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ON A GRILLAGE SUPPORTED AT FOUR CORNERS

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ABSTRACT: A matrix analysis of a grillage simply supported at four corners under joint loadings is presented with a new approach. The beams of the grillage may be of variable stiffness with unequal spacing, and their intersections need not be orthogonal. However, the stiffness of all transverse or longitudinal beams must be the same. The joints of the grillage are assumed to transmit only compression and tension. The matrix equation of the grillage leads to the form $AX+XB=E$, in which matrices A and B may be singular, and the unknown matrix X is rectangular. A method of solution is developed and illustrated by a numerical example.

I. INTRODUCTION

Grillages are practical structures of interest. Many papers have been written to determine deflections and internal forces of grillages under different boundary conditions and various external loadings. S.P. Timoshenko[1] employed trigonometric series to express elastic curves of individual beams of a grillage. W. W. Ewell, S. Okubo, and J. I. Abrams[2] used moment and torque distribution process to obtain solutions of rigidly connected grillages. J. Szabo[3] calculated deflections and internal forces of a grillage in a bridge structure. C. T. F. Rose[4] treated a deep girder grillager under uniform load. T. Yamanchi[5] experimented with electric analogy in the analysis of a grillage. F. S. Shaw[6] used moment-blancing process to find a lower bound of the collapse load of a rigid gridwork. T. Wah[7] used finite difference calculus to solve a simply-supported gridwork. His method leads to an analog with the classical theory of plates. D. L. Dean[8] treated lamella grids by using an artificial vector to represent scalar mode quantities. F. D. DiMaggio and W. R. Spillers[9] applied network theory to analyze structures.

This paper presents a matrix analysis of a grillage simply supported at four corners under joint loadings. The beam of the grillage may be of variable stiffness and spaced unequally. The intersections of the beams are not required to be orthogonal. However, the stiffness of all transverse or longitudinal beams must be the same. The joints of the grillage are assumed to transmit only compression and tension. The governing matrix equation of the grillage leads to the form AX+XB=E, where A and B are singular matrices. The unknown quantities, the internal forces and deflections of the joints, are expressed in rectangular matrices rather than column vectors in other matrix analyses. A method of solution is developed and illustrated by a numerical example.

II. DERIVATION OF EQUATIONS

A grillage simply supported at four corners under normal joint loads is shown in Fig. 1. There are n transverse beams and m longitudinal beams arranged in parallel in each direction, and their intersections need not be orthogonal. The analysis is approached by considering the grillage being simply supported at all joints along the edges, then the deflections of these joints are superposed to give the governing matrix equation of the system.

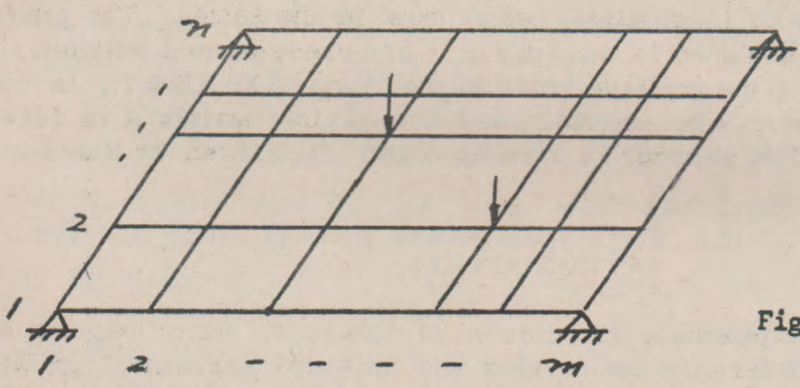

Figure 1

Assume that the grillage in Fig. 1 is simply supported at all joints on the boundary. Let (α) and (β) be the influence matrices of transverse and longitudinal beams respectively, (δ) and (Δ) be their respective joint deflection matrices, and (f) and (F) be their respective matrices of internal forces distributed to the joints due to external loads. For a given grillage, the influence matrices (α) and (β) are known and different for different stiffness and spacings of the beams. The matrix equations of transverse and longitudinal beams are

$$\underset{(m,m)}{(\alpha)}\ \underset{(m,n)}{(f)} = \underset{(m,n)}{(\delta)} \tag{1}$$

and

$$\underset{(n,n)}{(\beta)}\ \underset{(n,m)}{(F)} = \underset{(n,m)}{(\Delta)} \tag{2}$$

where

$$(\alpha) = \begin{bmatrix} \circ\circ & \cdots & \circ \\ \circ & & \circ \\ \vdots & \alpha_{ij} & \vdots \\ \circ\circ & \cdots & \circ \end{bmatrix} \qquad (\beta) = \begin{bmatrix} \circ\circ & \cdots & \circ \\ \circ & & \circ \\ \vdots & \beta_{ij} & \vdots \\ \circ\circ & \cdots & \circ \end{bmatrix}$$

$$(\delta) = \begin{bmatrix} oo \cdots o \\ \delta_{ij} \\ oo \cdots o \end{bmatrix} \qquad (\Delta) = \begin{bmatrix} oo \cdots o \\ \Delta_{ij} \\ oo \cdots o \end{bmatrix}$$

$$(f) = \begin{bmatrix} f^{(1)} \\ --- \\ f_{ij} \\ --- \\ f^{(m)} \end{bmatrix} \qquad (F) = \begin{bmatrix} F^{(1)} \\ --- \\ F_{ij} \\ --- \\ F^{(n)} \end{bmatrix}$$

The dimensions of the matrices are shown under Eq. (1) and (2). $f^{(1)}$, $f^{(m)}$, $F^{(1)}$ and $F^{(n)}$ are the first and the last rows of (f) and (F) respectively. Hereafter, a notation $f^{(i)}$ denotes the i^{th} row of (f), and $f_{(i)}$ means the i^{th} column of (f).

Let (G) of dimension (m,n) be the matrix of external loads including the reactions at the corner supports. Equilibrium equations at each joint leads to the following relation between the internal and external force matrices

$$(f)+(F)^T = (G) \tag{3}$$

where $(F)^T$ denotes the transpose of (F). Eq. (3) implies

$$f^{(1)}+F^T_{(1)} = G^{(1)} \tag{4}$$

$$f^{(m)}+F^T_{(m)} = G^{(m)} \tag{5}$$

and

$$f_{(1)} + F^{(1)^T} = G_{(1)} \tag{6}$$

$$f_{(n)} + F^{(n)^T} = G_{(n)} \tag{7}$$

The row matrices $f^{(1)}$ and $f^{(m)}$ can be expressed in terms of (f) by using moment equations of all transverse beams with respect to both ends. Similarly, expressions can be obtained for $F^{(1)}$ and $F^{(n)}$ in terms of (F).

$$f^{(1)} = -(0,\ c_2,\ c_3 \dots c_{m-1}, 0) = -C^T_{(1)}\ (f) \tag{8}$$

$$f^{(m)} = -(0,\ c_{m-1},\ c_{m-2} \dots c_2, 0)\ (f) = -C^T_{(2)}\ (f) \tag{9}$$

$$F^{(1)} = -(0,\ d_2,\ d_3 \dots d_{n-1}, 0)\ (F) = -D^T_{(1)}\ (F) \tag{10}$$

and

$$F^{(n)} = -(0,\ d_{n-1},\ d_{n-2} \dots d_2,\ 0)\ (F) = -D^T_{(2)}\ (F) \tag{11}$$

where the row matrices $C^T_{(1)} = (0,\ c_2,\ c_3 \dots c_{m-1},\ 0)$ etc. The scalar quantities c_i and d_i depend on the spacing between beams. For an equally-spaced grillage, the following relations hold.

$$C^T_{(1)} = (0, \frac{m-2}{m-1}, \frac{m-3}{n-1}, \cdots \frac{1}{m-1}, 0) \tag{12}$$

$$D^{T}_{(1)} = \left(0, \frac{n-2}{n-1}, \frac{n-3}{n-1}, \cdots \frac{1}{n-1}, 0\right) \qquad (13)$$

In considering joint deflections, the deflection at each joint of individual beam due to the deflections at both ends must be added to the deflection of that joint obtained under the assumption of simply supported ends. For compatibility, the joint deflection in a transverse beam must be equal to the corresponding joint deflection on a longitudinal beam. This reasoning leads to the following deflection relation.

$$(\Delta)^{T} + \delta_{(1)} D^{T}_{(3)} + \delta_{(n)} D^{T}_{(4)} = (\delta) + C_{(3)} \Delta^{T}_{(1)} + C_{(4)} \Delta^{T}_{(m)} \qquad (14)$$

where

$$D^{T}_{(3)} = (1, d_2, d_3 \ldots d_{n-1}, 0) \qquad (15)$$

$$D^{T}_{(4)} = (0, d_{n-1}, d_{n-2} \ldots d_2, 1) \qquad (16)$$

$$C_{(3)} = (1, c_2, c_3 \ldots c_{m-1}, 0)^{T} \qquad (17)$$

and

$$C_{(4)} = (0, c_{m-1}, c_{m-2} \ldots c_2, 1)^{T} \qquad (18)$$

The scalar quantities c_i and d_i, depending on the spacing of the grillage, are identical with those in Eqs.(8), (9), (10), and (11).

The column matrix $\delta_{(1)}$ can be expressed in terms of (α), $G_{(1)}$, $(F)^{T}$ and $D_{(1)}$ by using Eqs. (1), (4) and (10). It yields

$$\delta_{(1)} = (\alpha)\, G_{(1)} + (\alpha)\, (F)^{T} D_{(1)} \qquad (19)$$

Similarly, the following expressions of $\delta_{(n)}$, $\Delta^{T}_{(1)}$ and $\Delta^{T}_{(m)}$ can be obtained.

$$\delta_{(n)} = (\alpha) G_{(n)} + (\alpha)\, (F)^{T} D_{(2)} \qquad (20)$$

$$\Delta^{T}_{(1)} = G^{(1)} (\beta)^{T} + C^{T}_{(1)} [(G) - (F)^{T}]\, (\beta)^{T} \qquad (21)$$

and

$$\Delta^{T}_{(m)} = G^{(m)} (\beta)^{T} + C^{T}_{(2)} [\, (G) - (F)^{T}]\, (\beta)^{T} \qquad (22)$$

Substituting Eqs.(19), (20), (21) and (22) into Eq. (14), the matrix equation of the grillage under consideration, after simplification, is given by

$$(I + C_{(3)} C^{T}_{(1)} + C_{(4)} C^{T}_{(2)})^{-1} (\alpha)(F)^{T} + (F)^{T} (\beta)^{T} (I + D_{(1)} D^{T}_{(3)} + D_{(2)} D^{T}_{(4)})^{-1}$$

$$= (I + C_{(3)} C^{T}_{(1)} + C_{(4)} C^{T}_{(2)})^{-1} \{ (\alpha) [\, (G) - G_{(1)} D^{T}_{(3)} - G_{(n)} D^{T}_{(4)}\,]$$

$$+ [\, C_{(3)} C^{T}_{(1)} (G) + C_{(4)} C^{T}_{(2)} (G) + C_{(3)} G^{(1)} + C_{(4)} G^{(m)}] (\beta)^{T} \}$$

$$(1 + D_{(1)} D^{T}_{(3)} + D_{(2)} D^{T}_{(4)})^{-1} \qquad (23)$$

In Eq. (23), the only unknown matrix is $(F)^{T}$. After solving $(F)^{T}$, the matrices (f), (δ) and (Δ) can be obtained from Eqs. (3), (1), and (2). The joint deflections of the grillage are given by the expression either before or after the equal sign in Eq. (14).

III. METHOD OF SOLUTION

The key of this analysis depends on the solution of Eq. (23) which can be reduced to the form

$$AX + XB = E \tag{24}$$

with $X = (F)^T$, where A and B are square matrices of dimension (m,m) and (n,n) respectively. X and E are rectangular matrices of dimension (m,n). Furthermore A and B are singular due to the singularity of (α) and (β). The solution of Eq. (24) has been discussed in a paper [10] by the author. Fortunately, the elementary divisions of the matrices A and B derived from this grillage analysis usually are linear and distinct, although a rigorous proof is not available. The discussion hereinafter will be limited under this assumption.

Assuming that the elementary divisors of the matrices A and B are linear and distinct, let them be denoted by

$$\lambda - \lambda_i \ , \ i = 1, 2, \ldots\ldots m$$

and

$$\mu - \mu_j \ , \ j = 1, 2, \ldots\ldots n$$

respectively. There exist nonsingular matrices U and V such that

$$A = U\widetilde{A}\ U^{-1} \text{ and } B = V\widetilde{B}\ V^{-1} \tag{25}$$

where $\widetilde{A}$ and $\widetilde{B}$ are Jordan normal forms of A and B respectively. In this case, $\widetilde{A}$ and $\widetilde{B}$ can be expressed as the following diagonal matrices.

$$\widetilde{A} = \text{diag}\ (\ \lambda_1,\ \lambda_2,\ \cdots \lambda_m\ ) \tag{26}$$

$$\widetilde{B} = \text{diag}\ (\ \mu_1,\ \mu_2,\ \cdots \mu_n\ ) \tag{27}$$

Substituting Eq. (25) into Eq. (24) yields the expression

$$\widetilde{A}X^* + X^*\widetilde{B} = E^* \tag{28}$$

where $X^* = U^{-1}XV$ and $E^* = U^{-1}EV$. The solution of Eq. (28) is given by

$$X^* = \left(\frac{e_{ij}}{\lambda_i + \mu_j}\right) \tag{29}$$

if $\lambda_i + \mu_j \neq 0$, where e_{ij} are the elements of E^*.

The solution of Eq. (24) can be obtained from

$$X = UX^*V^{-1} \tag{30}$$

IV. NUMERICAL EXAMPLE

In Fig. 2, a grillage simply supported at four corners under a concentrated load P at joint (3.3) is shown. The transverse beams are equally spaced with variable stiffness $K\sqrt{1+x}$, where K is a constant. The longitudinal beams are unequally spaced with constant stiffness K. The dimensions of the grillage are indicated in Fig. 2.

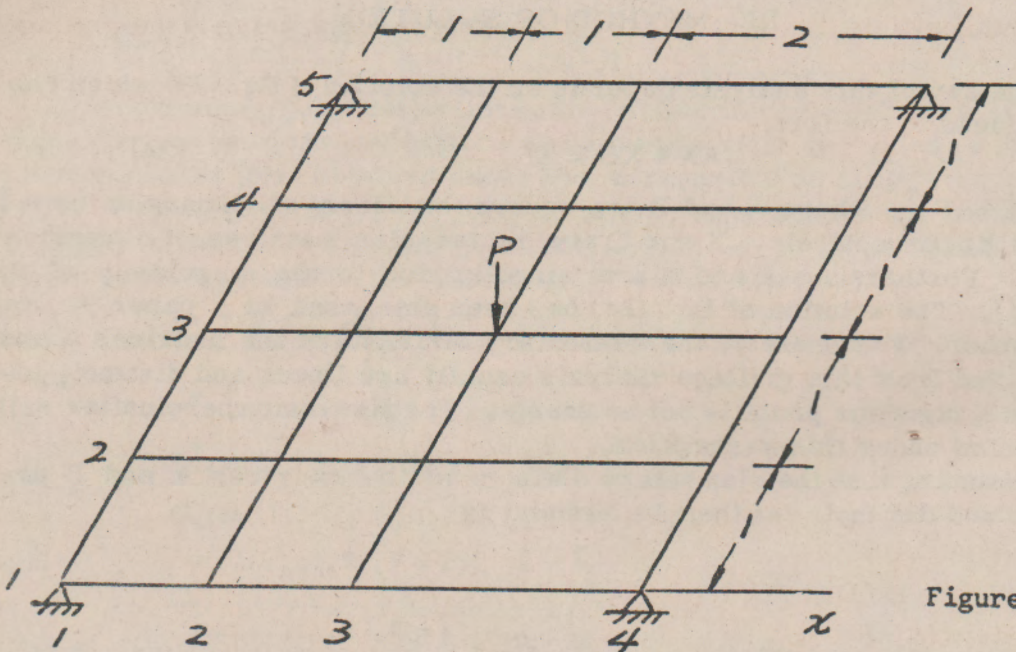

Figure 2

The influence matrices of the transverse and longitudinal beams are

$$(\alpha) = \frac{1}{15\mathrm{K}} \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 7.3037 & 8.4586 & 0 \\ 0 & 8.4586 & 11.7607 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(\beta) = \frac{1}{12\mathrm{K}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 11 & 7 & 0 \\ 0 & 11 & 16 & 11 & 0 \\ 0 & 7 & 11 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The external loading matrix is

$$(\mathrm{G}) = \begin{bmatrix} \mathrm{R}_1 & 0 & 0 & 0 & \mathrm{R}_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{P} & 0 & 0 \\ \mathrm{R}_3 & 0 & 0 & 0 & \mathrm{R}_4 \end{bmatrix}$$

where R_1, R_2, R_3 and R_4 are reactions at the corners.

The coefficient row matrices, depending on the spacing of the grillage are:

$$C^T_{(1)} = (0, 3/4, 1/2, 0)$$

$$C^T_{(2)} = (0, 1/2, 3/4, 0)$$

$$C^T_{(3)} = (1, 3/4, 1/2, 0)$$

$$C^T_{(4)} = (0, 1/2, 3/4, 1)$$

$$D^T_{(1)} = (0, 3/4, 1/2, 1/4, 0)$$

$$D^T_{(2)} = (0, 1/4, 1/2, 3/4, 0)$$

$$D^T_{(3)} = (1, 3/4, 1/2, 1/4, 0)$$

$$D^T_{(4)} = (0, 1/4, 1/2, 3/4, 1)$$

Substituting above matrices into Eq.(23) and simplifying yields

$$\begin{bmatrix} 0 & -0.3075 & -0.3824 & 0 \\ 0 & 0.2050 & 0.2075 & 0 \\ 0 & 0.3076 & 0.4535 & 0 \\ 0 & -0.2051 & -0.2786 & 0 \end{bmatrix} (F)^T$$

$$+ (F)^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.4833 & 0.2833 & 0.4667 & 0.1500 & -0.4167 \\ -0.6333 & 0.2833 & 0.7000 & 0.2833 & -0.6333 \\ -0.4167 & 0.1500 & 0.4667 & 0.2833 & -0.4833 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= P \begin{bmatrix} -0.0502 & 0.1332 & -0.1659 & 0.1332 & -0.0502 \\ -0.1863 & 0.0233 & 0.3260 & 0.0233 & -0.1863 \\ -0.2536 & -0.0178 & -0.5428 & -0.0178 & -0.2536 \\ -0.1433 & 0.1447 & -0.0029 & 0.1447 & -0.1433 \end{bmatrix}$$

The Jordan normal forms of matrices A and B are

$$\widetilde{A} = \mathrm{diag}\ (\ 0,\ 0.0477,\ 0.6108,\ 0\ )$$

and

$$\widetilde{B} = \mathrm{diag}\ (\ 0,\ 0.0354,\ 0.1333,\ 1.0979,\ 0\ )$$

respectively. The nonsingular matrices U and V associated with matrices A and B respectively are

$$U = \begin{bmatrix} 1 & -0.3690 & -0.8834 & 0 \\ 0 & 1 & 0.5113 & 0 \\ 0 & -0.7580 & 1 & 0 \\ 0 & 0.0128 & -0.6279 & 1 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.7497 & 0.3128 & 0.5000 & 0.1873 & 0.2501 \\ 0.5001 & -0.2666 & 0 & 0.2664 & 0.5001 \\ 0.2501 & 0.3128 & -0.5000 & 0.1873 & 0.7497 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

After transformation, the matrix E^* in Eq.(28) has the following form

$$E^* = U^{-1} E V = P \begin{bmatrix} 0 & -0.0047 & 0 & 0.1464 & 0 \\ 0 & 0.0053 & 0 & 0.0181 & 0 \\ 0 & -0.1519 & 0 & 0.1518 & 0 \\ 0 & -0.0045 & 0 & 0.1485 & 0 \end{bmatrix}$$

From Eq. (29) the solution of Eq. (28) is given by

$$X^* = P \begin{bmatrix} K_1 & -0.1330 & 0 & 0.1333 & K_2 \\ 0 & 0.0638 & 0 & 0.0158 & 0 \\ 0 & -0.2350 & 0 & 0.0888 & 0 \\ K_3 & -0.1270 & 0 & 0.1353 & K_4 \end{bmatrix}$$

where K_1, K_2, K_3 and K_4 are arbitrary constants obtained from the indeterminate form o/o using Eq. (29). These constants will be determined later by considering the equilibrium equations for the longitudinal beams.

The transpose of the internal force matrix for the longitudinal beams is obtained from

$$(F)^T = UX^*V^{-1} = P\begin{bmatrix} -0.1215+K_1 & 0.0995 & 0.0442 & 0.0995 & -0.1215+K_2 \\ -0.1162 & 0.0048 & 0.2228 & 0.0048 & -0.1162 \\ -0.0826 & -0.2065 & 0.5782 & -0.2065 & -0.0826 \\ -0.1798+K_3 & 0.1023 & 0.1552 & 0.1023 & -0.1798+K_4 \end{bmatrix}$$

Equilibrium conditions of the first and the fourth longitudinal beams yield

$$K_1 = K_2 = K_3 = K_4 = -0.0001$$

Therefore, the internal force matrix of the longitudinal beams is given by

$$(F) = P\begin{bmatrix} -0.1216 & -0.1162 & -0.0826 & -0.1799 \\ 0.0995 & 0.0048 & -0.2065 & 0.1023 \\ 0.0442 & 0.2228 & 0.5782 & 0.1552 \\ 0.0995 & 0.0048 & -0.2065 & 0.1023 \\ -0.1216 & -0.1162 & -0.0826 & -0.1799 \end{bmatrix}$$

The internal force matrix of the transverse beams is obtained from Eq. (3).

$$(f) = P\begin{bmatrix} 0.1216+\frac{R_1}{P} & -0.0995 & -0.0442 & -0.0995 & 0.1216+\frac{R_2}{P} \\ 0.1162 & -0.0048 & -0.2228 & -0.0048 & 0.1162 \\ 0.0826 & 0.2065 & 0.4218 & 0.2065 & 0.0826 \\ 0.1799+\frac{R_3}{P} & -0.1023 & -0.1552 & -0.1023 & 0.1779+\frac{R_4}{P} \end{bmatrix}$$

Equilibrium conditions of the first and the fifth transverse beams yield

$$R_1 = R_2 = R_3 = R_4 = -0.2501\,P$$

therefore

$$(f) = P\begin{bmatrix} -0.1285 & -0.0995 & -0.0442 & -0.0995 & -0.1285 \\ 0.1162 & -0.0048 & -0.2228 & -0.0048 & 0.1162 \\ 0.0826 & 0.2065 & 0.4218 & 0.2065 & 0.0826 \\ -0.0702 & -0.1023 & -0.1152 & -0.1023 & -0.0702 \end{bmatrix}$$

The matrix of the joint deflections of the grillage is given by Eq. (14)

$$(\delta) + C_{(3)}\,\Delta^T_{(1)} + C_{(4)}\,\Delta^T_{(4)} = \frac{P}{K}\begin{bmatrix} 0 & 0.1732 & 0.2414 & 0.1732 & 0 \\ 0.1032 & 0.3137 & 0.4090 & 0.3137 & 0.1032 \\ 0.1303 & 0.3851 & 0.5230 & 0.3851 & 0.1303 \\ 0 & 0.2787 & 0.3945 & 0.2787 & 0 \end{bmatrix}$$

V. SUMMARY

A grillage simply supported at four corners under normal joint loads has been analyzed with a new approach. This analysis also applies to applies to grillages with different boundary conditions such as all edges simply supported, three edges simply supported, and one edge free, and two opposite edges simply supported and the other two edges free. In all cases, the matrix equation of the grillage can be reduced to the form

$$AX + XB = E$$

in which matrices A and B may be singular, and the unknown matrix is rectangular rather than a column vector obtained by other matrix analyses. A general method of solution for this type matrix equation is developed and illustrated by a numerical example. Further work can be done to extend this analysis to grillages with rigid intersections and multi-layer grillages.

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NOTATIONS

A, B	General square matrices
$\widetilde{A}$, $\widetilde{B}$	Jordan normal forms of A and B
$C_{(i)}$, $D_{(i)}$	Coefficient column matrices
c_i , d_i , k_i	Scalar quantities
E, E*, X, X*	General rectangular matrices
(f), (F)	Internal force matrices of transverse and longitudinal beams
(G)	Matrix of external loads
K	Stiffness constant
P	Concentrated force
U, V	Nonsingular matrices associated with $\widetilde{A}$ and $\widetilde{B}$
(α), (β)	Influence matrices of transverse and longitudinal beams
(δ), (Δ)	Joint deflection matrices of transverse and longitudinal beams

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