

# On Asymmetrical Derivates of Non-Differentiable Functions

Tsui-Hwa Yang\*

## 1. Introduction

First we introduce some main definitions in the following two paragraphs:

### (i) Definition of Dini Derivates

We consider real-valued functions defined on the real line  $R$ . We shall associate four functions, which are called the Dini derivates of  $f$ , with every function  $f$ . The Dini derivates may have either real values or the values  $+\infty$  and  $-\infty$ .

We define the Dini derivates as follows:

$$D^+f(\xi) = \overline{\lim}_{h \rightarrow 0^+} \frac{f(\xi + h) - f(\xi)}{h}$$

$$D_+f(\xi) = \underline{\lim}_{h \rightarrow 0^+} \frac{f(\xi + h) - f(\xi)}{h}$$

$$D^-f(\xi) = \overline{\lim}_{h \rightarrow 0^-} \frac{f(\xi + h) - f(\xi)}{h}$$

$$D_-f(\xi) = \underline{\lim}_{h \rightarrow 0^-} \frac{f(\xi + h) - f(\xi)}{h}$$

\* The author is associated with College of Engineering, National Chiao-Tung University.

$D^+f(\xi)$ ,  $D_+f(\xi)$ ,  $D^-f(\xi)$  and  $D_-f(\xi)$  are respectively called the upper right, lower right, upper left and lower left Dini derivatives of  $f$  at  $\xi$ .

(ii) Definitions of Asymmetrical derivatives and knot points

Let  $f$  be a non-differentiable function in  $R$ , i.e., a real-valued continuous function defined on a linear interval which has nowhere a finite or infinite derivate. We shall say that  $f$  has symmetrical derivatives at a point  $x$  if the four Dini derivatives of  $f$  at  $x$  satisfy the relations

$$D^+f(x) = D^-f(x), \quad D_+f(x) = D_-f(x)$$

Otherwise, we shall say that  $f$  has asymmetrical derivatives at  $x$ . Let  $A$  denote the set of points where  $f$  has asymmetrical derivatives and let  $K$  denote the set of knot points of  $f$ , viz the points where

$$D^+f = D^-f = +\infty, \quad D_+f = D_-f = -\infty.$$

Clearly  $K \subset C(A)$ , where  $C(A)$  denote the complementary set of  $A$ , and so  $A \subset C(K)$ .

In the present paper we first prove the existence of non-differentiable functions by giving some examples. Then we study in detail the properties of the derivatives of the general non-differentiable functions; the set  $K$  is residual, the set  $A$  is therefore of the first category. We further prove that  $A$  is necessarily non-empty and is everywhere dense. The set  $A$  also has the power  $c$  and therefore  $C(K)$  has the power  $c$ .

## 2. Definitions and Terminologies

In this section we collect some more definitions and terminologies which will be used in the subsequent analysis.

The first category: A set  $S$  is of the first category in a set  $T$  if it can be represented as a countable union of sets each of which is nowhere dense in  $T$ .

Residual set: The complement of a set  $S$  of the first category in  $T$  is called a residual set of  $T$ .

Power of the continuum (or power  $c$ ): If the set  $A$  is equivalent to the segment  $U = [0, 1]$ ,  $A \sim U$ , then  $A$  is said to have the power of the continuum.

Exterior and Interior measure: Let  $E$  be a set of points and  $S$  be a finite or countably infinite set of intervals (open or closed) such that each point of  $E$  belongs to at least one of the intervals. The exterior measure of  $E$  is the greatest lower bound of the sum of the measure of the intervals of  $S$  for all such sets  $S$ . Let  $E$  be contained in an interval  $I$  and let  $E'$  be the complement of  $E$  in  $I$ . Then the interior measure of  $E$  is the difference between the measure of  $I$  and the exterior measure of  $E'$ .

Semi-continuous: Let  $f$  be defined on a set  $S$ , then  $f$  is said to be lower semi-continuous at  $\xi \in S$  relative to  $S$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $f(\xi) < f(x) + \varepsilon$  whenever  $x \in S$  and  $|x - \xi| < \delta$ .  $f$  is said to be upper semicontinuous at  $\xi \in S$  relative to  $S$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $f(\xi) > f(x) - \varepsilon$  whenever  $x \in S$  and  $|x - \xi| < \delta$ .

### 3. General Non-differentiable Functions

(i) The existence of non-differentiable functions

(a) Let  $G(x)$  be the distance from the real number  $x$  to the nearest integer. The graph of  $G$  is shown in Fig. 1.

Let  $H(x) = \sum_{n=0}^{\infty} 2^{-n} G_n(x)$  where  $G_n(x) = G(2^n x)$ . The continuity of  $H$  follows from the continuity of  $G$  and the uniform convergence of the series.

Ralph P. Boas proved that  $H$  has no finite derivative at any point [I, 115-116].

(b) A. S. Besicovitch gave in 1915 the first example of a continuous function which has nowhere a unilateral derivative. [VII, 244-253].

(c) Weierstrass gave in 1874 on the function  $\sum a^n \cos b^n x$  without bothsided finite or infinite derivatives in any point. [VI, 177-178].

(ii) The derivatives of a general non-differentiable function

If a function  $f$  is non-differentiable in an interval, then,

(a) the derivatives of  $f$  at a point  $x$  containing in certain set satisfy one of the following relations.

$$1) D^+f(x) \neq D^-f(x),$$

$$2) D_+f(x) \neq D_-f(x),$$

$$3) D^+f(x) \neq D^-f(x) \text{ and } D_+f(x) \neq D_-f(x);$$

(b) at the remaining points of the interval both upper derivatives are equal and both lower derivatives are equal. i.e.,

$$D^+f = D^-f(x), D_+f(x) = D_-f(x) \text{ and } D^+f \neq D_+f.$$

Lemma 1.

The derivatives of a non-differentiable function are unbounded both above and below in every interval.

Proof:

Case 1:

Let the derivatives have both a finite upper bound and a finite lower bound in the interval considered.

By Lebesgue's theorem [V, 95], a differentiable coefficient exists at every point of a set whose measure is that of the interval considered.

## Case 2:

Let only one of the bounds be finite, say the lower bound  $-L (L > 0)$ .

Let  $y = f(x) + (1 + L)x$ ; then, the derivatives of  $y$  with respect to  $x$  are obtained by adding  $(1 + L)$  to the corresponding derivatives of  $f$ , and are therefore always positive, being  $\geq +1$ . It is clear that  $x$  is a monotone increasing function of  $y$  and the various derivatives of  $x = F(y)$  are reciprocals of derivatives of  $y$  with respect to  $x$ . It follows that the derivatives of  $x$ , i.e.,  $F(y)$  with respect to  $y$  are all positive or zero and are  $\leq 1$ . Then by Case 1, a value of  $y$  exists in the intervals of values of  $y$  considered at which a differential coefficient  $F'(y)$  exists. Let this value be  $y_0$  then

$$\frac{x - x_0}{y - y_0}$$

has a single finite limit which is either zero, or positive and  $\leq 1$ . This implies that

$$\frac{y - y_0}{x - x_0}$$

has a definite infinite limit or a limit  $\geq 1$ .

That is to say  $y$  has a differential coefficient with respect to  $x$ . Thus in this case  $y$ , and therefore also  $f$ , is not a non-differentiable function of  $x$ .

Similarly, we can prove that the derivative of a non-differential function are unbounded from above.

Thus we complete the proof of the lemma.

This at once follows that statement:

## Corollary 1:

If  $f$  is a non-differentiable function, whose upper derivatives are equal at a point, then  $D^+f(x) = D^-f(x) = -\infty$ . Similarly if both lower derivatives are equal, then  $D_+f(x) = D_-f(x) = -\infty$ .

By Corollary 1, we show that the only possibility of the case (ii), (b) is

$$D^+f = D^-f = +\infty, \quad D_+f = D_-f = -\infty.$$

(iii)  $K$  is residual and  $A$  is of the first category

W. H. Young proved the following theorem [IX, 305]

## Theorem 1.

There is no distinction of right and left with respect to the Dini derivatives except at points of a set of the first category.

This at once yields

Theorem 2.

If  $f$  is a non-differentiable function, every point of the interval considered which does not belong to a certain set of the first category is the knot point.

Proof:

From theorem 1, we know that the two upper derivatives are equal and the two lower derivatives are equal except at a set of the first category, and therefore, by Corollary 1, both upper derivatives are  $+\infty$  and both lower derivatives are  $-\infty$ .

The conclusion is that every point of the interval considered is the knot point except at a set of the first category.

Since the set  $K$  of the knot points is the complement of a set of the first category, so  $K$  is residual.

Moreover,

$$K \subset C(A), \quad A \subset C(K).$$

Thus,  $A$  is of the first category.

(iv)  $A$  is non-empty and is everywhere dense

Theorem 3.

Let  $f$  is a non-differentiable function. The exceptional set of the first category at which there is a distinction of right and left exists and is dense everywhere.

Proof:

To prove this theorem we require the first Theorem of the mean in an extended form:

"If there is no distinction of right and left with regard to the derivatives of  $f$ , then there is a point in the completely open interval  $(a, b)$  at which  $f$  has a differential coefficient, and the value of that differential coefficient is precisely

$$m(a, b) = \frac{f(b) - f(a)}{b - a}$$

That is, for some value of  $Q$

$$m(a, b) = f'(a + \theta(b - a)) \quad (0 < \theta < 1)."$$

Thus,  $f$  would not be a non-differentiable function.

Now, if the exceptional set were not dense everywhere, there would in some interval be no distinction of right and left, and therefore, by the First Theorem of the Mean in the extended form, there would be a point at which there was a differential coefficient, so that  $f$  would not be a non-differentiable function. Thus the exceptional set certainly exists and is dense everywhere.

From this theorem, we know that the set  $A$  is non-empty and

is dense everywhere.

Theorem 4.

The derivatives of a non-differentiable function at the point of the exceptional set at which there is a distinction of right and left have no finite upper or lower bound.

Proof:

Suppose that the derivatives at the points of the exceptional set have a finite upper bound  $B$ , and let  $C$  be greater than  $B$ . Then, by Lemma 1, the upper bound of the derivatives in the whole interval considered is  $+\infty$  and therefore by a well-known theorem this is same as upper bound of the incrementary ratio

$$m(x, x') = \frac{f(x) - f(x')}{x - x'}$$

where  $x$  and  $x'$  are restricted to lie in the same interval. But the incrementary ratio assumes every value between its upper and lower bounds, so that we must be able to find a pair of point  $(x, x')$ , such that

$$m(x, x') = C.$$

By the following theorem [VIII, 68],

"There is in the complete interval  $(x, x')$  a point at which one of the lower derivatives is  $\geq m(x, x')$  while the other upper derivatives is  $\leq m(x, x')$ ."

The point referred to must be one at which there is no distinction of right and left, since  $m(x, x') = C > B$ , and therefore both derivatives must be  $\geq C$  and both upper derivatives  $\leq C$ , which is only possible if all the derivatives are  $= C$ . However, this is not possible, since there is no point at which all the derivatives are equal,  $f$  being a non-differentiable function. Thus the original supposition is untenable, which prove that the upper bound of the values of the derivatives at points of the exceptional set is  $+\infty$ . Similarly, the lower bound is  $-\infty$ , which prove the theorem.

Thus for any non-differentiable function  $f$  the derivatives of  $f$  are unbounded from above and below at the points of  $A$ .

(v) The set  $A$  has the power of continuum

K. M. Garg has proved the following theorem called "Denjoy analogue" [II, 9-14]:

"Given an arbitrary finite real function  $f$  defined in a linear interval  $I$ , the Dini derivatives of  $f$  at each point  $x$ , except possible at a set  $E \subset I$  for which  $\mu f(E) = 0$  (where  $\mu f(E)$  denotes the Lebesgue measure of the image of  $f$  over the set  $E$ ), satisfy one of the following four relations [IV, 183]:

- (a)  $D_+f(x) = D_-f(x) = D^+f(x) = D^-f(x) \neq 0$  ;  
 (b)  $D_+f(x) = D^-f(x) \neq 0$ ,  $D^+f(x) = +\infty$ ,  $D_-f(x) = -\infty$  ;  
 (c)  $D^+f(x) = D_-f(x) \neq 0$ ,  $D_+f(x) = -\infty$ ,  $D^-f(x) = +\infty$  ;  
 (d)  $D^+f(x) = D^-f(x) = +\infty$ ,  $D_+f(x) = D_-f(x) = -\infty$ ."

Now, we only discuss the non-differentiable function, so that the relation (a) is removed. Then combining this with the three relations of Dini derivatives in (ii), except the set  $K$ , we obtain thereby the following distribution of every possible set of points of the set of the first category:

If  $f$  is a non-differentiable function defined in an interval  $I$  and for every real number  $r \geq 0$ , we have

$$S_{1r}\{S_{2r}\} = \{x: D^+f = D_-f = \text{finite} > r\{<r\}, D_+f = -\infty, D^-f = +\infty\};$$

$$S_{3r}\{S_{4r}\} = \{x: D_+f = D^-f = \text{finite} > r\{<r\}, D^+f = +\infty, D_-f = -\infty\};$$

$$S_{1\infty}\{S_{2\infty}\} = \{x: D^+f = D_-f = +\infty\{-\infty\}, D_+f = -\infty, D^-f = +\infty\};$$

$$S_{3\infty}\{S_{4\infty}\} = \{x: D_+f = D^-f = +\infty\{-\infty\}, D^+f = +\infty, D_-f = -\infty\};$$

K. M. Garg [III, 666] established that if  $f$  is non-differentiable in  $I$ , then each interval  $J \subset I$  contains, for each of  $i = 1$  to  $4$ , either a subset of  $S_{i0}$  of positive measure, or a subset of  $S_{i\infty}$  of power  $c$ , which is mapped by  $f$  into a set of positive interior measure.

For any  $r \geq 0$  and for each of  $i = 1$  to  $4$ , at a point  $x$  belonging to any of the sets  $S_{ir}$  and  $S_{i\infty}$  the derivatives of  $f$  are asymmetrical and, moreover, at least one derivate of  $f$  is  $+\infty$  whereas at least one derivate of  $f$  is  $-\infty$ . We therefore have

Theorem 5.

If  $f$  is a non-differentiable function in  $I$ , there exists a set  $E$  which has the power of the continuum in every subinterval of  $I$  and is mapped by  $f$  into a set of positive interior measure, such that at each point of  $E$  the derivatives of  $f$  are asymmetrical and two of them are  $+\infty$  and  $-\infty$  respectively.

Thus for any non-differentiable function  $f$  the set  $A (\supset E)$  has the power  $c$  in every interval.

(vi)  $C(K)$  has the power of the continuum

After we have proved that  $A$  has the power  $c$  in every interval, of course, directly from this we can conclude that  $C(K)$  has the power  $c$  in every interval since  $A \subset C(K) \subset R$ .

#### 4. Discussion

This paper is only a general discussion on the asymmetrical derivatives of the general non-differentiable function. Concerning this subject, we can know somemore details by studying the special examples of the non-differentiable functions such as Besicovitch and Weiertrass function.

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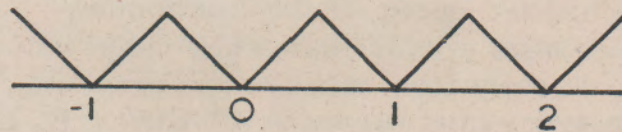


FIG. I