

STABILITY EQUATION METHOD

PART II: COMPUTATION ROUTINES FOR SOLVING POLYNOMIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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and

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Abstract : *Methods for computing the roots of polynomial equations with complex coefficients by use of the stability equations are presented in this paper. Based upon the presented methods, computation routines are written, which can be used to solve polynomial equations with complex coefficients by dealing with real numbers only. In comparison with current methods the advantages and disadvantages of the presented methods are given, and applications to the analysis and design of control systems are considered*

INTRODUCTION

Control systems with mathematical models having complex numbers or parameters are quite common [1, 2, 3]. In order to analyze such kind of systems, it is necessary to solve system characteristic equations having complex coefficients. For doing this kind of calculations several computation routines are available, such as HELP, CPOLRT and MULLP prepared by the Control Data Corporation (CDC) for the CDC-cyber-72-14 digital computers[4]. Although each of these methods has a particular approach for calculating the roots, all of them are dealing directly with complex numbers; thus several particular subroutines are required, and usually these subroutines are not suitable for small digital computers.

The general approach of the methods presented in this paper is to deal with two stability equations with only real coefficients, and only the real roots of stability equations are required for finding all the characteristic roots of a polynomial equation with complex coefficients[5]. Therefore a small computer can be used to solve high order polynomial equations with complex coefficients.

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Besides, the presented method is useful for the analysis of damping characteristics of control systems, such as finding the characteristic roots in a particular sector of the S -plane. The basic approach is presented in the following two sections.

I. EVALUATION OF ROOTS BY ROTATING THE COORDINATE AXES

In this section, a computation routine for solving polynomial equations by rotating the coordinate axes is presented. The n -th order polynomial equation in S -domain is given as

$$F(S) = \sum_{i=0}^n A_i S^i = 0 \quad A_n \neq 0 \quad (1)$$

where A_i are the complex coefficients. For ease of presentation, let $S = j\lambda$ then

$$F(j\lambda) = A(\lambda) = \sum_{i=0}^n a_i \lambda^i = 0 \quad (3)$$

where

$$a_i = \alpha_i + \beta_i j \quad \text{and} \quad a_n \neq 0 \quad (4)$$

Eq. (3) can be separated as

$$A(\lambda) = A_R(\lambda) + jA_I(\lambda) = 0 \quad (5)$$

where

$$A_R(\lambda) = \sum_{i=0}^n \alpha_i \lambda^i \quad (6)$$

$$A_I(\lambda) = \sum_{i=0}^n \beta_i \lambda^i = 0 \quad (7)$$

which are called stability equations. [5]. If λ_0 is a real root of Eq. (3), then from Eq. (5) one has

$$A(\lambda_0) = A_R(\lambda_0) + jA_I(\lambda_0) = 0 \quad (8)$$

Since the coefficients of $A_R(\lambda)$ and $A_I(\lambda)$ are all real, Eq. (8) implies that

$$A_R(\lambda_0) = 0 \quad (9)$$

$$A_I(\lambda_0) = 0 \quad (10)$$

In other words, if Eq. (3) has a real root λ_0 , then λ_0 is the common real root of the stability equations. If the axes of λ -plane are rotated by an angle E to form a λ_E -plane [Fig. 1], then the equation $A_E(\lambda_E)$ in λ_E -plane relative to $A(\lambda)$ in λ -plane is given by

$$A_E(\lambda_E) = A(\lambda_E e^{jE}) = \sum_{i=0}^n a_i \lambda_E^i e^{j(iE)} \quad (11)$$

Therefore any complex root of Eq. (3) can be transformed to a real root of Eq. (11) by properly choosing of rotation angle E .

If λ_R is one of the real roots of $A_R(\lambda) = 0$, which is quite close to one of the real roots of $A_I(\lambda) = 0$ such as λ_I , and assume that the exact root of $A(\lambda) = 0$, which is near λ_R and λ_I , is

$$\lambda = \lambda_0 + j\omega \quad (12)$$

where ω is a small quantity, then

$$A_R(\lambda_0 + j\omega) = (\lambda_0 + j\omega - \lambda_R) A_{R1}(\lambda_0 + j\omega) \quad (13)$$

$$A_I(\lambda_0 + j\omega) = (\lambda_0 + j\omega - \lambda_I) A_{I1}(\lambda_0 + j\omega) \quad (14)$$

where

$$A_{R1}(\lambda) = \frac{A_R(\lambda)}{\lambda - \lambda_R} \quad (15)$$

$$A_{I1}(\lambda) = \frac{A_I(\lambda)}{\lambda - \lambda_I} \quad (16)$$

If ω is very small, then

$$A_{R1}(\lambda_0 + j\omega) = A_{R1}(\lambda_0) \quad (17)$$

$$A_{I1}(\lambda_0 + j\omega) = A_{I1}(\lambda_0) \quad (18)$$

Substituting Eqs. (17) and (18) into Eqs. (13) and (14), and from Eq. (5), one gets

$$\begin{aligned} & A_R(\lambda_0 + j\omega) + jA_I(\lambda_0 + j\omega) \\ &= (\lambda_0 + j\omega - \lambda_R)A_{R1}(\lambda_0) + j(\lambda_0 - \lambda_I + j\omega)A_{I1}(\lambda_0) \\ &= 0 \end{aligned} \quad (19)$$

$$\text{or } (\lambda_0 - \lambda_R)A_{R1}(\lambda_0) - \omega A_{I1}(\lambda_0) + j[A_{R1}(\lambda_0) + (\lambda_0 - \lambda_I)A_{I1}(\lambda_0)] = 0 \quad (20)$$

Eq. (20) implies that

$$(\lambda_0 - \lambda_R)A_{R1}(\lambda_0) - \omega A_{I1}(\lambda_0) = 0 \quad (21)$$

$$(\lambda_0 - \lambda_I)A_{I1}(\lambda_0) + \omega A_{R1}(\lambda_0) = 0 \quad (22)$$

$$\text{or } \lambda_0 - \lambda_R = \frac{\omega A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)} \quad (23)$$

$$\lambda_0 - \lambda_I = -\frac{\omega A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} \quad (24)$$

From Eqs. (23) and (24), one obtains

$$\lambda_R - \lambda_I = -\omega \left[\frac{A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} + \frac{A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)} \right] \quad (25)$$

$$\text{or } \omega = -\frac{\lambda_R - \lambda_I}{\frac{A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} + \frac{A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)}} \quad (26)$$

$$\begin{aligned} \text{But } \left| \frac{A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} + \frac{A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)} \right| &= \left| \frac{A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} \right| + \left| \frac{A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)} \right| \\ &\geq 2\sqrt{\left| \frac{A_{R1}(\lambda_0)}{A_{I1}(\lambda_0)} \right| \cdot \left| \frac{A_{I1}(\lambda_0)}{A_{R1}(\lambda_0)} \right|} = 2 \end{aligned} \quad (27)$$

Therefore

$$|\omega| \leq \left| \frac{\lambda_R - \lambda_I}{2} \right| = \frac{1}{2} |\lambda_R - \lambda_I| \quad (28)$$

In order to find the root $\lambda = \lambda_0 + j\omega$ one should choose

$$|\omega| = \frac{1}{2} |\lambda_R - \lambda_I| \quad (29)$$

for the next iteration. The sign of ω can be justified by the aid of Index of IRSSS (Irreducible real single singularity sequence, see Eq. (60) in Reference 5.); e.g. if $I_s(i)$ of the pole (λ_R) is 0, then this means that the root locus from λ_R will get into the DHP (UHP) of λ -plane if $K_T > 0$ ($K_T < 0$) and ω should be negative (positive) in this case. If $I_s(i)$ of λ_R is 1, then the root locus from λ_R will get into UHP (DHP) of λ -plane if $K_T > 0$ ($K_T < 0$) and ω should be positive (negative). In general

$$\omega = \frac{1}{2} K \omega |\lambda_R - \lambda_I| \quad (30)$$

where

$$K \omega = \text{Sign} [K_T] \times [2I_s(i) - 1] \quad (31)$$

Multiplying Eq.(23) to Eq. (24), one has

$$(\lambda_0 - \lambda_R) (\lambda_0 - \lambda_I) = -\omega^2 \quad (32)$$

Substituting Eq. (29) into Eq. (32), yields

$$\left(\lambda_0 - \frac{\lambda_R + \lambda_I}{2} \right)^2 = 0 \quad (33)$$

$$\text{or } \lambda_0 = \frac{\lambda_R + \lambda_I}{2} \quad (34)$$

Therefore, in the next iteration, one should choose

$$\lambda = \frac{\lambda_R + \lambda_I}{2} + j \frac{1}{2} K \omega |\lambda_R - \lambda_I| \quad (35)$$

where λ_R and λ_I are the real roots of the stability equations of the present iteration.

Based upon the presented analyses, an iterative process can be programmed as follows:

- (i) Transform $F(S)$ to $A(\lambda)$ by the substitution of $S = j\lambda$
- (ii) Solve the stability equations of the present iteration and find the real roots only.

(iii) Calculate $\lambda_{Ri} + \lambda_{Ii}$ and $|\lambda_{Ri} - \lambda_{Ii}|$, where λ_{Ri} and λ_{Ii} are poles and zeros that are occurred subsequently.

(iv) Find the minimal one of $\frac{|\lambda_{Ri} - \lambda_{Ii}|}{|\lambda_{Ri} + \lambda_{Ii}|}$ and get

$$\delta = \min \left(\left| \frac{\lambda_{Rj} - \lambda_{Ij}}{\lambda_{Rj} + \lambda_{Ij}} \right|, j = 1, 2, \dots \right) \quad (36)$$

(v) If

$$\delta = \left| \frac{\lambda_{RK} - \lambda_{IK}}{\lambda_{RK} + \lambda_{IK}} \right| \quad (37)$$

then the rotation angle for the next iteration is given by

$$E_j = \tan^{-1} \frac{K \omega |\lambda_{RK} - \lambda_{IK}|}{\lambda_{RK} + \lambda_{IK}} \quad (38)$$

- (vi) Transform the present equation by rotating an angle E_i [Eq. (11)] and form a new equation for the next iteration, then repeat step (ii) [see Fig.1] until the difference between the roots of stability equations, λ_{RK} and λ_{IK} is small enough; i.e.,

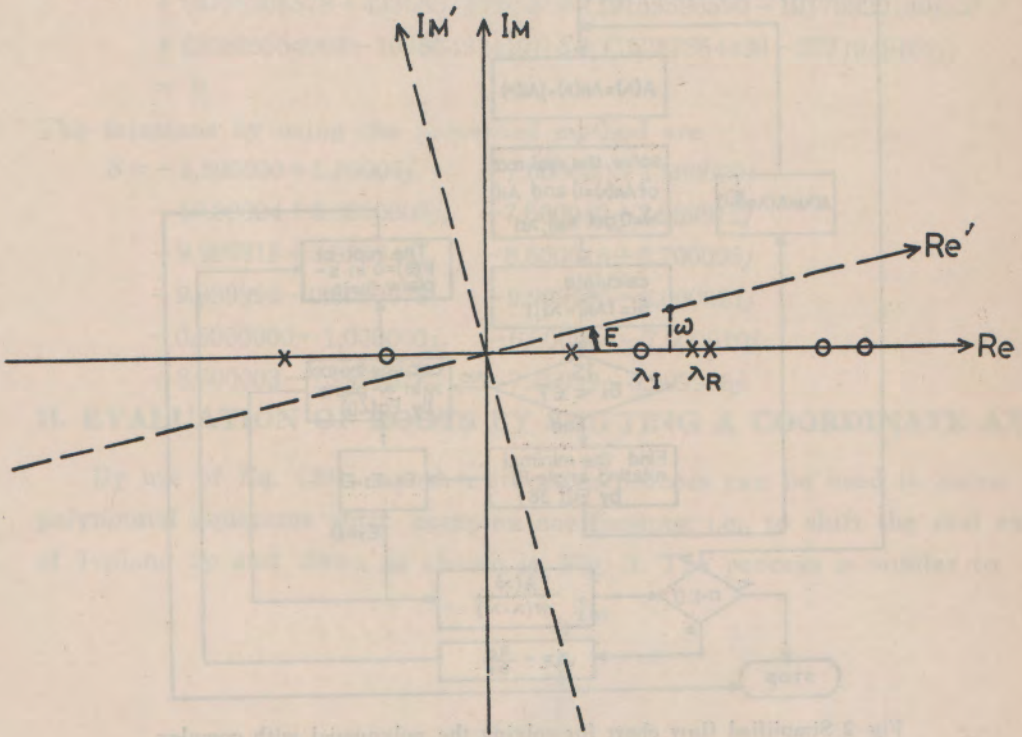


Fig. 1 λ -plane and λ_E -plane

$$|\lambda_{RK} - \lambda_{IK}| < \epsilon \quad (39)$$

where ϵ is the tolerable error from Eq. (29). If Eq. (39) is satisfied, the considered root of the original equation will be in the circle with center at $(\lambda_{RK} + \lambda_{IK})/2$ and radius $\epsilon/2$ of the λ_E -plane, or the root in the λ -plane is given by

$$\lambda_0 = [(\lambda_{RK} + \lambda_{IK})/2][\cos E_T + j \sin E_T] \quad (40)$$

$$\text{where } E_T = E_i \quad (41)$$

Note that if there are many pairs of poles and zeros satisfied Eq. (39), then one has obtained many roots in the same time. This is a powerful merit for equations with multiple roots or with many roots having the same damping ratio.

- (vii) Remove the root (or roots) of Eq. (40) from Eq. (3) by synthetic division, and repeat the process of (ii) to (vi) to get all the roots of Eq. (3).

A flow chart for solving a polynomial equation with complex coefficients is given in Fig. 2; a computer subroutine has been written and applied

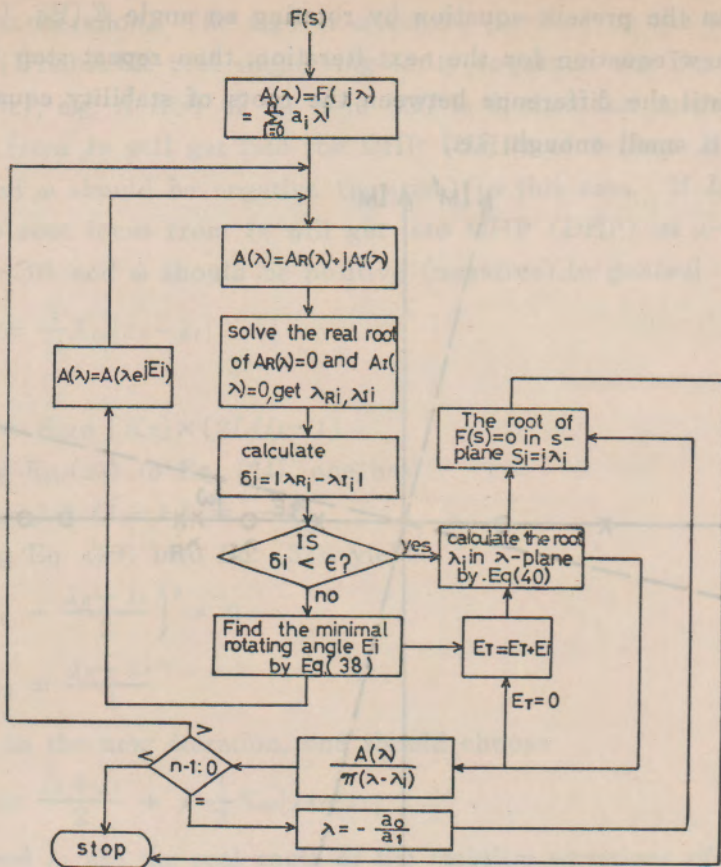


Fig. 2 Simplified flow chart for solving the polynomial with complex coefficient by rotating process.

for solving the following example.

Example 1. Solve the following equation

$$S^3 + (3 + 15.4j)S^2 + 21S + (6 + j)S^3 + 9jS^2 + (5 + 0.6j)S + 0.3 - 0.1j = 0$$

The roots by using the presented method are

$$\begin{aligned} S &= 0.163000 + 1.37475j, & -0.0619636 + 0.0211536j \\ & -0.758897 + 0.0188006j, & -2.77744 - 16.6200j \\ & 0.0875300 + 0.390694j, & 0.347771 - 0.585542j \end{aligned}$$

Example 2. Solve the following polynomial equation. (With double roots)

$$\begin{aligned} S^6 + (130 - 18.0j)S^5 - (61.78 + 163.2j)S^4 - (563.87 + 95.28j)S^3 \\ + (413.6279 + 496.968j)S^2 + (141.78 - 22.695j)S + 99.5301 + 134.442j = 0 \end{aligned}$$

The roots solved by the presented method are

$$\begin{aligned} S &= 0.500000j, & -0.500000j \\ & -1.20000 - 3.00000j, & -1.20000 - 3.00000j \\ & -5.30018 - 6.00014j, & -5.29983 - 5.99986j \end{aligned}$$

Example 3. The given equation is

$$S^{12} + (82.7 - 0.2j)S^{11} + (3196.1 - 26.77j)S^{10}$$

$$\begin{aligned}
 &+ (75906.973 - 1652.818j)S^9 + (1228191.956 - 57132.7343j)S^8 \\
 &+ (14197412.47 - 122989.052j)S^7 + (119581693.8 - 17447459.29j)S^6 \\
 &+ (734182643.9 - 166680037.8j)S^5 + (3227890174 - 1063544148j)S^4 \\
 &+ (9776205578 - 4336857929j)S^3 + (19159596590 - 10179920140j)S^2 \\
 &+ (22626554960 - 10788494410j)S + (15287864430 - 2771049162j) \\
 &= 0
 \end{aligned}$$

The solutions by using the presented method are

$$\begin{aligned}
 S = &-1.500000 + 5.00000j, & -2.000000 + 1.500000j \\
 &-10.000004 + 6.3000008j, & -7.500040 + 3.999972j \\
 &-9.999918 + 4.999941j, & -8.600016 + 2.700095j \\
 &-9.999996 - 0.8000579j, & -9.999986 - 6.000021j \\
 &-0.5000000 - 1.000000j, & -6.500004 - 7.500010j \\
 &-8.600003 - 1.999952j, & -7.499999 - 6.999976j
 \end{aligned}$$

II. EVALUATION OF ROOTS BY SHIFTING A COORDINATE AXIS

By use of Eq. (30), another iterative process can be used to solve polynomial equations with complex coefficients; i.e., to shift the real axis of λ -plane up and down as shown in Fig. 3. The process is similar to

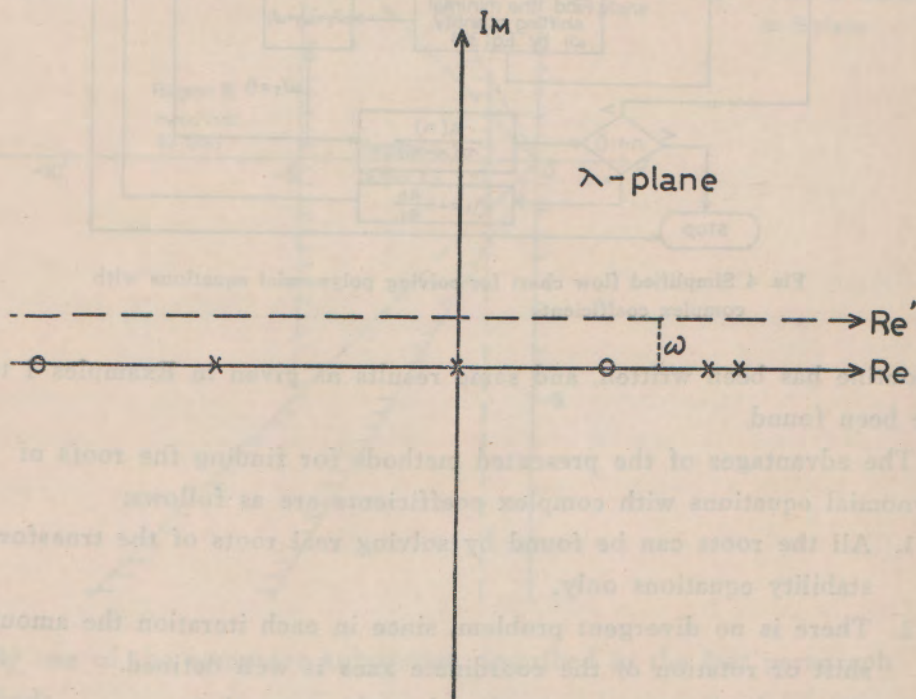


Fig. 3 λ -plane with a shifted coordinate

that of rotating the axes of λ -plane, and the root in λ -plane is given by

$$\lambda_0 = \frac{\lambda_{RK} + \lambda_{IK}}{2} + j\omega_T \quad (42)$$

where

$$\omega_T = \sum \omega_i \quad (43)$$

$$\text{and } |\lambda_{RK} - \lambda_{IK}| < \epsilon \quad (44)$$

A flow chart for this shifting process is given in Fig. 4; a computer

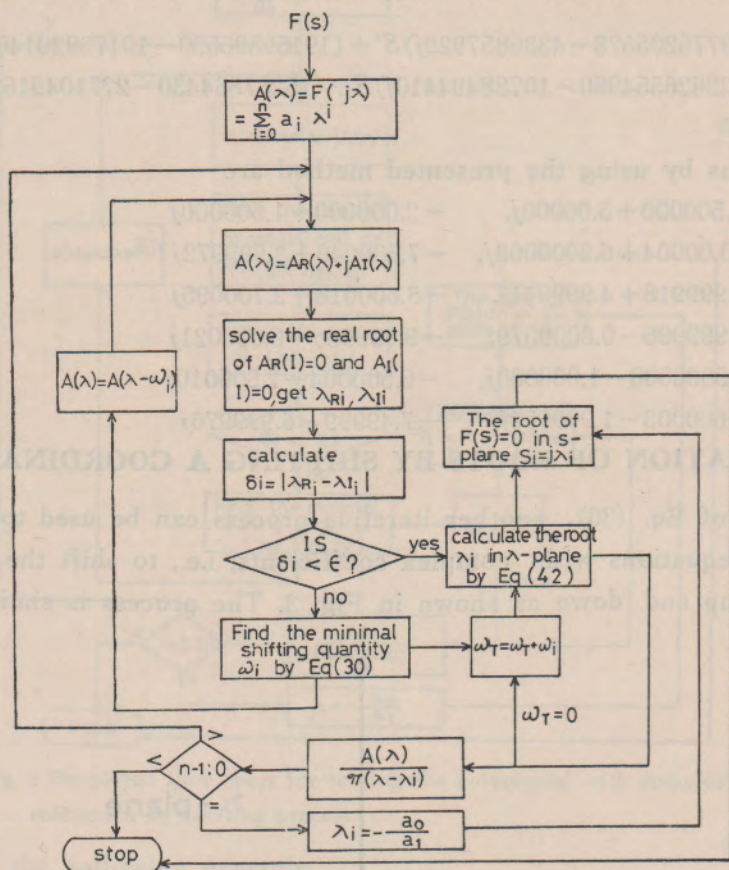


Fig. 4 Simplified flow chart for solving polynomial equations with complex coefficients

subroutine has been written, and same results as given in Examples 1 to 3 have been found.

The advantages of the presented methods for finding the roots of polynomial equations with complex coefficients are as follows:

1. All the roots can be found by solving real roots of the transformed stability equations only.
2. There is no divergent problem, since in each iteration the amount of shift or rotation of the coordinate axes is well defined.
3. The multiple roots can be found at the same time.
4. In the analysis of large scale control systems, usually not all the characteristic roots of the system are important, since only the roots in certain regions of the S -plane are interesting. For example, in a high order system, it might be that only the roots

with $\zeta \leq 0.5$ and $\sigma \leq -1.0$ are required to find the accurate values (i.e. a small ϵ_1 is tolerable); the roots with $0.5 < \zeta \leq 0.7$ and $-1 < \sigma \leq -5.0$ are required to find the approximate values (i.e. larger ϵ_2 is tolerable) for they are not sensitive to transient responses; and that the roots in the remainder region need not be found [Fig. 5]. By the rotating, shifting and iterative processes given in this paper, the requirement described above can be achieved without expanding too much computer time. A computer subroutine for doing this has been prepared and tested by the authors using the following example.

Example 4. For the equation in Example 3, find the roots which are in the regions specified in Fig. 5. for $\epsilon_1 = 1.0 \times 10^{-8}$ and $\epsilon_2 = 1.0 \times 10^{-5}$.

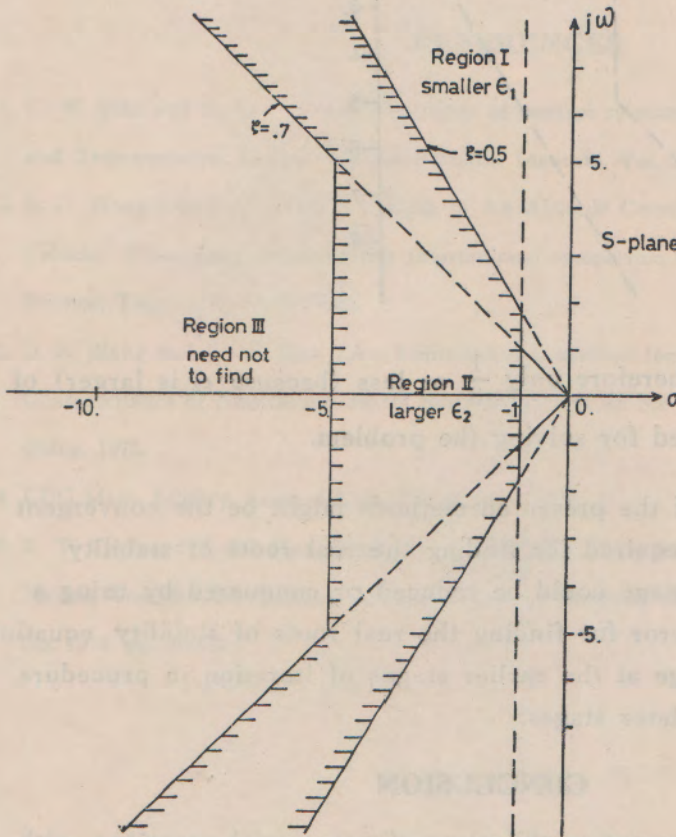


Fig. 5 Various regions in S-plane

By use of the computer subroutine described in the last paragraph, one finds

$$\begin{aligned}
 S = & -0.500000000 - 1.0000000 j, & -1.5000000 + j 5.0000000j \dots \dots \dots & \text{in Region I} \\
 & -2.0000 + 1.5000 j, & -6.5000 - 7.5000j \dots \dots \dots & \text{in Region II}
 \end{aligned}$$

There are eight roots in Region III (Refer to Example 3 and Fig. 6),

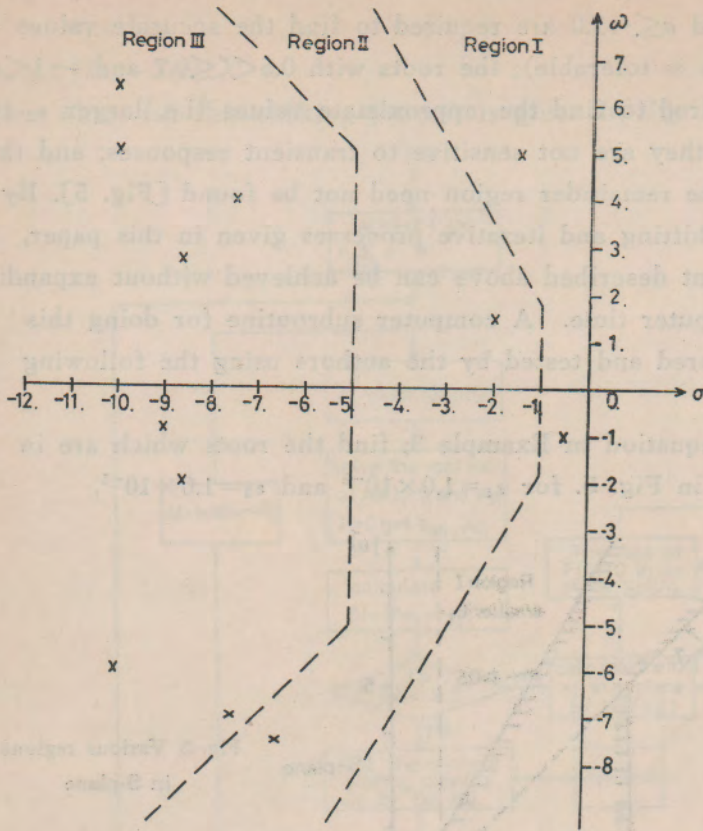


Fig. 6 Specified regions in S-plane

but need not to find. Therefore only $\frac{1}{3}$ or less (because ϵ_2 is larger) of computer time is required for solving the problem.

The disadvantage of the presented methods might be the convergent speed due to the time required for finding the real roots of stability equations. This disadvantage could be reduced or conquered by using a programmed tolerable error for finding the real roots of stability equations; i.e., the tolerance is large at the earlier stages of iteration in procedure (ii) and smaller at the later stages.

CONCLUSION

New methods for computing the roots of polynomial equations with complex coefficients have been presented, and computer flow charts as well as numerical examples for applying the presented methods have been given. The unique approach of the presented method, by only considering the real roots of two stability equations, might bring up some topics for further research, and the authors of this paper are looking forward to finding engineering problems to test the applicability of the presented methods.

LIST OF SYMBOLS

$a_i = \alpha_i + \beta_i$	Complex coefficients
$A_R(\lambda), A_I(\lambda)$	Stability equations
$I_s(i)$	Index of IRSSS
IRSSS	Irreducible real single singularity sequence
DHP	Down half plane
UHP	Up half plane
ω	A small distance
λ_0	A real root of $A(\lambda)$
S	Laplace operator
$\lambda = -jS$	Operator

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