

Analysis of Linear Control Systems with Transport Lags

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1. Introduction

A control system, containing a transport lag is usually difficult for analysis, due to the existence of a transcendental function in the system characteristic equation. Yaohan Chu had presented a phase-angle-loci method, but it becomes increasingly difficult for higher order systems.(1) Eisenberg had given a fairly nice method, but it has been applied only to systems containing one transport lag.(2) In this paper, the root locus method is applied to the analysis of linear control systems with transport lags. The basic approach is to use the special property of zero-order holding circuit to represent a linear control system with transport lag by an equivalent sampled-data control system, then the system stability is analyzed in Z-domain using root-locus method.

2. Analysis of Linear Control Systems with One Transport Lag

In reference 3, it had been proved that a linear control system may be approximated by a sampled-data control system; i.e., to modify a continuous system into an equivalent sampled-data system. In a sampled-data control system, usually a zero-order holding circuit is used to remove the complementary components resulting from the sampling process. Since the special property of a zero-order holding circuit is to introduce a time delay of $T/2$ (T is the sampling period) to the signal passing through the sampler, by a suitable selection of T , one can use a zero-order holding circuit to replace a delay element $e^{-\tau s}$ and to find a sampled-data system which is equivalent to a continuous system with a transport lag. The following two examples are used as illustrations.

Example 1. Consider the system in Fig. 1. Assume $\tau = 1$ se-

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cond and $T = 2$ seconds, then an equivalent sampled-data system can be obtained as in Fig. 2(a), where the forward transfer function is

$$G(s) = \frac{K(1 - e^{-Ts})(s + 0.3)}{s^3} \quad (1)$$

which gives

$$G(Z) = \frac{2.6K(Z - 0.538)}{(Z - 1)^2} \quad (2)$$

Using root-locus method, the results of analysis are shown in Fig. 2(b). Since the root-locus gets through the unit cycle at $Z = -1$, the maximum value of open loop gain for system stability is found to be $K_m = 1$.

The same example had been analyzed by Chu and Eisenberg; the value of K_m is 1.32 in both papers. [1,2] The erroneous result obtained in the foregoing example is due to that the selected sampling period (T) is too long for the considered system.

In case the sampling period (T) is reduced to 0.8 second, since the holding circuit can only delay the signal by $T/2$, an additional delay element should be added to the forward path. The equivalent sampled-data system is shown in Fig. 3(a), which gives [4,5]

$$G(s) = \frac{K(1 - e^{-0.8})(s + 0.3)e^{-0.68}}{s^3} \quad (3)$$

and

$$G(Z) = \frac{0.206K(Z - 0.79)(Z + 3.37)}{Z(Z - 1)^2} \quad (4)$$

The results of root-locus analysis are shown in Fig. 3(b), which indicate that the root-loci get through the unit cycle at $Z = 0.47 \pm j0.9$. At these two points the value of K_m is 1.33, which is approximately the same as that in both Chu's and Eisenberg's papers.

Example 2. Consider the system in Fig. 4(a). After a sampling period (T) of two seconds is chosen, the equivalent sampled-data system is shown in Fig. 4(b); thus

$$G(s) = \frac{K(1 - e^{-0.2s})}{s^2(s + 4)(s + 2)} \quad (5)$$

and

$$G(Z) = \frac{0.06K(Z + 0.145)(Z + 3.425)}{64(Z - 1)(Z - 0.69)(Z - 0.45)} \quad (6)$$

The root-loci of this system are shown in Fig. 5, which indicates that the maximum value of K_m is 31.2. This result has been checked using Eisenberg's method. [2].

In the foregoing examples, it can be seen that the results of analysis will be more accurate if the chosen sampling period T is smaller, but for small value of T the equivalent transfer function (in Z domain) will become more complex. So a compromise between these two cases should be decided by the designer.

3. Control Systems with more than one Transport Lag

The method presented in the last section can be applied to the analysis of control systems containing multi-transport lags. This is illustrated by the following example.

Example 3. Consider the system in Fig. 6(a). Choosing a sampling period (T) of one second, the equivalent sampled-data system is shown in Fig. 6(b); thus

$$G(s) = \frac{K(s+0.3)(1-e^{-s})(e^{-s})}{s^3 + K(s^3 + 0.3s^2)e^{-2s}} \quad (7)$$

Since it is difficult to find the Z transform of Eq. (9) one may approximate it using Z -form method. [6] The result is

$$G(Z) = \frac{K(Z^3 + 10.8Z^2 - 7.2Z - 1)}{6[2Z^4 - 4Z^3 + (2.3K + 2)Z^2 - 4KZ + 1.7K]} \quad (8)$$

Thus the characteristic equation is

$$F(Z) = 12Z^4 + (K - 24)Z^3 + (12 + 24.6K)Z^2 - 31.2KZ - 9.2 = 0 \quad (9)$$

Using K as a variable parameter, the root-loci are shown in Fig. 7, which indicate that the maximum open loop gain (K_m) is 0.62. This result has been checked using Eisenberg's method also.

4. Conclusions

A method is presented whereby the stability characteristics of linear control systems containing transport lags can be determined. Using the special property of zero-order holding circuit and a suitable sampling period, one can modify a linear control system with transport lag into an equivalent sampled-data control system and then analyze system stability using the root-locus method.

References

1. Y. Chu, "Feedback Control System with Deat-time Lag or Distributed Lag by Root-locus Method", AIEE Trans. Vol. 71, pp. 291-297, Nov. (1952).
2. L. Eisenberg, "Stability of Linear System with Transport Lag", IEEE Trans. on Automatic Control, April (1966) .
3. Boxer R. and S. Thaler, "A Simplified Method of Solving Linear and Nonlinear Systems", Proc, IRE, Vol, 44. pp. 89, (1956).
4. J. T. Tou, "Digital and Sampled-data Control System", McGraw-Hill Book Company, (1958).
5. I. Jury, "Theory and Application of the Z-transform Method ", McGraw-Hill Book Company, (1963).

List of symbols

| | |
|--------|----------------------|
| s | Laplace operator |
| τ | transport lag |
| G(s) | transfer function |
| Z | Z-transform operator |
| K | open loop gain |
| a, b | constants |
| R | input quantity |
| C | output quantity |
| T | sampling period |

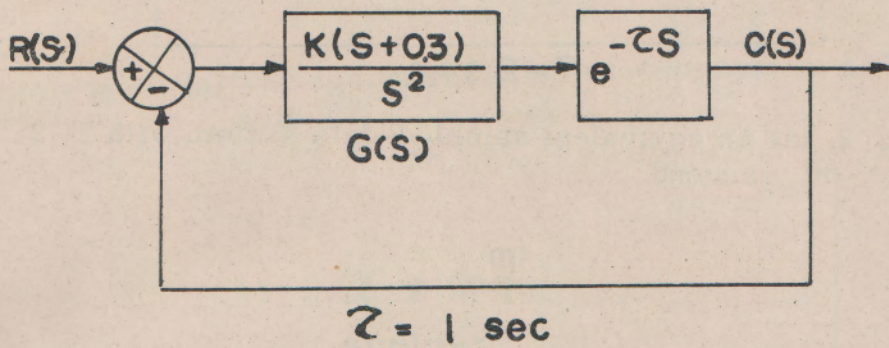


Fig. 1. Block diagram of a linear system with a transport lag.

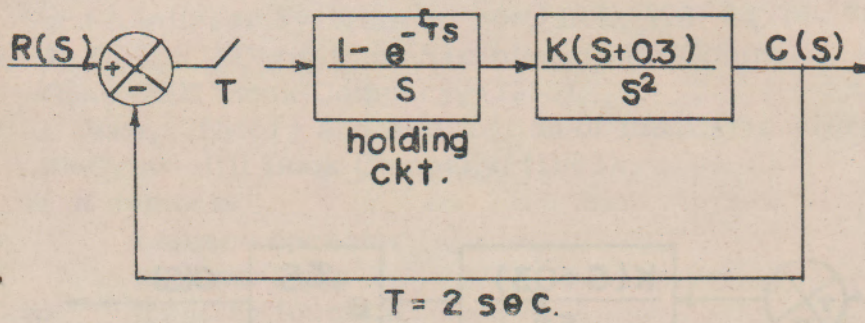


Fig. 2. (a) An equivalent sampled-data system with $T = 2$ second.

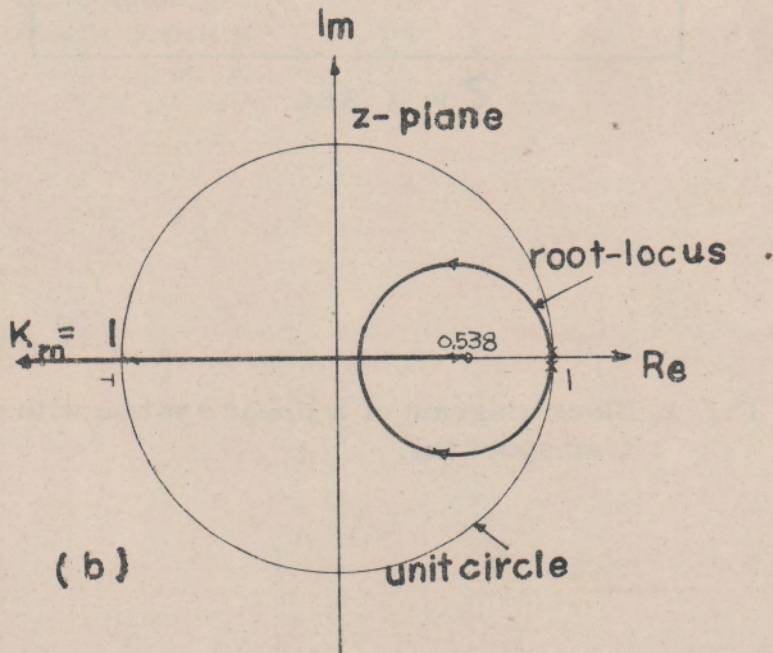


Fig. 2. (b) The root-loci of the system in Fig. 2(a).

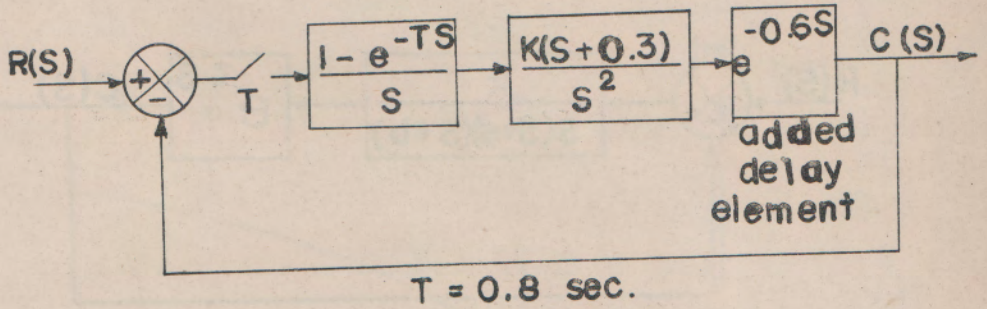


Fig. 3. (a) The equivalent sampled-data system of Fig. 1 with $T = 0.8$ second.

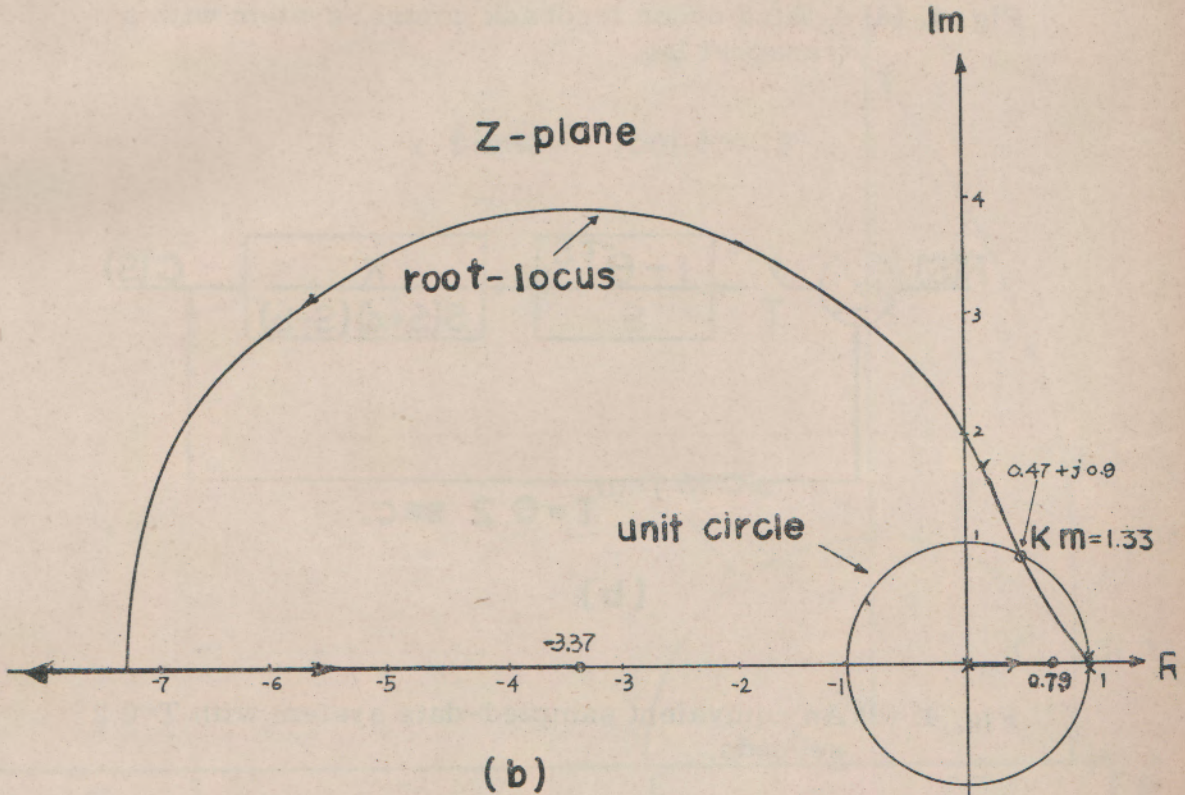
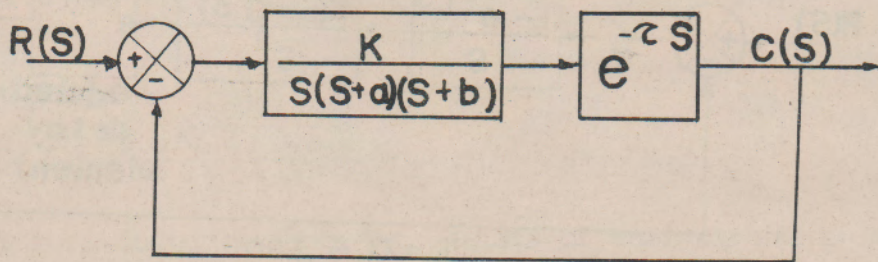


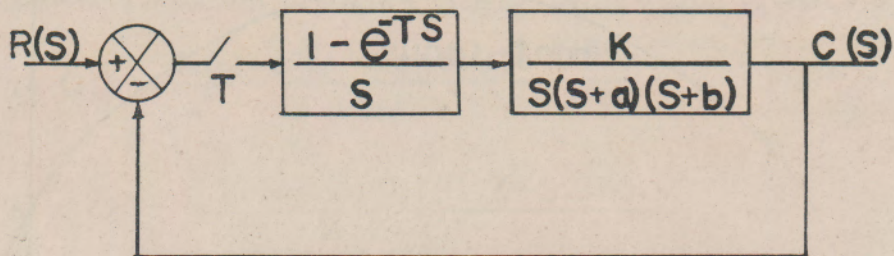
Fig. 3. (b) The root-loci of the system in Fig. 3(a).



$$\tau = 0.1 \text{ sec.}$$

(a)

Fig. 4. (a) A third order feedback control system with a transport lag.



$$T = 0.2 \text{ sec.}$$

(b)

Fig. 4. (b) An equivalent sampled-data system with $T=0.2$ seconds.

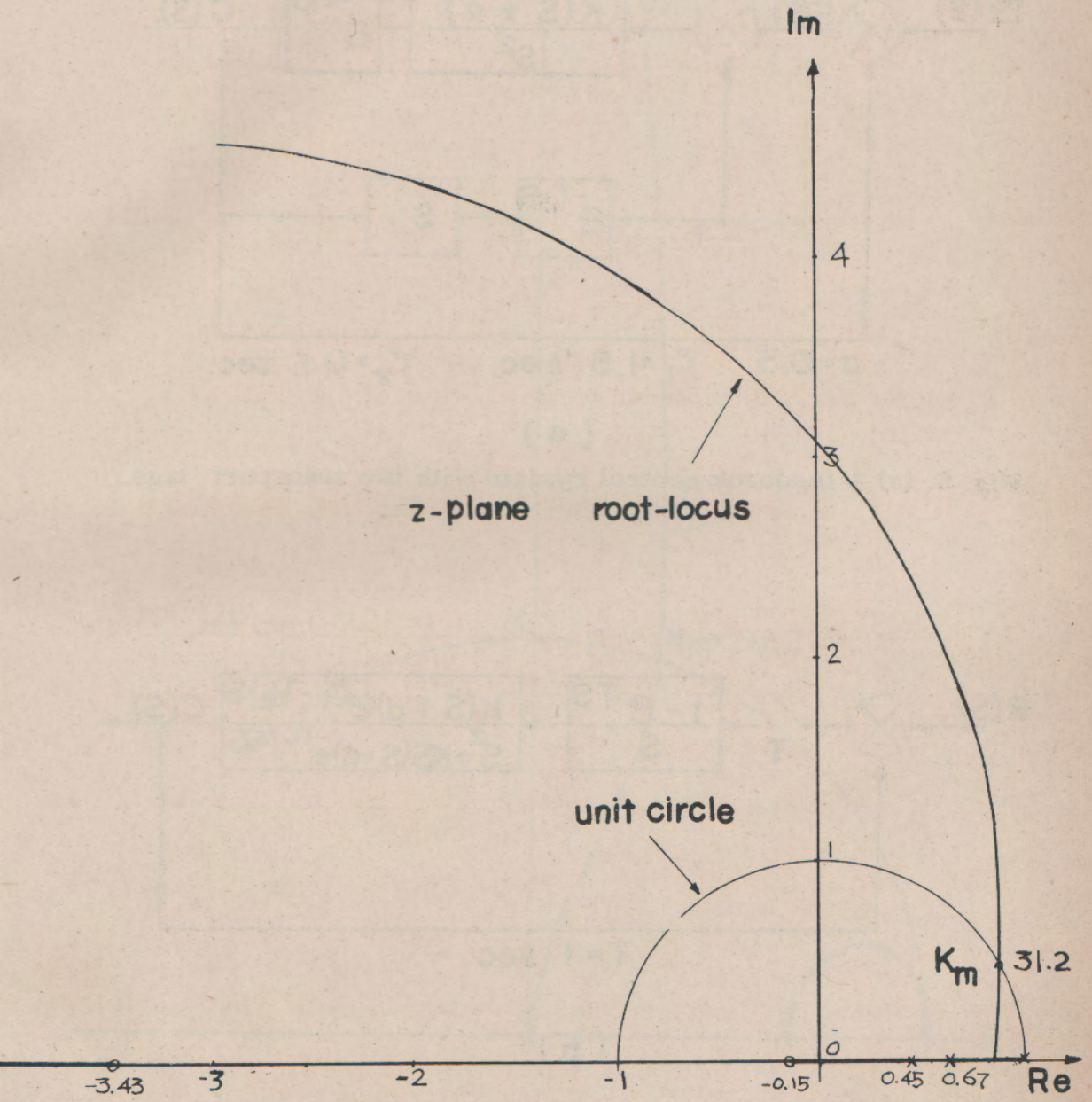
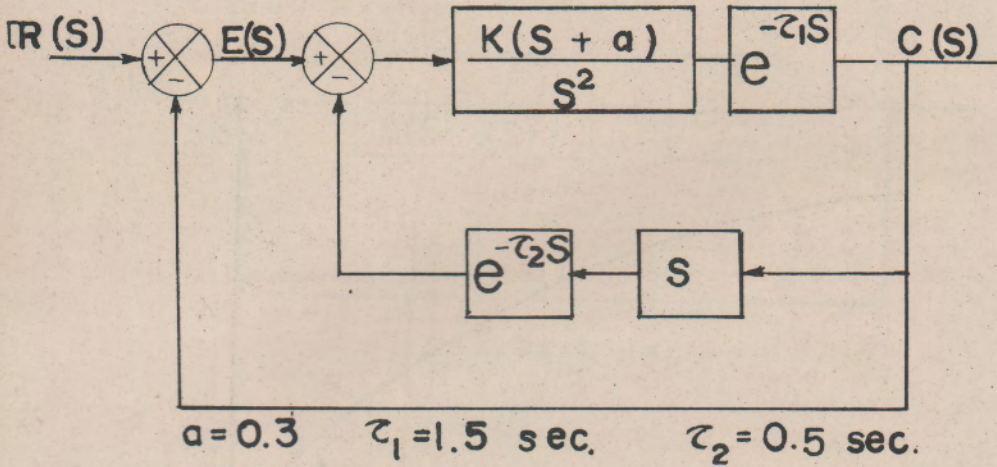
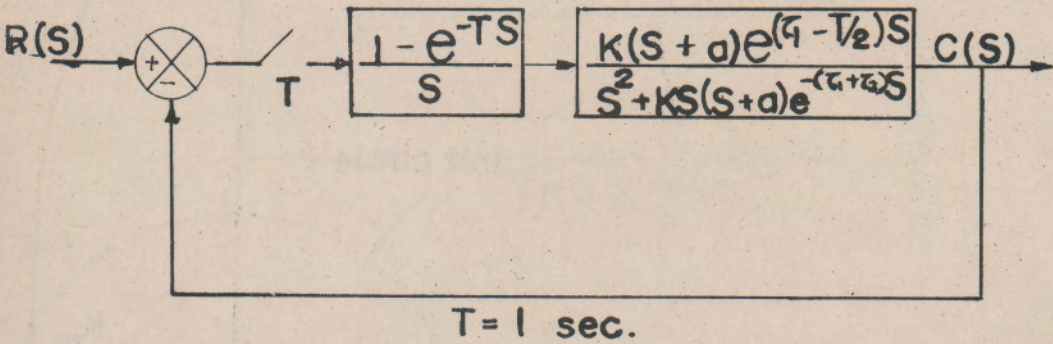


Fig. 5. The root-loci of the system in Fig. 4(b).



(a)

Fig. 6. (a) A feedback control system with two transport lags.



(b)

Fig. 6. (b) The equivalent sampled-data system with $T = 1$ second.

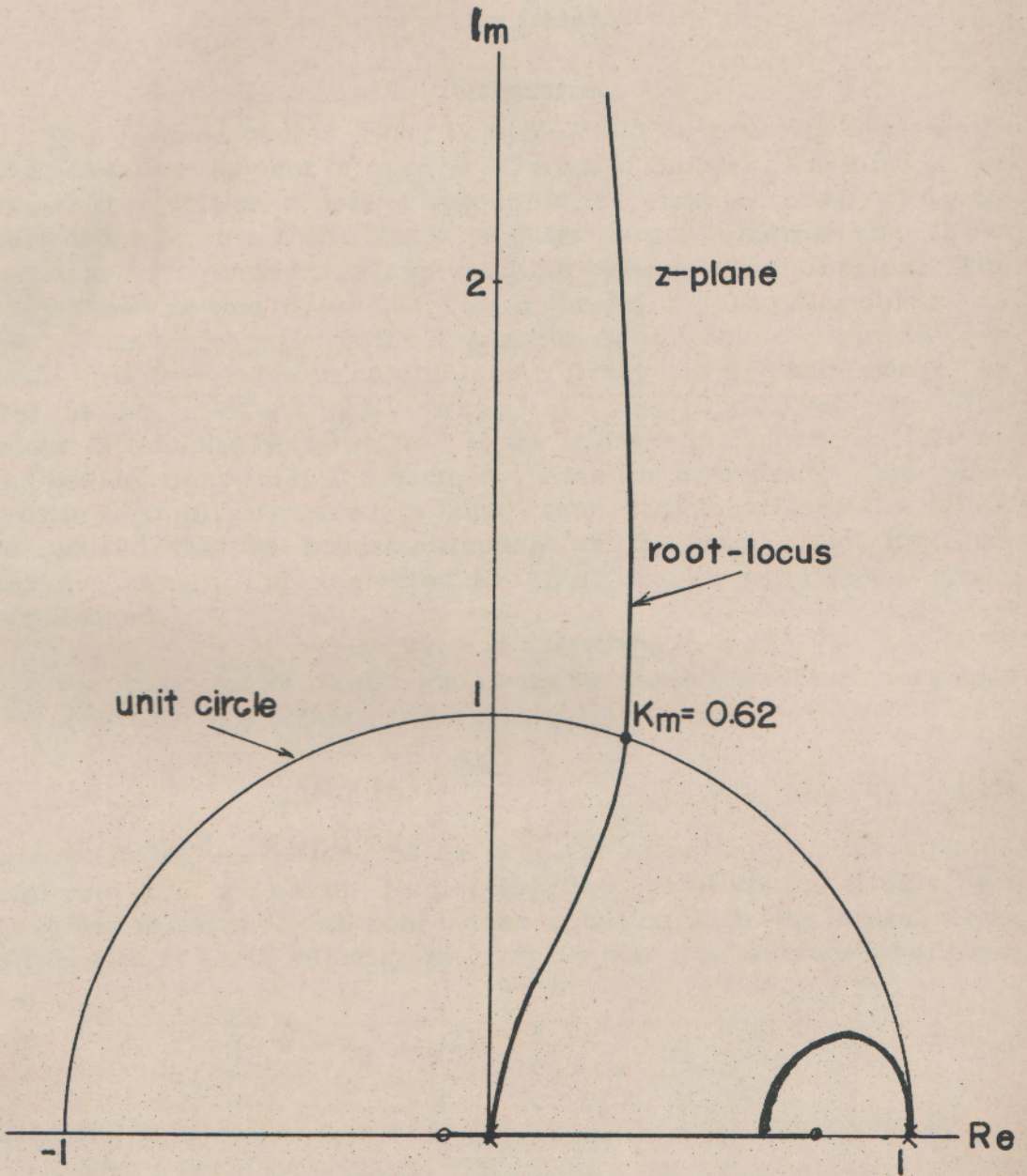


Fig. 7. The root-loci of the system in Fig. 6(b).