Stability Analysis of Nonlinear Reactor Control Systems

Y. L. Chen* (陳英亮)　　　　K. W. Han** (韓光渭)

SUMMARY: A method for testing control system stability is presented, which is useful ror finding the limit cycles of high order systems with multiple nonlinearities. The presented method is applied to the analysis of the control system of a material testing reactor, and a comparison with the method in a current literature is given.

I. INTRODUCTION

It is well known that a nuclear reactor control system usually consists of several nonlinearities such as saturation, bachlash etc. The commonly used method for analysis is Nyquist diagram, which is not suitable for analyzing control systems with multiple nonlinearities, and also not suitable for finding the effects of the adjustable paramters. The main purpose of this paper is to present a stability-equation method and to apply this method to the analysis of a nonlinear reactor control system.

II. STABILITY-EQUATION METHOD FOR NONLINEAR SYSTEM ANALYSIS

The method proposed in this section is based upon a linearization technique; i. e., to replace the nonlinearities by their describing functions, and then a linearized characteristic equation can be obtained. After the characteristic equation is separated into two parts as

$$F(s)=F_R+F_I=0 \tag{1}$$

where F_R and F_I are the real and imaginary parts of $F(s)$ respectively (after the substitution $s=jw$), then a standard root-locus form can be written as

$$F_R/E_I=-1 \tag{2}$$

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* Y. L. Chen is formerly a graduate student at the Institute of Electronics of Chiao-tung University Hsinchu, Taiwan.

** K. W. Han is with the Chung-Shan Institute, and adjunct Professor at the Chiao-tung University.

It is readily shown that for all the characteristic roots in the left half of s-plane, all the poles (p_i) and zeros (z_i) of Eq. (2) must lie on the imaginary axis of s-plane and their absolute values are related as

$$\cdots p_{-1} < z_{-1} < p_0 < z_1 < p_1 < z_2 \cdots \tag{3}$$

From Eqs. (2) and (3), it can be seen that the stability limit is reached when a pole is equal to a zero; thus to test system stability becomes simply to find the real roots of F_R and F_I. (In a later part of this paper, these two equations are called stability equations.) The following example is used as an illustration.

Example 1. Consider the system in Fig. 1, where N_1 and N_2 are the describing functions of the nonlinearities, the characteristic equation is

$$s^3+s^2+N_1\ (g-jb)\ s+0.05N_1\ (g-jb)=0 \tag{4}$$

After the substitution $s=jw$, the stability equations are

$$F_R=-w^2+N_1bw+0.05N_1g=0 \tag{5}$$

$$F_I=-w^3+N_1gw-0.05N_1b=0 \tag{6}$$

which gives

$$N_1=\frac{(0.05+w^2)\ w^2}{g(0.0025+w^2)}=\frac{0.95w^3}{b\ (0.0025+w^2)} \tag{7}$$

Let $R=\frac{\beta}{2|\delta_k'|}$ where β is the amount of backlash and $|\delta_k'|$ is the magnitude of the input signal to the backlash, then for each value of β, the corresponding values of w, R, g, b, and E for making the system have a limit cycle can be found, and their relations can be represented by various curves. For example, the relations among E, w and β are given in Fig. 2, which indicates that, for the considered system, if the amount of backlash is increased, the frequency of the limit cycle will be reduced and its magnitude will be increased.

Fig. 2 also indicates that the presented method is useful for adjusting parameters. In the considered system, if the open loop gain (k) is adjustable, then for any fixed value of β the relations among w, E and k can be found using the same method

LIMIT CYCLE STABILITY ANALYSIS

After a limit cycle is found, its stability characteristics should be defined. The main purpose of this section is to present a method for finding the stability characteristics of a limit cycle using the stability-equation method.

As mentioned before, a limit cycle exists whenever there is a pole (p_i) equal to a zero (z_i); thus the stability characteristics of a limit cycle can be defined if the variations of the pole and zero in the neighbourhood of the limit cycle (due to the variation of E) can be defined. On the other hand, since the stability equations are functions of frequency, the effect of w should be considered also. A method of using partial derivatives to find the effects of the variations of E and w upon the real roots of stability squations is given as follows:

Using Taylor's series, for incrementals of E (ΔE) and w (Δw), the stability equations can be written approximately as

$$F_R(w+\Delta w, |E|+\Delta E)=F_R(w, |E|)+\frac{\partial F_R}{\partial w}\Delta w+\frac{\partial F_R}{\partial E}\Delta E=0 \tag{8}$$

$$F_I(w+\Delta w, |E|+\Delta E)=F_I(w, |E|)+\frac{\partial F_I}{\partial w}\Delta w+\frac{\partial F_I}{\partial E}\Delta E=0 \tag{9}$$

Since $F_R(w, |E|)=0$ and $F_I(w, |E|)=0$ at the limit cycle, thus

$$\Delta w_R=-\frac{\partial F_R}{\partial E}\Delta E\Big/\frac{\partial F_R}{\partial w} \tag{10}$$

$$\Delta w_I=-\frac{\partial F_I}{\partial E}\Delta E\Big/\frac{\partial F_I}{\partial w} \tag{11}$$

where Δw_R & Δw_I represent the variations of the roots of F_R and F_I in the neighbourhood of the limit cycle respectively.

For the example in the last section, if $\beta=0.2075$, then as indicated by the dotted lines in Fig. 2, a limit cycle with w=3.14 R=0.2573 and $|E|=0.1033$ exists. The partial derivatives are found as

$$\frac{\partial F_R}{\partial w}=-2w+N_1b+N_1w\frac{\partial b}{\partial w}+0.05N_1\frac{\partial g}{\partial w} \tag{12}$$

$$\frac{\partial F_I}{\partial w}=-3w^2+N_1g+N_1w\frac{\partial g}{\partial w}-0.05N_1\frac{\partial b}{\partial w} \qquad (13)$$

$$\frac{\partial F_R}{\partial|E|}=bw\frac{\partial N_1}{\partial|E|}+N_1w\frac{\partial b}{\partial|E|}+0.05g\frac{\partial N_1}{\partial|E|}+0.05N_1\frac{\partial g}{\partial|E|} \qquad (14)$$

$$\frac{\partial F_I}{\partial|E|}=gw\frac{\partial N_1}{\partial|E|}+N_1w\frac{\partial g}{\partial|E|}-0.05N_1\frac{\partial b}{\partial|E|}-0.05b\frac{\partial N_1}{\partial|E|} \qquad (15)$$

where
$$\frac{\partial g}{\partial w}=\frac{\partial g}{\partial R}\ \frac{\partial R}{\partial w},\quad \frac{\partial g}{\partial E}=\frac{\partial g}{\partial R}\ \frac{\partial R}{\partial E} \qquad (16)$$

$$\frac{\partial b}{\partial w}=\frac{\partial b}{\partial R}\ \frac{\partial R}{\partial w},\quad \frac{\partial b}{\partial|E|}=\frac{\partial b}{\partial R}\ \frac{\partial R}{\partial|E|} \qquad (17)$$

and where
$$R=\frac{\beta}{2|\delta'_k|}=\frac{\beta\pi}{8w\sqrt{1-\frac{\epsilon_0^2}{4|E|^2}}} \qquad (18)$$

$$\frac{\partial R}{\partial w}\doteqdot-\frac{\beta\pi}{8w^2} \qquad (19)$$

$$\frac{\partial R}{\partial|E|}\doteqdot-\frac{\pi\beta\,\epsilon_0^2}{32w\,|E|^3} \qquad (20)$$

For the considered limit cycle, the results are

$$\frac{\partial g}{\partial R}\doteqdot-1.15,\quad \frac{\partial b}{\partial R}\doteqdot+0.6,\quad \frac{\partial R}{\partial w}\doteqdot-0.00826,$$

$$\frac{\partial R}{\partial E}\doteqdot-0.00236\quad \frac{\partial N_1}{\partial E}\doteqdot-120$$

$$\frac{\partial g}{\partial w}=9.5\times10^{-3},\quad \frac{\partial g}{\partial E}=2.72\times10^{-3},\quad \frac{\partial b}{\partial w}=-4.95\times10^{-3}$$

$$\frac{\partial b}{\partial E}=-1.41\times10^{-3}$$

thus
$$\frac{\partial F_R}{\partial w}=-3.49,\quad \frac{\partial F_R}{\partial|E|}\doteqdot-95$$

$$\frac{\partial F_I}{\partial w}\doteqdot-19.2,\quad \frac{\partial F_I}{\partial|E|}=-303$$

which give

$$\Delta w_R=-27{,}2\ \Delta E \qquad (21)$$

$$\Delta w_I=-15.8\ \Delta E \qquad (22)$$

Hence for a positive incremental Δ E in E, the pole and zero distribution of the stability equations becomes

$$p_{-1}=-3.165,\ z_{-1}=-0.16,\ p_0=-0.025,\ z_1<3.14,\ p_1>3.14$$

which represents a stable system; i. e., E will be brought back to its original value (0.1033). Therefore, the limit cycle is stable.

The considered example has been simulated with a digital computer, and the results are checked with those obtained from theoretical analysis.

STABILITYANALYSIS OF A NONLINEAR REACTOR CONTROL SYSTEM

The design of a nonlinear reactor control systm has been stated to some detail in reference 3, where a system with two nonlinearities separated by a linear section has been analyzed using Nyquist diagram and describing function methods; and for avoiding the complexiety in analysis, the authors always approximate the ON-OFF element with a pure gain. Even so, the analyses show great complexiety, especially when the effeccts of the adjustable parameters are considered.

In this section, the stability-equation method is applied for the analysis of the aforementioned system.

The considered system is given in Fig. 3, where the transfer functions are

(a) Reactor:

$$\frac{1}{n_0}R_R G_R=\frac{1}{n_0}\frac{\delta n}{\delta k}=\frac{E}{\delta k}$$

$$=\frac{104(s+3.01)(s+1.14)(s+0.301)(s+0.111)(s+0.0305)(s+0.0124)}{s\,(s+64.4)\,(s+2.9)\,(s+1.02)\,(s+0.195)\,(s+0.0681)\,(s+0.0143)} \tag{23}$$

(b) Motor:

$$K_D G_D=\frac{\delta'_K}{\mu/\mu_0}=\frac{M_0}{s(s+\tau s)} \tag{24}$$

where M_0 is the reactivity ramp for the motors and $\tau=0.05$ second is the motor time constant.

(c) The ON-OFF error unit

$$K_C G_c=\frac{\mu/\mu_0}{E}=\frac{4}{g_0\pi}R'\sqrt{1-R'^2} \tag{25}$$

where ϵ_0 is the dead zone amplitude, and $R'=\epsilon_0/|E|$ is a parameter.

(d) The backlash

$$K_BG_B=g-jb \tag{26}$$

where g and b are the real and imaginary parts of the describing function of the backlash respectively.

The characteristic equation is

$$1+\frac{1}{n_0}K_RG_RK_CG_CK_DG_DK_BG_B=0 \tag{27}$$

Let $K=M_0K_CG_C\times10^4$, then Eq. (27) gives

$$\begin{aligned}&s^9+88.6s^8+1647s^7+(20Kg+5750-j20Kb)s^6+(92.2Kg\\&+5310-j92.2Kb)s^5+(107.4Kg+1146-107.4Kb)s^4\\&+(35.5Kg+65.9-j35.5Kb)s^3+(3.67Kg+0.72\\&-j3.67Kb)s^2+(0.1098Kg-j0.1098Kb)s+(0.000866Kg\\&-j0.000866Kb)=0\end{aligned} \tag{28}$$

Thus the stability equations are

$$\begin{aligned}F_I=&w^9-1467w^7+20Kbw^6+(92.2Kg+5310)w^5-107.4Kbw^4\\&-(3.5Kg+66)w^3+3.67Kbw^2+0.1098Kgw-0.000866Kb\\&=0\end{aligned} \tag{29}$$

$$\begin{aligned}K_R=&88.6w^6-(20Kg+5750)w^6+92.2Kbw^5+(107.4Kg+1146)w^4\\&-35.5Kbw^3-(3.67Kg+0.72)w^2+0.1098Kbw+0.000866Kg\\&=0\end{aligned} \tag{30}$$

In order to determine a limit eycle, Eqs. (29) and (30) should be solved simultaneously. Rewrite these two equations as

$$\begin{aligned}-K&=\frac{w^9-1467w^7+5310w^5-66w^3}{(92.2w^5-35.5w^3+0.1098w)g+(20w^6-107.4w^4+3.67w^2-0.000866)b}\\&=\frac{A}{Cg+Db}\end{aligned} \tag{31}$$

and

$$\begin{aligned}-K&=\frac{88.6w^8-5750w^6+1146w^4-0.72w^2}{-(20w^6-107.4w^4+3.67w^2-0.000866)g+(92.2w^5-35.5w^3+0.1098w)b}\\&=\frac{B}{Cb-Dg}\end{aligned} \tag{32}$$

Since g and b are functions of $|\delta'_k|$; i. e.,[2]

$$g=g(\ |\delta'_k|\ ) \tag{33}$$

$$b=b(\ |\delta'_k|\ ) \tag{34}$$

where $$|\delta'_k| = \frac{10^{-4}K}{w\sqrt{1+\tau^2 w^2}}\,|E| \tag{35}$$

and $$K=10^4 M_0 K_C G_C = 10^4 M_0 \frac{4}{|E|\,\pi}\sqrt{1-\left(\frac{\epsilon_0}{|E|}\right)^2} \tag{36}$$

thus for different values of M_0, E_0 and β, the values of w, |E| , g, b and K which satisfy Eqs. (31) to (36) simultaneously can be found using a digital computer. A flow chart for doing this is given in Fig. 4, and some of the computed results are given in Fig. 5.

For example, if $M_0=1.5\times10^{-4}$, $\epsilon_0=0.005$, $\beta=0.23°$, which define point M in Fig. 5, then the corresponding values of w, |E| , g, b and K are

$$w=0.066,\ |E|=0.43,\ K=4.44,\ g=0.258, \text{ and } b=0.269$$

The pole-zero bistribution of the stability equations is given in Fig. 6, which indicates that the considered system has a limit cycle at w=0.066. In order to test stability of this limit cycle, the method presented in the last section is applied. The calculations of the partical derivatives are given in the appendix, and the results are

$$1.211\ \Delta w_R - 0.1391\ \Delta E = 0 \tag{37}$$

$$-4.417\ \Delta w_I - 0.0288\ \Delta E = 0 \tag{38}$$

It can be seen that, for an incremental of |E| , the zero of $F_R=0$ (originally at w=0.066) tends to increase, and the zero of $F_I=0$ (originally at w=0.66) tends to decrease; thus the pole and zero distribution becomes alternative in sequence, and the system is stable. Therefore, the considered limit cycle is stable.

Since the effects of the adjustable parameters on system stability can be found using a digital computer and the stability characteristics of any limit cycle can be defined, the presented method is useful for the analysis and design of control systems with a digital computer.

CONCLUSIONS

The method of testing stability in high order systems with multiple nonlinearities, presented in this paper, has the advantage of reducing the required work for finding the limit cycles and its stability characteristics. The presented method is suitable for use with a digital computer to find the effects of various adjustable parameters on system stability. In comperasion with the commonly used graphical method,[3] the superior characteristics of the presented method are quite evident.

Since the basic approach used in this paper is linearization, same limitations are uncountered as the use of describing function method; i e., the nonlinearities must be separated by linear transfer functions with enough low pass characteristics.

APPENDIX

From Eqs. (8) and (9)

$$F_R(w_0+\Delta w, |E|_0+\Delta E) \doteq F_R(w_0, |E|_0) + \frac{\partial F_R}{\partial w}\Delta w + \frac{\partial F_R}{\partial |E|}\Delta E = 0 \qquad (A)$$

$$F_I(w_0+\Delta w, |E|_0+\Delta E) \doteq F_I(w_0, |E|_0) + \frac{\partial F_I}{\partial w}\Delta w + \frac{\partial F_I}{\partial |E|}\Delta E = 0 \qquad (B)$$

The partial derivatives are

$$\begin{aligned}\frac{\partial F_R}{\partial w} &= 8\times 88.6w^7 - 6(20Kg+5750)w^5 + 5\times 92.2Kbw^4 + 4(107.4Kg \\ &\quad +1146)w^3 - 3\times 35.5Kbw^2 - 2(3.67Kg+0.72)w + 0.1098Kb \\ &\quad -20w^6\frac{\partial Kg}{\partial w} + 92.2w^5\frac{\partial Kb}{\partial w} + 107.4w^4\frac{\partial Kg}{\partial w} - 35.5w^3\frac{\partial Kb}{\partial w} \\ &\quad -3.67w^2\frac{\partial Kg}{\partial w} + 0.1098w\frac{\partial Kb}{\partial w} + 0.000866\frac{\partial Kg}{\partial w} \qquad (C)\end{aligned}$$

$$\begin{aligned}\frac{\partial F_R}{\partial |E|} &= -20w^6\frac{\partial Kg}{\partial |E|} + 92.2w^5\frac{\partial Kb}{\partial |E|} + 107.4w^4\frac{\partial Kg}{\partial |E|} - 35.5w^3\frac{\partial Kb}{\partial |E|} \\ &\quad -3.67w^2\frac{\partial Kg}{\partial |E|} + 0.1098w\frac{\partial Kb}{\partial |E|} + 0.000866\frac{\partial Kb}{\partial |E|} \qquad (D)\end{aligned}$$

$$\frac{\partial F_I}{\partial w}=9w^8-7\times 1467w^6+6\times 20Kbw^5+5(92.2Kg+5310)\,w^4-4\times 107.4Kbw^3-3(35.5Kg+66)w^2+3.67\times 2Kbw+0.1098Kg$$
$$+20w^6\frac{\partial Kb}{\partial w}+92.2w^5\frac{\partial Kg}{\partial w}-107.4w^4\frac{\partial Kb}{\partial w}-35.5w^3\frac{\partial Kg}{\partial w}$$
$$+3.67w^2\frac{\partial Kb}{\partial w}+0.1098\,w\frac{\partial Kg}{\partial w}-0{,}000866\frac{\partial Kb}{\partial w} \qquad (E)$$

$$\frac{\partial F_I}{\partial |E|}=20w^6\frac{\partial Kb}{\partial |E|}+92.2w^5\frac{\partial Kg}{\partial |E|}-107.4w^4\frac{\partial Kb}{\partial |E|}-35.5w^3\frac{\partial Kg}{\partial |E|}$$
$$+3.67w^2\frac{\partial Kb}{\partial |E|}+0.1098\,w\frac{\partial Kg}{\partial |E|}-0.000866\frac{\partial Kb}{\partial |E|} \qquad (F)$$

For the considered limit cycle at M (i. e., $w=0.066$, $|E|_0=0.43$, $\beta=0.23^0$, $M_0=1.5\times 10^{-4}$, $K=4.44$, $g=0.258$, $b=0.269$),

$$R=\frac{\beta}{2(\delta'_k)}\doteq\frac{10^4 w\beta}{2\,|E|\,K} \qquad (G)$$

then $$\frac{\partial R}{\partial w}=\frac{10^4\beta}{2\,|E|\,K}=10.52$$

$$\frac{\partial g}{\partial w}=\frac{\partial g}{\partial R}\ \frac{\partial R}{\partial w}=-1.3\times 10.52=-13.7$$

$$\frac{\partial b}{\partial w}=\frac{\partial b}{\partial R}\ \frac{\partial R}{\partial w}=-0.5\times 10.52=-5.26$$

Since $$K=10^4M_0\frac{4}{|E|\pi}\sqrt{1-R'^2},$$

$$\doteq 10^4M_0\frac{4}{\pi\,|E|}$$

thus $$R\doteq\frac{\beta w\varkappa}{8M_0}$$

whioh gives $$\frac{\partial R}{\partial |E|}=0,\quad \frac{\partial g}{\partial |E|}=\frac{\partial g}{\partial R}\ \frac{\partial R}{\partial E}=0$$

and $$\frac{\partial K}{\partial |E|}=-\frac{4\times 10^4\times 1.5\times 10^{-4}}{\pi\,|E|^2}=-10.37,\quad \frac{\partial K}{\partial w}=0$$

similarily $\frac{\partial b}{\partial |E|}=0$

Thus $\frac{\partial Kg}{\partial w}=K\frac{\partial g}{\partial w}=-13.7K$

$$\frac{\partial Kb}{\partial w}=K\frac{\partial b}{\partial w}=-5.26K$$

$$\frac{\partial Kg}{\partial |E|}=g\frac{\partial K}{\partial |E|}=-10.37g$$

$$\frac{\partial Kb}{\partial |E|}=b\frac{\partial K}{\partial |E|}=-10.37b$$

Substituting into Eqs. (C) and (F), then

$$\frac{\partial F_R}{\partial w}=1.211,\ \frac{\partial F_R}{\partial |E|}=-0.139,\ \frac{\partial F_I}{\partial w}=-4.417,\ \frac{\partial F_I}{\partial |E|}=-0.0288$$

Hence Eqs. (37) and (38) are obtained.

SYMBOLS

s Laplace operator

w frequency

c output quantity

r input quantity

K gain of transfer function

p zero of E_I

z zero of F_R

F_R real part of characteristic equation

F_I imaginary part of characteristic equation

M_0 the velocity ramp of motors

ϵ_0 dead-zone amplitude of the ON-OFF error unit

$R=\beta/2\,|\delta'_k|$ parameter

$R'=\epsilon_0/\,|E|$

g-jb describing function of a backlash-type nonlinearity

N describing function of ON-OFF nonlinearity

γ_k input signal to backlash

E input signal to the ON-OFF element, E= |E| sinwt

n_0 steady-state power of reactor

β magnitude of backlash

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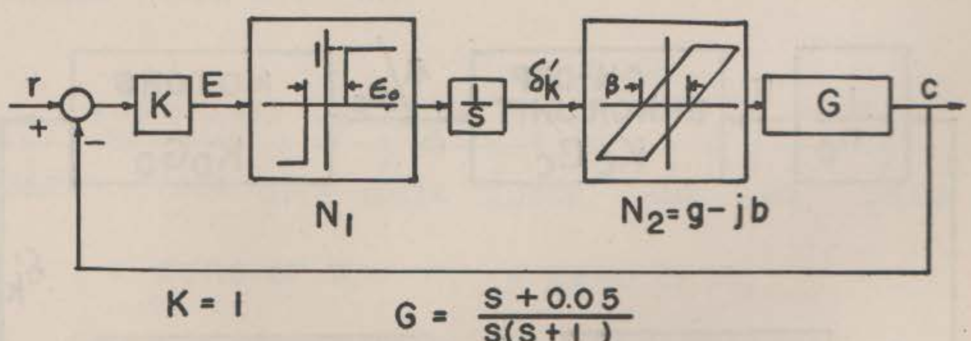

FIG. 1 BLOCK DIAGRAM OF A NONLINEAR SYSTEM

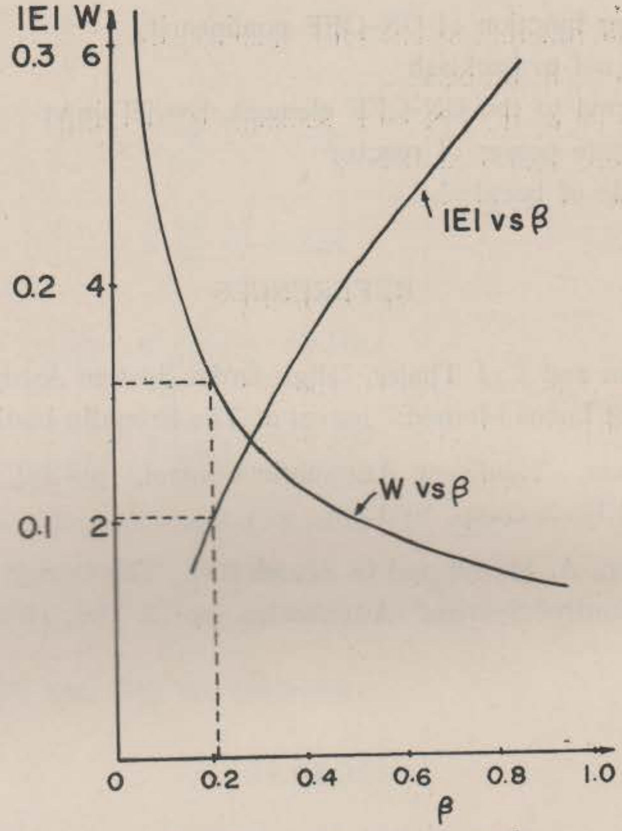

FIG.2 RELATIONS AMONG |E|, W AND β

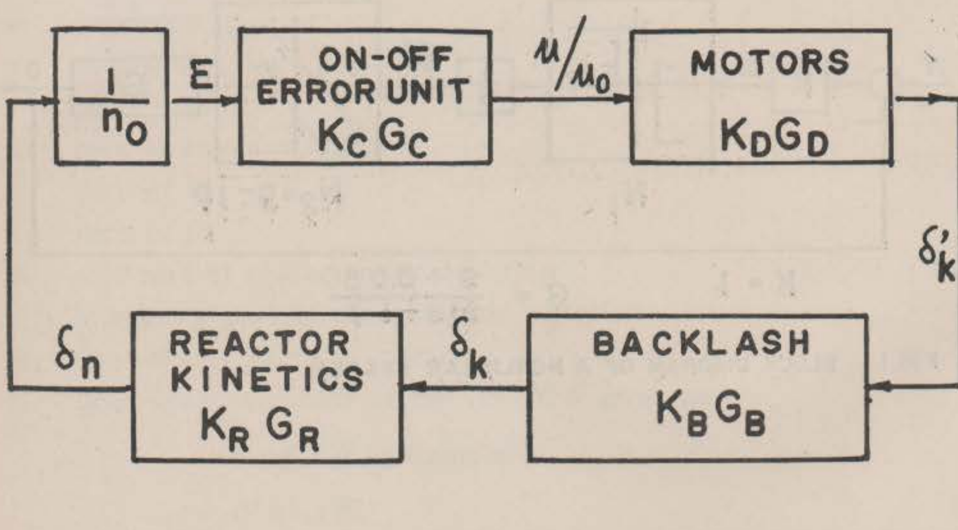

FIG. 3 BLOCK DIAGRAM OF A NONLINEAR REACTOR CONTROL SYSTEM

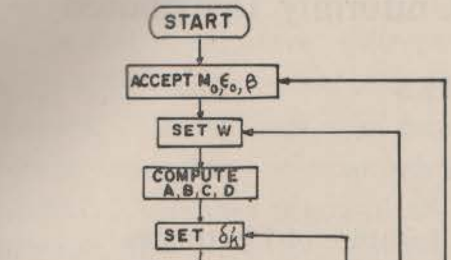

FIG. 4 FLOW CHART FOR FINDING THE LIMIT CYCLES

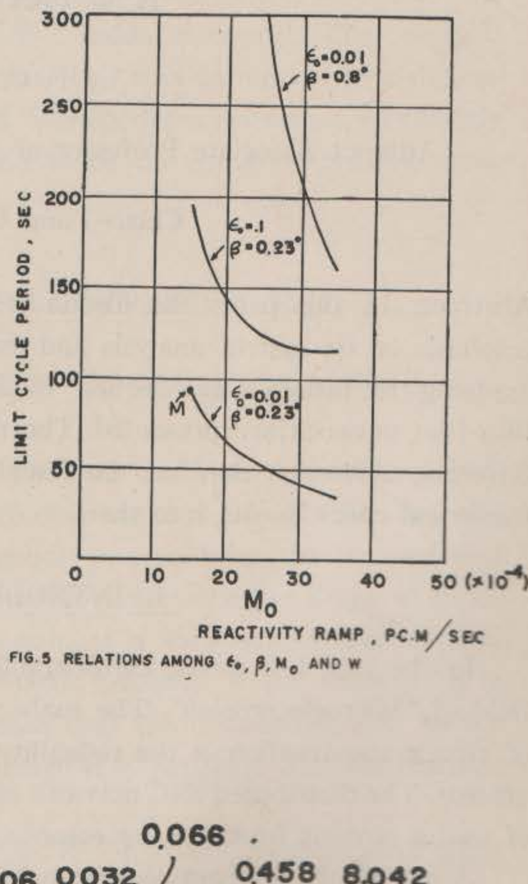

FIG. 5 RELATIONS AMONG ϵ_0, β, M_0 AND W

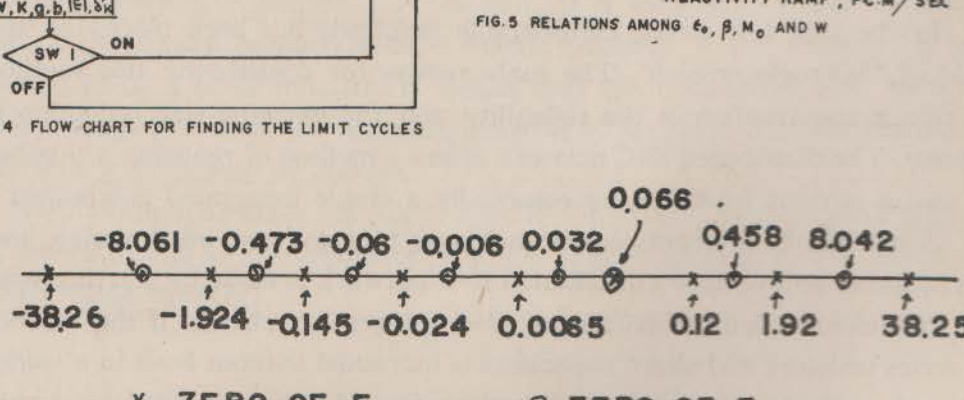

FIG. 6 AN ILLUSTRATION OF POLE-ZERO DISTRIBUTION