, then $\{\lambda : A(\lambda) = 0\} \in RL(K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$. Since $RL(K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$ is separable, therefore $\delta_{\updownarrow}\ (A,\ \lambda) = B_{\updownarrow}\ (K,\ T^*(\lambda),\ \theta)|_{k1}^{k2}$.

Lemma 2 gives the relation between root distribution of a polynomial $A(\lambda)$ and the distribution of root locus of a rational function $T(\lambda)$.

Definition 6. Positive trend branch distribution index of $RL(K,\ T^*(\lambda),\ \theta)$ is defined as $\beta_{\updownarrow}\ (0^+,\ T^*(\lambda),\ \theta) = \lim_{KT \to 0+} \beta_{\updownarrow}\ (K,\ T^*(\lambda),\ \theta)|_0^{KT}$. Negative trend branch distribution index is defined as $B_{\updownarrow}\ (0^-,\ T^*(\lambda),\ \theta) = \lim_{KT \to 0-} \beta_{\updownarrow}$ $(K, T^*(\lambda),\ \theta)|_{KT}^{0}$.

Lemma 3. For given $T^*(\lambda)$ and $A(\lambda)$ as defined in Eq. (5) and Lemma 2, if $RL(K,\ T^*(\lambda),\ \theta)|_{0}^{0-} = \lim_{k1 \to 0-} RL(K,\ T^*(\lambda),\ \theta)|_{k}^{0}$. and $RL(K,\ T^*(\lambda),\ \theta)|_0^{0+}$ $= \lim_{k2 \to 0+} RL(K,\ T^*(\lambda),\ \theta)|_0^{k2}$ are separable, then

$$\delta_{\updownarrow}\ (A,\ \lambda) = \beta_{\updownarrow}\ (0^+,\ T^*(\lambda),\ \theta) \qquad \text{for } K_T > 0$$
$$= B_{\updownarrow}\ (0^-,\ T^*(\lambda),\ \theta) \qquad \text{for } K_T < 0$$

proof:

For $RL(K,\ T^*(\lambda),\ \theta)|_0^{0-}$ is separable, then

$$\beta \updownarrow (K,\ T^*(\lambda),\ \theta)|_{-\infty}^{0} = \beta \updownarrow (0^-,\ T^*(\lambda),\ \theta). \tag{6}$$

Since $A(\lambda) = A_R^*(\lambda) - K_T\ e^{-j\theta} A_I^*(\lambda)$, if $K_T < 0$, then

$K_T \in [0,\ -\infty]$

From Lemma 3 and Eq. (6), one gets

$$\delta_{\updownarrow}(A, \lambda)=\beta_{\updownarrow}(0^-, T^*(\lambda), \theta), \text{ for } K_T<0.$$

By the same reasoning, if $RL(K, T^*(\lambda), \theta)|_0^{0+}$ is separable then

$$\delta_{\updownarrow}(A, \lambda)=\beta_{\updownarrow}(0^+, T^*(\lambda), \theta), \text{ for } K_T>0.$$

Lemma 4. $RL(K, T^*(\lambda), \frac{\pi}{2})|_0^{0+}$ and $RL(K, T^*(\lambda), \frac{\pi}{2})|_0^{0-}$ are all separable and only singular branches can exist on real axis

Proof: From Eq. (1), the coefficients of $T(\lambda)$ are all real. Therefore, if λ (on the real axis) is neither a pole nor a zero then

$$K_T\ T^*(\lambda) \neq j \qquad \text{for real } K_T,$$

$$\text{or } T(\lambda) \neq i$$

so the real axis does not belong to the $RL(K, T^*(\lambda), \frac{\pi}{2})|_{k1}^{k2}$ except the poles or zeros of $T^*(\lambda)$. This shows that $RL(K, T^*(\lambda), \frac{\pi}{2})|_{k2}^{k1}$ is separable, and therefore $RL(K, T^*(\lambda), \frac{\pi}{2}|_0^{0+}$ and $RL(K, T^*(\lambda), \frac{\pi}{2})|_0^{0-}$ are separable.

The branch on the real axis is only the singular case $p_i=z_i$.

Theorem 1 If $A(\lambda)=\sum_{i=0}^{n} a_i\lambda^i$ where $a_i=\alpha_i+j\beta_i$ and $\alpha_n \neq 0$, and $T(\lambda)$ is defined as

$$T(\lambda)=\frac{\sum_{i=0}^{m}\beta_i\lambda^i}{\sum_{i=0}^{n}\alpha_i\lambda^i}=K_T T^*(\lambda)$$

where

$$T^*(\lambda)=\frac{\prod_{i=1}^{m}(\lambda-z_i)}{\prod_{i=1}^{n}(\lambda-p_i)} \qquad K_T=\beta_m/\alpha_n, \text{ and } m\leq n,$$

then

$$\delta_{\updownarrow}(A, \lambda)=\beta_{\updownarrow}(0^+, T^*(\lambda)\tfrac{\pi}{2}) \text{ if } K_T>0$$

$$=\beta_{\updownarrow}(0^-, T^*(\lambda), \tfrac{\pi}{2}), \text{ if } K_T<0$$

The proof is directly from Lemmas 3 and 4.

Lemma 5. If $\beta_{\updownarrow}(0^+, T^*(\lambda), \frac{\pi}{2})=(n_1, n_2, n_3)$, then

$$\beta\updownarrow(0^-, T^*(\lambda), \tfrac{\pi}{2})=(n_3, n_2, n_1)$$

Proof: By Definition 5, one gets

$$RL(K, T^*(\lambda), \theta)|_{-K_T}^{0}=RL(K, T^*(\lambda), \pi+\theta)|_0^{K_T}$$

therefore

$$\beta_{\updownarrow}(0^-, T^*(\lambda), \theta)=\beta_{\updownarrow}(0^+, T^*(\lambda), \pi+\theta) \tag{7}$$

If $\bar{\lambda}$ represents the conjugate of λ, then

$$\mathrm{DHP}(\bar{\lambda}) = \mathrm{UHP}(\lambda) \tag{8}$$

$$\mathrm{UHP}(\bar{\lambda}) = \mathrm{DHP}(\lambda) \tag{9}$$

Since $T^*(\lambda)$ is a real function, so

$$\overline{T^*}(\bar{\lambda}) = T^*(\bar{\lambda})$$

where $\overline{T^*}(\bar{\lambda}) = \prod_{i=1}^{m} (\lambda - \bar{z}_i) \;/\; \prod_{i=1}^{n} (\lambda - \bar{p}_i)$

If $\lambda \varepsilon\ RL(K, T^*(\lambda), \theta)|^{0}_{-K_T}$ then

$$KT^*(\lambda) = e^{j\theta} \leftrightarrow \overline{KT^*}(\lambda) = e^{-j\theta}$$

$$\leftrightarrow KT^*(\bar{\lambda}) = e^{-j\theta} \tag{10}$$

therefore

$$\beta_{\updownarrow}\ (0^+, T^*(\lambda), \theta) = \beta_{\updownarrow}\ (0^+, T^*(\bar{\lambda}), -\theta) \tag{11}$$

From Eqs. (7) and (10), one gets

$$\beta_{\updownarrow}\ (0^-, T^*(\lambda), \theta) = \beta_{\updownarrow}\ (0^+, T^*(\bar{\lambda}), -\pi-\theta)$$

$$= \beta_{\updownarrow}\ (0^+, T^*(\bar{\lambda}), \pi-\theta) \tag{12}$$

For $\theta = \frac{\pi}{2}$, Eq. (11) can be rewritten as

$$\beta_{\updownarrow}\ (0^-, T^*(\lambda), \frac{\pi}{2}) = \beta_{\updownarrow}\ (0^+, T^*(\lambda), \frac{\pi}{2}) \tag{13}$$

From Eqs. (8), (9) and (12), one can conclude that

$$\beta_{\updownarrow}\ (0^-, T^*(\lambda), \frac{\pi}{2}) = (n_3, n_2, n_1)$$

$$\text{if} \quad \beta_{\updownarrow}\ (0^+, T^*(\lambda), \frac{\pi}{2}) = (n_1, n_2, n_3). \tag{14}$$

Corollary 1. If $\beta_{\updownarrow}\ (0^+, T^*(\lambda), \frac{\pi}{2}) = (n_1, n_2, n_3)$, then

$$\delta_{\updownarrow}\ (A, \lambda) = (n_1, n_2, n_3), \text{ for } K_T > 0 \tag{15}$$

$$= (n_3, n_2, n_1), \text{ for } K_T < 0 \tag{16}$$

Now the relation between root distribution of a polynomial $A(\lambda)$ and the branch distribution of the testing function $T(\lambda)$ is formulated.

II. THE POSITIVE-TREND BRANCH DISTRIBUTION INDEX

In order to find the criterion of the root distribution index of a polynomial, the general formula of $\beta_{\updownarrow}\ (0^+, T^*(\lambda), \frac{\pi}{2})$ will be presented in this section.

Definition 7. The multiplicity of the singularity q_i (pole or zero) is defined by $m_i(q_i)$, if q_i is a $|m_i|$ − multiple pole or zero fo $T^*(\lambda)$. $m_i(q_i) = |m_i|$ if q_i is a zero; $m_i(q_i) = -|m_i|$ if q_i is a pole. If q_i is a simple zero or pole, $m_i = 1$ or -1.

Definition 8. The irreducible real singularity r_i is defined by the combination of the real poles and zeros of $T^*(\lambda)$. The multiplicity of r_i is defined by $m_{ri}=m_i(p_i)+m_i(z_i)$, if $p_i=z_i=r_i$ (common root); $m_{ri}=m_i(p_i)$ or $m_{ri}=m_i(z_i)$, if no common poles and zeros occurred.

Definition 9. Common multiplicity of the real pole is defined by $m_0 i=\min[-m_i(p_i),\ m_i(z_i)]$, if $p_i=z_i$; $m_{c1}=0$ if no zero is identical to this pole.

Lemma 6. If there are ℓ distinct poles on the real axis and if $\beta\updownarrow(0^+,\ T^*(\lambda),\ \pi 2)\triangleq(n_u,\ n_0,\ n_d)$, then

$$n_0 = \sum_{1=1}^{\ell} m_{ci} \tag{17}$$

Proof: From Lemma 4, there are only singular branches on the real axis, so n_0 is the number of the singular branches from the real poles. If the common multiplicity of a pole p_i is m_{ci} then there are m_{ci} singular branches of root locus from p_i to z_i (where $z_i=p_i$). Therefore

$$n_0 = \sum_{1=1}^{\ell} m_{ci} \tag{18}$$

Lemma 7. If there are ℓ distinct poles on the real axis and C pairs of complex poles with multiplicity $m_i(p_i)$ for $T^*(\lambda)$, and if $\beta\updownarrow[0^+,\ T^*(\lambda),\frac{\pi}{2}]\triangleq(n_u,\ n_0,\ n_d)$, $m_i^*(p_i)=-m_i(p_i)$, $m_{ri}^*=-m_{ri}$, then

$$n_d = \sum_{1}^{\ell} n_i(\pi,\ 2\pi)+\sum_{1}^{C} m_i^*(p_i) \tag{19}$$

$$n_u = \sum_{1}^{\ell} n_i(0,\ \pi\ )+\sum_{1}^{C} m_i^*(p_i) \tag{20}$$

where $n_i(n,\ 2\pi) = \dfrac{m_{ri}^*+e_i}{2}$ for r_i to be a pole (21)

$n_i(0,\ \pi) = \dfrac{m_{ri}^*-e_i}{2}$ for r_i to be a pole (22)

$$e_i=(m_{ri}^*)_{mod}\ {}^{*}_{2}(-1)^{\alpha} \tag{23}$$

$$\alpha = (\sum_{k=1}^{i-1} m_{ri})_{mod\ 2} \tag{24}$$

Proof: Consider the pole p_i of $T^*(\lambda)$. If p_i is complex with multiplicity $m_i(p_i)$, then no matter there is a zero z_i equal to p_i or not, the root loci $RL(K,\ T^*(\lambda),\ \frac{\pi}{2})|_0^{0+}$ from p_i are all in the UHP (DHP) if p_i is in the UHP (DHP). And the number of root loci from p_i is equal to the multiplicity $m_i(p_i)$. Therefore-n_u and n_d due to the complex poles are

$$n_{dc} = n_{uc} = \sum_{1}^{C} m_i^*(p_i) \tag{25}$$

for $T^*(\lambda)$ has real coefficients and the complex poles should occur in conjugate pairs.

※ $(m_{ri}^*)_{mod_2}$ means $(m_{ri}^*)_{mod_2}=1$ for m_{ri}^* is odd, $(m_{ri}^*)_{mod_2}=0$ for m_{ri}^* is even.

If a pole is on the real axis with the irreducible multiplicity m_{ri}, then assume $\lambda = r_i + \varepsilon e^{j\phi}$ for $RL(K, T^*(\lambda), \frac{\pi}{2})|_0^{0+}$ from r_i, one has

$$\lim_{k\to 0+} KT^*(\lambda) = \lim_{k\to 0+} KT_i^*(\lambda)(\lambda - r_i)m_{rj}$$

$$= [\lim_{k\to 0+} KT_i^*(\lambda)\varepsilon^{m_{ri}}]\, e^{jm_{ri}\phi} = j \tag{26}$$

where $T_i^*(\lambda) = T^*(\lambda)(\lambda - r_i)^{-m_{ri}}$ (27)

Assume

$$\lim_{k\to 0+} KT_i^*(\lambda)\varepsilon^{m_{ri}} = e^{j\psi} \tag{28}$$

then Eq. (26) can berewritten as

$$e^{-jm_{r_i}^*\phi} = e^{j(\frac{\pi}{2}-\phi)} \tag{29}$$

The solution for ϕ is

$$\phi = \frac{1}{m_{ri}^*}(2n\pi - \frac{\pi}{2} + \psi) \qquad n = 1, 2 \cdots\cdots\cdots m_{ri}^* \tag{30}$$

Since r_i is real, the principal value of ψ can be taken as

$$\psi = ARG\left[\frac{\prod_{k=1}^{m1}(r_i - z_k)^{m_{zk}}}{\prod_{\substack{k=1\\k\neq i}}^{nu}(r_i - p_k)^{-m_{pk}}}\right] \tag{31}$$

where z_k and p_k are real, since the complex poles and zeros have no contribution to the angle ψ. Eq. (31) can be written as

$$\psi = ARG \prod_{\substack{k=1\\k\neq i}}^{Md} (r_i - r_k)^{m_{rk}} \tag{32}$$

if the number of real distinct singularities is M_d. From Eq. (32)

$$\psi = \sum_{\substack{k=1\\k\neq i}}^{Md} ARG(r_i - r_k)^{m_{rk}} = \sum_{\substack{k=1\\k\neq i}}^{Md} m_{rk} ARG(r_i - r_k) \tag{33}$$

But $ARG(r_i - r_k) = 0$ for $r_i > r_k$ (34)

$ARG(r_i - r_k) = \pi$ for $r_i < r_k$ (35)

therefore

$$\psi = \sum_{k=1}^{i-1} m_{rk}\pi = \alpha^*\pi \tag{36}$$

where

$$\alpha^* = \sum_{k=1}^{i-1} m_{rk} \tag{37}$$

If ψ is the principal value then

$\psi = \pi$ if α^* is odd (38)

$\psi = 0$ if α^* is even (39)

Let's define $\alpha = (\alpha^*)_{mod\ 2}$ (40)

then $\psi = \alpha\pi$ (41)

From Eqs. (30) and (41)

$$\phi = \frac{1}{m_{ri}^*}\,(2n+\alpha-\frac{1}{2})\pi \qquad n=1,\ 2,\cdots\cdots m_{ri}^* \tag{42}$$

Consider the following cases

(i) If m_{ri}^* is even then

$$0<\phi<\pi \quad \text{for } n=1,\ 2,\cdots\cdots\frac{m_{ri}^*}{2} \tag{43}$$

$$\pi<\phi<2\pi \quad \text{for } n=\frac{m_{ri}^*}{2}+1,\cdots\cdots m_{ri}^* \tag{44}$$

(ii) If m_{ri}^* is odd

(a) if $\alpha=0$

$$0<\phi<\pi \quad \text{for } n=1,\ 2\cdots\cdots\frac{m_{ri}^*}{2}+1 \tag{45}$$

$$\pi<\phi<2\pi \quad \text{for } n=\frac{m_{ri}^*+1}{2}+1,\cdots\cdots m_{ri}^* \tag{46}$$

(b) if $\alpha=1$

$$0<\phi<\pi \quad \text{for } n=1,\ 2,\cdots\cdots\frac{m_{ri}^*-1}{2} \tag{47}$$

$$\pi<\phi<2\pi \quad \text{for } n=\frac{m_{ri}^*+1}{2}\cdots\cdots m_{ri}^* \tag{48}$$

From Equations (43) to (48), one can conclude that

$$n_i(0,\ \pi)=\tfrac{1}{2}[m_{ri}^*-e_i] \tag{49}$$

$$n_i(\pi,\ 2\pi)=\tfrac{1}{2}[m_{ri}^*+e_i] \tag{50}$$

$$\left.\begin{array}{l}\text{where } n_i(0,\ \pi) \text{ is the number of possible } \phi \text{ for } 0<\phi<\pi \\ n_i(\pi,\ 2\pi) \text{ is the number of possible } \phi \text{ for } \pi<\phi<2\pi\end{array}\right\} \tag{51}$$

$$\left.\begin{array}{ll}e_i=(-1)^{\alpha} & \text{if } m_{ri}^* \text{ is odd} \\ \quad =0 & \text{if } m_{ri}^* \text{ is even}\end{array}\right\} \tag{52}$$

$$e_i=(-1)^{\alpha}\,(m_{ri}^*)_{mod\ 2} \tag{53}$$

Since the root loci $RL(K,\ T^*(\lambda), \frac{\pi}{2})|_{0-}^{0+}$ from r_i is described by $\lambda=r_i+\varepsilon e^{j\phi}$, $n_i(0,\ \pi)$ is equal to the number of root loci from r_i into UHP, and $n_i(\pi,\ 2\pi)$ is the number of root loci from r_i into DHP. So n_d and n_u due to the real poles are

$$n_{dr} = \sum_{1}^{\ell} n_i(\pi,\ 2\pi) \tag{54}$$

$$n_{ur} = \sum_{1}^{\ell} n_i(0,\ \pi) \tag{55}$$

From Equations (25), (54) and (55) one has

$$n_d=n_{dc}+n_{dr}=\sum_{i}^{\ell} n_i(\pi,\ 2\pi)+\sum_{i}^{c} m_i^*(p_i) \tag{56}$$

$$n_u=n_{uc}+n_{ur}=\sum_{i}^{\ell} n_i(0,\ \pi)+\sum_{i}^{c} m_i^*(p_i) \tag{57}$$

Definition 10. The irreducible real single singularity sequence is defined by the combination of the real poles and zeros of $T^*(\lambda)$; the multiple poles or zeros with multiplicity m must be treated as m single poles or zeros having the same position, and the sequence can be described as

$$r_1 \geq r_2 \geq \cdots\cdots\cdots \geq r_M \tag{58}$$

where

$$M = \sum_{1=1}^{Md} |m_{ri}| \tag{59}$$

and M_d is the total number of irreducible real distinct singularities.

Definition 11. The index of IRSSS [irreducible real single singularity sequence] is defined by

$$I_s(i) = \frac{1+(-1)^i}{2} \qquad \text{if } r_i \text{ is a pole} \tag{60}$$

$$I_s(i) = 0 \qquad \text{if } r_i \text{ is a zero} \tag{61}$$

The complement of the index of IRSSS is defined by

$$I_s^*(i) = 0 \qquad \text{if } r_i \text{ is a zero} \tag{60-a}$$

$$I_s^*(i) = \frac{1-(-1)^i}{2} \text{ if } r_i \text{ is a pole} \tag{61-a}$$

Lemma 8. If the total number of irreducible real singularities is M, then

$$n_{ur} = \sum_{1}^{\ell} n_i(0, \pi) = \sum_{1=1}^{M} I_s(i) \tag{62}$$

$$n_{dr} = \sum_{1}^{\ell} n_i(\pi, 2\pi) = \sum_{1=1}^{M} I_s^*(i) \tag{63}$$

and $n_{ur} + n_{dr} = M_p$ is equal to the number of irreducible real poles.

The proof of Lemma 8 is directly from Lemma. 7.

Theorem 2. The rational function $T^*(\lambda)$ has M irreducible real singularities, C pairs of complex poles, and $\beta_{\updownarrow}\ (0^+, T^*(\lambda), \frac{\pi}{2}) \stackrel{\triangle}{=} (n_u, n_0, n_d)$, then

$$n_u = \sum_{1=1}^{M} I_s(i) + \sum_{1}^{C} m_i^*(p_i) \tag{64}$$

$$n_0 = \sum_{1=1}^{\ell} m_{ci} \tag{65}$$

$$n_d = \sum_{1=1}^{M} I_s^*(i) + \sum_{1}^{C} m_i^*(p_i) \tag{66}$$

The proof of Theorem 2 is Lemmas 6, 7, and 8.

Theorem 3. if an n-th order polynomial is given by

$$A(\lambda) = \sum_{1=0}^{n} a_i \lambda^i \tag{67}$$

where $a_i = \alpha_i + j\beta_i$, and $\alpha_n \neq 0$, The testing function $T(\lambda)$ is defined by

$$T(\lambda) = \frac{\sum_{1=0}^{m} \beta_i \lambda^i}{\sum_{1=0}^{n} \alpha_i \lambda^i} = \frac{\beta_m \prod_{1=1}^{m} (\lambda - z_i)}{\alpha_n \prod_{1=1}^{n} (\lambda - p_i)} = K_T\ T^*(\lambda),\ m \leq n \tag{68}$$

then the root distribution index of $A(\lambda)=0$ is

$$\delta_{\updownarrow}(A, \lambda)=(n_+, n_0, n_-) \quad \text{if } K_T>0 \tag{69}$$

$$\delta_{\updownarrow}(A, \lambda)=(n_-, n_0, n_+) \quad \text{if } K_T<0 \tag{70}$$

where $n_+ = \sum_{1=1}^{M} I_s(i) + \sum_{1}^{C} m_i{*}(p_i)$ (71-a)

$$n_0 = \sum_{1=1}^{\ell} m_{ci} \tag{71-b}$$

$$n_- = \sum_{1=1}^{M} I_s{*}(i) + \sum_{1}^{C} m_i{*}(p_i) \tag{71-c}$$

and M is the irreducible real singularities of $T^*(\lambda)$.

C is the number of pairs of the complex poles of $T^*(\lambda)$.

$I_s(i)$ is the index of IRSSS, $I_s{*}(i)$ is the complement index of IRSSS.

ℓ is the number of distinct real poles,

m_{ci} is the common multiplicity of the real pole of $T^*(\lambda)$,

$m_i{*}(p_i)$ is the negative multiplicity of the complex pole p_i.

Theorem 3 is the combination of theorem 2 and corrollary 1.

III. ROOT DISTRIBUTION IN THE S-PLANE

The main purpose of this section is to present the criterion for the root distribution of $F(S)=0$ in the RHP, LHP, and on the imaginary axis of S-plane. If the S-plane is rotated $\pi/_2$clockwise, a new plane (or coordinate system) is the λ-plane. Then the UHP of λ is the LHP of S, and DHP of λ is the RHP of S, also the real axis of λ is the imaginary axis of S. So $F(j\lambda)$ is the transformation of $F(S)$ to λ-plane.

Consider an n-th order polynomial of S given by

$$F(S) = \sum_{1=0}^{n} f_i S^i = 0 \tag{72}$$

where $f_i = g_i + jh_i$ (73)

Substituting $S=j\lambda$ into Eq.(72) vields

$$F(j\lambda) = A(\lambda) = \sum_{1=0}^{n} (g_i + jh_i) j^i \lambda^i \tag{74}$$

$$= \sum_{1=0}^{n} (\alpha_i + j\beta_i)\lambda^i \tag{75}$$

where $\alpha_i = g_i j^i$ for i even (76)

$= h_i j^{i+1}$ for i odd (77)

$\beta_i = h_i j^i$ for i even (78)

$= g_i j^{i-1}$ for i odd (79)

Definition 12. Roots distribution index $\delta_{\leftrightarrow}(A, \lambda)$ of an algebraic equation $A(\lambda)=0$ is an ordered set or vector (n_l, n_i, n_r), if there are n_l roots of $A(\lambda)=0$ in LHP of λ, n_i roots on the imaginary axis and n_r roots in RHP of λ.

Lemma 9. If an n-th order polynomial of S is given by Eq. (72), and $A(\lambda) \triangleq F(j\lambda)$, then

$$\delta_{\updownarrow}(A, \lambda) = \delta_{\leftrightarrow}(F, S) \tag{80}$$

Lemma 9 is a mathematical expression of the statement in the beginning of this section.

Theorem 4. If an n-th order polynomial is given by

$$F(S) = \sum_{i=0}^{n} f_i S^i = 0,\ f_i = g_i + jh \text{ and } g_n \neq 0 \text{ or } h_i \neq 0$$

and $\quad A(\lambda) = F(j\lambda) = A_R(\lambda) + jA_I(\lambda)$

$$T(\lambda) = \frac{A_I(\lambda)}{A_R(\lambda)} = \frac{K_I A_I^*(\lambda)}{K_R A_R^*(\lambda)} \triangleq K_T\ T^*(\lambda)$$

where $\quad K_T = K_I / K_R$

$$A_I^*(\lambda) = \prod_{i=1}^{n} (\lambda - z_i)\ , \tag{81}$$

and

$$A_R^*(\lambda) = \prod_{i=1}^{n} (\lambda - p_i),\ m \leq n \tag{82}$$

then the root distribution index of $F(S) = 0$ is

$$\delta_{\leftrightarrow}(F, S) = (n_+, n_0, n_-) \quad \text{if } K_T > 0 \tag{83}$$

$$= (n_-, n_0, n_+) \quad \text{if } K_T < 0 \tag{84}$$

where

$$n_+ = \sum_{i=1}^{M} I_s(i) + \sum_{1}^{C} m_i^*(p_i)$$

$$n_0 = \sum_{i=1}^{\ell} m_{ci}$$

$$n_- = \sum_{i=1}^{M} I_s^*(i) + \sum_{1}^{C} m_i^*(p_i)$$

All the symbols M, C, $Mi^*(p_i)$, $I_s^*(i)$, ℓ and m_{ci} are the same as those defined in Theorem 3.

Theorem 4 is the direct result of Theorem 3 and Lemma 9. Eqs. (81) and (82) are called stability equations.

Theorem 5. If an n-th order polynomial is define as in Theorem 4, then $F(S)$ has no root in the RHP of S-plane if and only if

(1) z_i $(i=1, m)$ and $p_i (i=1, n)$ are all real.

(2) The irreducible real singularities of $T^*(\lambda)$ are simple and related in an alternative sequence.

(3) The largest singularity is a pole if $K_R K_I < 0$
The largest singularity is a zero if $K_R K_I > 0$.

Proof: Assume that $n \geq m$. From Theorem 4 $F(S)$ has no root in the RHP if and only if the third component of $\delta_{\leftrightarrow}(F, S)$ is equal to zero; or $n_- = 0$ if $K_T > 0$, $n_+ = 0$ if $K_T < 0$. Now, consider $K_T > 0$ or $K_I K_R > 0$. In this case

$$n_- = \sum_{i=1}^{M} I_s^*(i) + \sum_{1}^{C} m_i^*(p_i) \tag{85}$$

Since the first and second summation of Eq. (85) are all nonnegative and independent, therefore $n_{-}=0$ if and only if

$$\sum_{i=0}^{M} I_s^*(i)=0 \tag{86}$$

and $$\sum_{1}^{C} m_i^*(p_i)=0 \tag{87}$$

Eq. (87) implies that $C=0$ or $T^*(\lambda)$ has no complex pole. From Eq. (86), one gets

$$I_s^*(i)=0 \tag{88}$$

From Eq. (61-a) one can obtain

$$I_s^*(i)=\frac{1-(-1)^i}{2}=0 \quad \text{For } r_i \text{ is a pole} \tag{89}$$

i.e., $$i=2k,\ k=1,\ 2,\ 3\cdots\cdots \quad \text{for } r_i \text{ is a pole} \tag{90}$$

From Eq. (90) and the basic assumption that the number of poles is great or equal to the number of zeros, one can conclude that the irreducible real singularities of $T^*(\lambda)$ must be simple and in alternative sequence, the zeros are at the odd position, and the largest singularity is a zero. Assume that the number of zeros identical to poles is ℓ_c, the number of irreducible poles is n_p, and the number of irreducible zeros is n_z, then

$$m=\ell_c+n_z+C_z \tag{91}$$

$$n=\ell_c+n_p \tag{92}$$

where C_z is the number of complex zeros. Since

$$n\geq m \tag{93}$$

therefore

$$n_p\geq n_z+C_z \tag{94}$$

But the real poles and zeros are alternative and the largest singularity is a zero, so

$$n_z=n_p \tag{95}$$

$$\text{or } n_z=n_p+1 \tag{96}$$

From Eqs. (94), (95), and (96), one gets

$$C_z\leq n_p-n_z\leq 0 \tag{67}$$

But $$C_z\geq 0 \tag{98}$$

Hence $C_z=0$ or there is no complex zero in $T^*(\lambda)$. Now, half of Theorem 5 is proved.

If $K_RK_I<0$, it is not difficult to prove that there is no complex pole in $T^*(\lambda)$, and the irreducible real singularities of $T^*(\lambda)$ are simple and in alternative sequence, and the largest singularity is a pole i.e. $n_p=n_z$ or $n_p=n_z+1$. From Eqs. (91), (92) and (94), one gets

$$0\leq C_z\leq n_p-n_z\leq 1 \tag{99}$$

or $$0\leq C_z\leq 1 \tag{100}$$

Since the stability-equation $A_I(\lambda)$ having real coefficients, if $A_I(\lambda)=0$ has complex roots, they must occur in pairs, so C_z is even. From Eq. (100), one can confirm $C_z=0$, or there is no complex zero in $T^*(\lambda)$. The same conclusion can be obtained for the case $n \leq m$.

Corollary 5. If an n-th order polynomial is given by

$$F(S) = \sum_{i=0}^{n} f_i S_i = 0$$

where $f_i = g_i + jh_i \qquad f_n \neq 0$

and if the stability equations in S-domain are chosen as

$$F_R(S) = g_n S^n + jh_{n-1} S^{n-1} + g_{n-2} S^{n-2} \cdots\cdots$$

$$F_I(S) = jh_n S^n + g_{n-1} S^{n-1} + jh_{n-2} S^{n-2} \cdots\cdots$$

then, $F(S)$ has no root in the RHP of S-plane, if and only if

(1) The roots of the stability equations $F_R(S)$ and $F_I(S)$ are all imaginary.

(2) The irreducible imaginary singularities of $F_R(S)$ and $F_I(S)$ are simple and in alternative sequence.

(3) a. The largest singularity is the root of $F_R(S)=0$ if $g_n h_n < 0$

b. The largest singularity is the root of $F_I(S)=0$ if $g_n h_n > 0$

c. $g_n g_n - 1 < 0 \qquad$ if $h_n = 0$

d. $h_n h_n - 1 > 0 \qquad$ if $g_n = 0$

Proof: Let $\qquad F_1(S) = j^{-n} F(S) = 0 \qquad (101)$

Since $F(S)=0$, the constant j^{-n} doesn't change the root distribution of $F(S)$

or $\qquad \delta_{\leftrightarrow}(F, S) = \delta_{\leftrightarrow}(F_1, S) \qquad (102)$

Thus $\qquad F_1(S) = j^{-n}[F_R(S) + F_I(S)] \qquad (103)$

or $\qquad F_1(j\lambda) = A_R(\lambda) + jA_I(\lambda) \qquad (104)$

where $\qquad A_R(\lambda) = j^{-n} F_R(j\lambda) \qquad (105)$

$$= g_n \lambda^n + h_{n-1}\lambda^{n-1} - g_{n-2}\lambda^{n-2} \cdots\cdots \qquad (106)$$

$$A_I(\lambda) = j^{-n} F_I(j\lambda) \qquad (107)$$

$$= h_n \lambda^n - g_{n-1}\lambda^{n-1} - h_{n-2}\lambda^{n-2} \cdots\cdots \qquad (108)$$

Based upon Eq. (104) and the statement of Theorem 5, $A_R(\lambda)$ should be taken as the denominator of the testing function $T(\lambda)$. From Theorem 5, the roots of $A_R(\lambda)=0$ and $A_I(\lambda)=0$ should be all real if $F(S)$ has no root in RHP. From Eqs. (105) and (107), one can confirm that the roots of $F_R(S)=0$ and $F_I(S)=0$ must be all imaginary. Also from the second condition of Theorem 5, the irreducible imaginary singulanities of $F_R(S)$ and $F_I(S)$ are simple and in alternative sequence. The condition (3) of Theorem 5 can be clarified by considering the following cases:

Case (i). If $g_n h_n \neq 0$, then comparing Eqs. (106) and (108) with the stability equations of theorem 5, gets

$$g_n = K_R \quad (108)$$
$$h_n = K_I \quad (109)$$

Therefore the condition (3) [(a) and (b)] of this corollary is an alternate of condition (3) of Theorem 5.

Case (ii) If $h_n=0$, $g_n g_{n-1} \neq 0$, then

$$g_n = K_R \quad (111)$$
$$-g_{n-1} = K_I \quad (112)$$

In this case, the order of $A_R(\lambda)$ is n and that of $A_I(\lambda)$ is n-1. Since the roots of $A_R(\lambda)$ and $A_I(\lambda)$ should be alternative, therefore the largest singularity should be that of $A_R(\lambda)$ or the pole of testing function, and from the first part of condition (3) of Theorem 5 one has

$$K_R K_I = -g_n g_{n-1} < 0 \quad (113)$$

or $$g_n g_{n-1} > 0 \quad (114)$$

Therefore condition (3)-(c) of this corollary is proved.

Case (iii). If $g_n=0$ and $h_n h_{n-1} \neq 0$, then

$$K_R = h_{n-1} \quad (115)$$
$$K_I = h_n \quad (116)$$

By the similar reasoning of case (ii) and from the second part of the condition (3) of Theorem 5, one has

$$K_R K_I > 0 \quad (117)$$

or $$h_n h_{n-1} > 0 \quad (118)$$

Thus the condition (3)-(d) of this corollary is proved.

If the difference of the order of $A_R(\lambda)$ and $A_I(\lambda)$ or $F_R(S)$ and $F_I(S)$ is greater than 1, the singularities of $A_R(\lambda)$ and $A_I(\lambda)$ can't be in alternative sequence, so do those of $F_R(S)$ and $F_I(S)$. Therefore no other case can exist except the above three cases if $F(S)$ has no root in the RHP.

IV. EXAMPLES

The examples given in this section are only concerning the roots of polynomials. The applications of the criterion to engineering problems will be presented in separated parts (PART II TO V) of this paper.

Example 1. [13] Find the root distribution of the following equation

$$F(S) = S^6 + S^5 + 3S^3 + 3S^2 + 2S + 1 \quad (119)$$

Solution: The stability equations in λ-domain are

$$A_R(\lambda) = -\lambda^6 + 3\lambda^4 - 3\lambda^2 + 1 \quad (120)$$
$$A_I(\lambda) = \lambda^5 - 3\lambda^3 + 2\lambda \quad (121)$$

From Eq. (120), one gets

$$K_R = -1 \quad (122)$$

and the poles of the testing function are

$$p_i = \pm 1,\ \pm 1,\ \pm 1 \quad (123)$$

From Eq. (121), one has

$$K_I = 1 \quad (124)$$

and the zeros of the testing function are

$$z_i = \pm 1, \ \pm\sqrt{2}, \ 0 \quad (125)$$

There is no complex pole, or

$$C = 0 \quad (128)$$

From Eqs. (123) and (125) $\ell=2$, $m_{c1}=m_{c2}=1$ (for $p=z=\pm 1$) and the irreducible singularities are 7 or $M=7$. The index of IRSSS according to Eq. (60) is

$$\{I_s(i)\,|\,i=1, 7\} = \{0, 1, 0, 0, 0, 1, 0\} \quad (127)$$

and the complement of the index of IRSSS is

$$\{I_s^*(i)\,|\,i=1, 7\} = \{0, 0, 1, 0, 0, 1, 0, 0\} \quad (128)$$

By Eqs. (69) to (71), one gets

$$n_+ = \sum_{1=1}^{7} I_s(i) + \sum_{1}^{C} m_i^*(p_i) = 2 \quad (129)$$

$$n_0 = \sum_{1=1}^{\ell} m_{ci} = 2 \quad (130)$$

$$n_- = \sum_{1=1}^{7} I_s^*(i) + \sum_{1}^{C} m_i^*(p_i) = 2 \quad (131)$$

$$K_T = K_I/K_R = -1 < 0 \quad (132)$$

By Eq. (68),

$$\delta_{\updownarrow}(A, \lambda) = (n_-, n_0, n_+) \quad (133)$$

$$= (2, 2, 2) \quad (134)$$

From Theorem 4

$$\delta_{\leftrightarrow}(F, S) = \delta_{\updownarrow}(A, \lambda)$$

$$= (2, 2, 2) \quad (135)$$

That means Eq. (119) has 2 roots in the LHP, 2 roots on the imaginary axis and 2 roots in the RHP. Solving the roots of $F(S)=0$, one gets

$S = 0.121744 - 1.30662j, \ 0.121744 + 1.30662j$ in RHP

$1.0j, \ -1.0j$ on I-axis

$-0.621744 - 0.440597j, \ -0.621744 + 0.440597j$ in LHP

Example 2. Find the root distribution of the following equation.

$$F(S) = (1+j)S^8 + 3S^7 + (1-9j)S^6 + (-27+2j)S^5 + (-79-16j)S^4$$
$$-48S^3 + (-81+144j)S^2 + (433-162j)S - 162 \quad (136)$$

Solution: The stability equations in λ-domain are

$$A_R(\lambda) = \lambda^8 - \lambda^6 - 2\lambda^5 - 79\lambda^4 + 81\lambda^2 + 162\lambda - 162 \quad (137)$$

$$A_I(\lambda) = \lambda^8 - 3\lambda^7 + 9\lambda^6 - 27\lambda^5 - 16\lambda^4 + 48\lambda^3 - 144\lambda^2 + 432 \quad (138)$$

Taking $A_R(\lambda)$ as the denominator of the testing function $T(\lambda)$, and from Eqs. (137) and (138), one can find the poles (p_i) and zeros (z_i) of the stability equations as

$$p_i = 1,\ 1,\ \pm 3,\ \pm 3j,\ -1 \pm j \tag{139}$$

$$z_i = 0,\ \pm 2,\ +3,\ \pm 3j,\ \pm 2j \tag{140}$$

and $$K_T = 1 > 0 \tag{141}$$

By Eqs. (139) and (140), one can obtain

$$\ell = 3$$

and $$\{m_{ci} \mid i=1,\ 2,\ 3\} = \{1,\ 0,\ 0\} \tag{142}$$

The irreducible singularities are 6 ($M=6$), thus

$$\{I_s\ (i) \mid i=1,\ 6\} = \{0,\ 1,\ 0,\ 0,\ 0,\ 1\} \tag{143}$$

$$\{I_s^*(i) \mid i=1,\ 6\} = \{0,\ 0,\ 1,\ 0,\ 0,\ 0\} \tag{144}$$

Also from Eq. (139)

$$C = 2 \tag{145}$$

$$m_1^*(p) = m_2^*(p) = 1 \tag{146}$$

By Eq. (141) to Eq. (146), one gets

$$\delta_{\leftrightarrow}(F,\ S) = \delta_{\updownarrow}\ (A,\ \lambda) = (4,\ 1,\ 3) \tag{147}$$

Solving the roots of $F(S)=0$, one gets

$S =$ 0.389075+0.133282j 1.116631−0.0733563j 3.0 } in RHP

−2.383833−1.160900j −0.280783+1.647869j −0.341089−2.046895j −3.0 } in LHP

3j ……………………………on the I-axis (148)

The numerical values of the roots show that the root distribution is that of Eq. (147).

The above two examples are the singular cases of the Routh Criterion [13]. It is impossible to use computer to find the root distribution of these equations directly if one uses Routh Criterion. But if one uses the stability-equation method there will be no trouble, since there is no difference between the singular and regular cases.

Example 3. Find the root distribution of the following equation.

$$F(S) = (1+2j)S^5 + (3-4j)S^4 + (5+6j)S^3 + (7-8j)S^2 + (9+10j)S + 11 - 12j = 0 \tag{149}$$

Solution: The stability equations in λ-domain are

$$A_R(\lambda) = -2\lambda^5 + 3\lambda^4 + 6\lambda^3 - 7\lambda^2 - 10\lambda + 11 \tag{150}$$

$$A_I(\lambda) = \lambda^5 - 4\lambda^4 - 5\lambda^3 + 8\lambda^2 + 9\lambda - 12 \tag{151}$$

By the same process of Examples 1 and 2 one gets

$$\ell = 1 \tag{152}$$

and $$m_{c1} = 0 \tag{153}$$

The irreducible singularities ($M=2$) for the real pole and zero are

$$p_1 = 1.684533 \tag{154}$$

$$z_1 = 4.641738 \tag{155}$$

and

$$\{I_s\ (i)\,|\,i=1,\ 2\}=\{0,\ 1\} \tag{156}$$

$$\{I_s{}^*(i)\,|\,i=1,\ 2\}=\{0,\ 0\} \tag{157}$$

There are 4 simple complex poles or $C=2$ and $m_1{}^*(p)=m_2{}^*(p)=1$. From Eqs. (150) and (151), $KT=-2$, therefore Eq. (84) gives

$$\delta_{\leftrightarrow}(F,\ S)=(2,\ 0,\ 3) \tag{158}$$

Solving Eq. (149), one gets

$S=\ -0.416098+1.06954j\quad -0.654659-1.30131j$ in LHP

$1.14398+2.55998j\qquad 0.627144-1.33660j$
$0.299214+1.00840j$ } in RHP

The numerical roots of Eq. (149) show that the root distribution index [Eq. (158)] is correct.

CONCLUSIONS

A criterion for testing the root distribution of a polynomial $F(S)$ with real or complex coefficients has been presented. The criterion is based upon two testing polynomials called stability equations, and only the real roots of the stability equations need to be considered, since the number of complex roots of stability equations can be calculated once the number of real roots is known.

The criterion can be operated by use of a digital computer for both the singular and regular cases of Routh Criterion, but the latter can't be operated by a digital computer once the singular case is occurred.

The presented criterion is useful for solving engineering problems since there are many mathematical models of physical systems having complex coefficients in system characteristic equations.

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LIST OF SYMBOLS

$a_i = \alpha_i + j\beta_i$	complex coefficients of $A(\lambda)$
$A(\lambda)$	Polynomial in λ domain
C	Number of pairs of complex poles
$f_i = g_i + jh_i$	Complex coefficients of $F(S)$
$F(S)$	Polynomial in S-domain
$IRSSS$	Irreducible real single singularity sequence
$I_s(i)$	Index of IRSSS
$I_s{*}(i)$	Complement index of IRSSS
I-$axis$	Imaginary axis of complex plane.
ℓ	Number of distinct real poles
$K = K_T = \frac{\beta_m}{\alpha_n}$	Constant gain of $T(\lambda)$
M	Total number of irreducible real singularities
m_u	Branches of root loci in UHP of λ
m_0	Branches of root loci on the real axis
m_d	Branches of root loci in DHP of λ
$m_i(q_i)$	Multiplicity of the singularity q_i
m_{ci}	Common multiplicity of real pole
m_{ri}	Multiplicity of irreducible real singularity
$m_i{*}(p_i)$	Negative multiplicity of pole p_i
M_p	Number of irreducible real poles
n	Order of polynomial
n_d	Roots of $A(\lambda)$ in DHP of λ
n_i	Roots of $A(\lambda)$ on imaginary axis
n_0	Roots of $A(\lambda)$ on the real axis
n_u	Roots of $A(\lambda)$ in UHP of λ
n_l	Roots of $A(\lambda)$ in LHP
n_r	Roots of $A(\lambda)$ in RHP
p_i	poles
r_i	Irreducible real singularity
R-$axis$	Real axis of complex plane
S	Laplace operator
$T(\lambda)$	Testing function of $\delta_{\updownarrow}(A, \lambda)$
$T{*}(\lambda)$	Normalized testing function
z_i	Zeros
λ	Operator $(S = j\lambda)$
$\overline{\lambda}$	Conjugate of λ
$\delta_{\leftrightarrow}(F, S)$	Root distribution index of $F(S)=0$ in LHP, I-axis and RHP of S-plane
$\delta_{\leftrightarrow}(A, \lambda)$	Poots distribution index of $A(\lambda)=0$ in DHP, UHP and Raxis of λ-plane
$\beta_{\updownarrow}(K, G(\lambda), \theta)\mid_{k1}^{k2}$	Branch distribution index
LHP	Left half plane
RHP	Right half plane
DHP	Down half plane
UHP	Up half plane

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