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On The Basics of Parametric Amplification

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Abstract

A quantitative analysis of parametric amplification was usually made from the viewpoint of voltage or power. This paper adopted electric charge as the quantity to be analyzed, since it reveals the physical insight of parametric amplification fully. The charge generation mechanism and the charge oscillation criteria have been established and discussed in depth. The simple case of pulsive pumping has been considered thoroughly, and the obtained results may be applied to wave propagation in time-variant media and to light reflection in periodic optical resonators. The approach developed here can be extended to more general cases such as linear pumping and sinusoidal pumping.

Introduction

A capacitor shunted by an inductor is shown in Fig. 1. If there is no ohmic loss and the radiation loss is neglected, the oscillation level sustained in the circuit will be a constant independent of time. If the bottom plate of the capacitor is fixed in position and the upper plate is moved up and down periodically, then, under certain conditions, the oscillation level of the tank circuit can be made larger than ever. This is the basic idea of parametric amplification[1]. Consider the simplest case for illustration:

If the upper plate is pulled up to a certain position at the maximum attraction force between the plates and is lowered down to its original position at zero force, the work done by pulling the plate further apart will become energy stored in the system such that the oscillation level will be increased. This is the viewpoint adopted by most authors in describing parametric amplification. However, if the restrictions of pulling at maximum force and restoring at zero force are relaxed, can the amplification still be obtained? If yes, what are the conditions? To the best knowledge of the authors, a systematic approach to investigate those problems has not been presented elsewhere. The main purpose of this paper is to investigate the basics of parametric amplification.

The oscillation level of a tank circuit may be described by the maximum amount of electric charge participating in the oscillation. Thus, the increment in oscillation level signifies that additional charges have been generated. The charge generation mechanism is to be considered first. A mathematical analysis of the general problem will be presented next, and it is then to be followed by discussions of the results.

Although, it is perfectly legitimate to find the net work done to a tank circuit by the pumping force for eacn pumping cycle and consequently determine the amount of incremental charge, the paper perfers another approach without involving pumping force in the formulation. This is accomplished by obtaining the equation of motion in the intervals of constant capacitance, where the pumping force is absent, and using the oscillation criteria to match the intervals.

Charge Generation Mechanism

If an ideal capacitor is suddenly shunted by an ideal inductor (the original energy stored in the capacitor is assumed to be $w_{e_1}=q_1^2/(2c_1)$ where q_1 is the initial charge on one of the capacitor plates, say the upper plate, and c the capacitance of the capacitor), then current starts to flow through the inductor and the amount of charge on the capacitor plates starts to decrease. In other words, the electrical energy of the system originally stored entirely in the capacitor is now being shifted to the inductor in the form of magnetic energy expressed as $w_m=LI^2/2$, where L is the inductance of the inductor and I is the current in the inductor. Notice that in this case both the charge on the plate and the current in the inductor are sinusoidal functions of time with an angular frequency $\omega_1 = (LC_1)^{-1/2}$ and that the total energy of the system w_{s1} at any instant is the sum of electric energy stored in the capacitor and the magnetic energy stored in the inductor, which is a constant, i.e., $w_{s1}=w_{e1}=q_1^2/(2c_1)$, if both the ohmic loss and radiation loss are absent. Since the charge on the upper plate, is a function of time, the attraction force between the plates is also a function of time. If the system is pumped by suddenly pulling the plates further apart at the maximum attraction force (or the maximum charge) such that the original capacitance C_1 is changed to C_2, the work done by the pulling force will be added instantaneously to be electric energy stored in the capacitor. The total energy of the system now becomes w_{s_2}, where $w_{s_2} > w_{s_1}$. The charge on the capacitor at that instant will still be the same as before, since the inductor prevents the current, and hence the charge on the plate, from a sudden change. The oscillation frequency now becomes $\omega_2=(LC_2)^{-1/2}$ ($\omega_2 > \omega_1$), and the total energy of the system is increased by an amount equal to the work done by the pumping source. Thus when the total energy of the system Ws_2 is entirely shifted to the inductor at a later time, the current in the inductor is a maximum and the charge on the capacitor plate is zero. Furthermore, the present value of the maximum current will be larger than the maximum value of the current in the inductor if the pumping were not taking place. Since the total amount of charge has not been changed by the sudden pulling, the increment of the maximum current in the inductor can be considered due to the faster rate of oscillation, namely $\omega_2 > \omega_1$. As a final step to understand the insight of the charge generation mechanism, let the capacitance be pumped back from C_2 to C_1 again at the instant when there is no charge on the plate. Obviously, no work is required for doing this since the attraction force between the plates is zero at that instant. When the total energy of the system Ws_2 is entirely shifted to the capacitor at a later time, it is seen that now the system energy expressed as $Ws_2 = q_2^2 / (2C_1)$ is larger than $Ws_1 = q_1^2 / (2C_1)$, hence $q_2 > q_1$. This means that additional charges have been generated at the beginning of the last half pump.ng cycle whenever the capacitance C_2 is pumped back to C_1.

The above discussion is summarized as follows:

1. A pumping cycle is defined as the capacitan.e is being changed from C_1 to C_2 ($C_2 > C_1$) and changed back from C_2 to C_1 again.
2. At the beginning of the cycle, namely C_1 to C_2, additional energy has been added to the system. However, the total amount of charge is unchanged.
3. At the beginning of the last half pumping cycle, namely C_2 back to C_1, additional charges have been created.

Based upon the above discussions, the following oscillation criteria are readily obtained.

Oscillation Criteria:

1. The charge on the capacitor plate is a continuous function of time.
2. The functional form of the charge on the plate consists of a sequence of connected segments of sinusoidal functions of time, which have angular frequencies ω_1 and ω_2 alternatively.
3. The first derivative of the charge on the plate with respect to time, namely the oscillation current, in general is a continuous function of time also, since the current in the inductor can not be changed abruptly.

Mathematical Analysis

To facilitate the analysis, define

$\omega_p = 2\pi/T_p$, where T_p =pumping period=time interval between the beginning of two successive pumping cycle.

ω_p=pumping angular frequency.

$\omega_1 = 2\pi/T_1$ $= (LC_2)^{-1\ 2}$ where T_1 =oscillation period of the tank circuit with capacitance C_1 and inductance L.

ω_1=oscillation angular frequency of the tank circuit.

2 $= 2\pi/T_2$ $= (LC_2)^{-1\ 2}$ where T_p =oscillation period of the tank circuit with capaci-

tance C_2 and inductance L.

ω_2 =oscillation angular frequency of the tank circuit.

and $\omega_q = 2\pi / T_q$ where T_q =oscillation period of the charge on the capacitor plate when the pumping is applied periodically.

ω_q =oscillation angular frequency of the charge on the capacitor plate.

Before the analysis, it should be bear in mind that the pumping period R_p can be completely arbitrary. However, in order to see the physical insight directly, the pumping cycle may be assumed either to start at the maximum charge (i.e., to pull the plates apart further at the maximum force) or to end at the zero charge (i.e., to restore the plates to original positions at zero force). Either assumption will lead to the conclusion that the pumping frequency ω_p is twice the charge oscillation frequency ω_q , or $T_p = T_q / 2$ is the relation between the periods. Thus, if the charge oscillates with the frequency ω_2 for a duration K ($K < T_p$), then it oscillates with the frequency ω_1 for a duration ($T_p - K$).

The assumption $T_p = T_q / 2$ and the definition of K will be adopted for the following analysis.

Consider an ideal tank circuit shown in Fig. 1, where the capacitor is pulsively modulated, see Fig. 2. The capacitance is expressed as C(t) and the inductance L is kept constant. The charge q on the capacitor plate is a function of time, oscillating with $\omega(t) = 1/\sqrt{LC(t)}$. The equation of motion for the intervals where the capacitance has constant values, e.g., $0 < t < T_p - K$, $T_p - K < t < T_p$, etc., (or where the pumping force is absent), may be derived from lagrange's equation as

$$\frac{d^2 q(t)}{dt^2} + \omega^2(t)\, q(t) = \pounds(t) q(t) = 0 \qquad (1)$$

where $\pounds(t) = \frac{d^2}{dt^2} + \omega^2(t)$ and $\pounds(t + T_p) = \pounds(t)$ due to periodicity. Equation (1) is a linear, homogeneous second-order differential equation with periodic coefficients of the Mathieu-Hill's type [2] . The exact solution of equation (1) is to be obtained by matrix method on the following, and the principle of parametric amplification will be classified.

Assuming that q(t) is a linear combination of cos t and sin t, then q(t) and q(t) may be expressed in the following forms:

$$q_1(t) = A\cos \omega_1 t + B\sin \omega_1 t \qquad (2)$$

$$q_1(t) = \omega_1 [-A\sin \omega_1 t + B\cos \omega_1 t] \quad \text{for } 0 \le t \le T_p - k$$

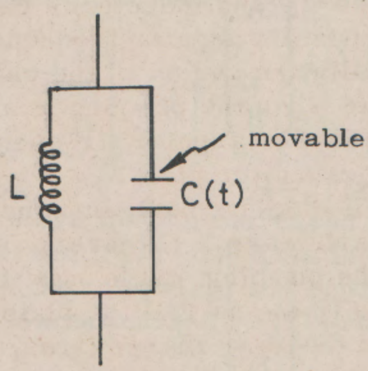

Fig. 1   An ideal tank circuit.

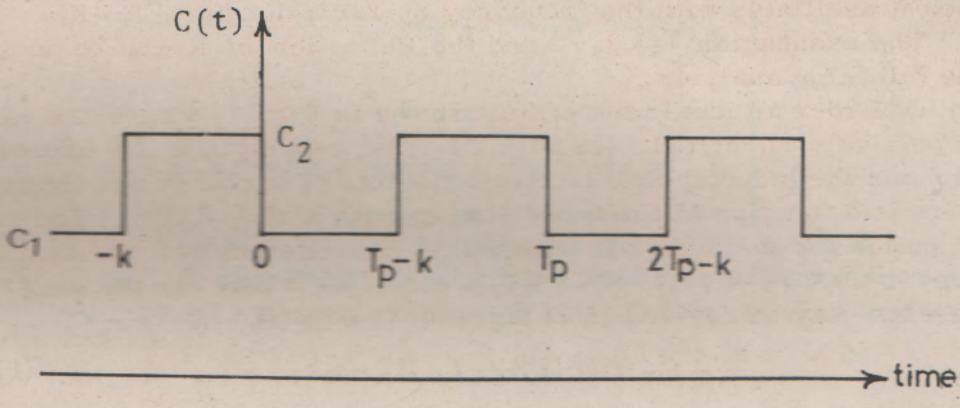

Fig. 2   The time variation of the capacitance.

$$q_2(t) = C\cos\omega_2 t + D\sin\omega_2 t$$

$$q_2(t) = \omega_2[-C\sin\omega_2 t + D\cos\omega_2 t] \quad \text{for } T_p - k \leq t \leq T_p$$

$$q_3(t) = E\cos\omega_1 t + F\sin\omega_1 t$$

$$q_3(t) = \omega_1[-E\sin\omega_1 t + F\cos\omega_1 t] \quad \text{for } T_p \leq t \leq 2T_p - k$$

Where A, B, C, D, E, and F are arbitrary constants. In matrix form, the above equations become

$$Q_1(t)=\begin{bmatrix} q_1(t) \\ \\ \dot{q}_1(t) \end{bmatrix}=U_1(t)\begin{bmatrix} A \\ \\ B \end{bmatrix}$$

$$\text{for} \quad 0 \le t \le T_p - k \tag{5}$$

$$Q_2(t)=\begin{bmatrix} q_2(t) \\ \\ \dot{q}_2(t) \end{bmatrix}=U_2(t)\begin{bmatrix} C \\ \\ D \end{bmatrix} \quad \text{for} \quad T_p - k \le t \le T_p$$

$$Q_3(t)=\begin{bmatrix} q_3(t) \\ \\ \dot{q}_3(t) \end{bmatrix}=U_1(t)\begin{bmatrix} E \\ \\ F \end{bmatrix} \quad \text{for} \quad T_p \le t \le 2T_p - k$$

$$J_i(t)=\begin{bmatrix} \cos\omega_i t & \sin\omega_i t \\ \\ -\omega_i \sin\omega_i t & \omega_i \cos\omega_i t \end{bmatrix} \quad i=1,2.$$

It is seen from the oscillation criteria 2 and 3 that Q_1 and Q_2 are continuous at $t=T_p-k$, and Q_2 and Q_3 are continuous at $t=T_p$, or

$$Q_1(T_p-k)=Q_2(T_p-k) \tag{9}$$

$$Q_2(T_p)=Q_3(T_p) \tag{10}$$

Substituting equations (5), (6) (7) into (9) (10), and eliminating Q_2, yield

$$\begin{bmatrix} E \\ F \end{bmatrix}=U_1^{-1}(T_p)U_2(T_p)U_2^{-1}(T_p-k)U_1(T_p-k)\begin{bmatrix} A \\ B \end{bmatrix}$$

Let

$$\begin{bmatrix} A \\ B \end{bmatrix}=U_1^{-1}(T_p-k)\begin{bmatrix} q_1 \\ \dot{q}_1 \end{bmatrix}, \quad \text{where} \quad U_1U_1^{-1}=I$$

then

$$Q_3(t)=U_1(t)\begin{bmatrix} E \\ F \end{bmatrix}=U_1(t)U_1^{-1}(T_p)U_2(T_p)U_2^{-1}(T_p-k)\begin{bmatrix} q_1 \\ \dot{q}_1 \end{bmatrix},$$

Due to periodicity, in general

$$\mathfrak{L}(t+T_p)=\mathfrak{L}(t)$$

and

$$\mathfrak{L}(t)q(t)=\mathfrak{L}(t+T_p)\,q(t+T_p)=\mathfrak{L}(t)\,q(t+T_p)=0$$

thus,

$$q(t+T_p)=\lambda q(t) \qquad \dot{q}(t+T_p)=\lambda\dot{q}(t)$$

Rewrite the above equation in matrix form and let $t=T_p-k$

$$\begin{bmatrix} q_3(2T_p-k) \\ \dot{q}_3(2T_p-k) \end{bmatrix} = \lambda \begin{bmatrix} q_1(T_p-k) \\ \dot{q}_1(T_p-k) \end{bmatrix}$$

It is obtained at this stage that an increased level of oscillation, namely $Q_3 > Q_1$, will be observed if $|\lambda|>1$. This is the essence of the parametric amplification. In the following, the condition where $|\lambda|>1$ will be carefully examined. Substituting Eq. (11) into (12) at $t=2T_p-k$

$$[U_1(2T_p-k)U_1^{-1}(T_p)U_2(T_p)U_2^{-1}(T_p-k)-\lambda I]\begin{bmatrix} q_1 \\ \dot{q}_1 \end{bmatrix}=0$$

Where I is the identity matrix.
The condition of non-trivial solution for q_1 and $\dot{q}_1$ is that the determinant of their coefficient matrix must vanish, which leads to

$$\lambda^2-2b\lambda+1=0 \qquad (13)$$

where

$$b=+\cos\omega_1(T_p-k)\cos\omega_2 k-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)\sin\omega_1(T_p-k)\sin\omega_2 k$$

To solve for λ from eq. (13), the pumping period T_p must be determined, since ω_1 and ω_2 are known and k (the duration of the system being pumped to ω_2 in one complete pumping cycle) is at our choice. Before any attempt is made to determine T_p, several points are to be classified first: (1) To simplify the discussion, the plates are assumed being separated farther always at the maximum attractional force (or they are restored to original positions always at zero force). Either assumption will lead to the same expression of T_p, since both assumptions ensure that the system can only take, never give away, energy from the pumping source. (2) In order to pump as much energy as possible into a system, the plates are pulled farther immediately at the maximum attractional force available (or they are restored immediately at the first zero attractional force available). Hence there are no skips of maxima (or minima) between two pulls. If parametric amplification is achieved according to the last two assumptions, it may be said to be in the "high mood", see Eig. 3 and 4. Otherwise, it is in the "low mood", see Fig. 5, 6 and 7.

If the pumping cycle always starts and ends at the same phase relative to the attractional force between the plates, it is called regular pumping, otherwise, it is called random pumping. It is immediately clear that for random pumping both h and k are arbitrary, see Fig. 8. Thus the pumping period $T_p=h+k$ is also arbitrary,

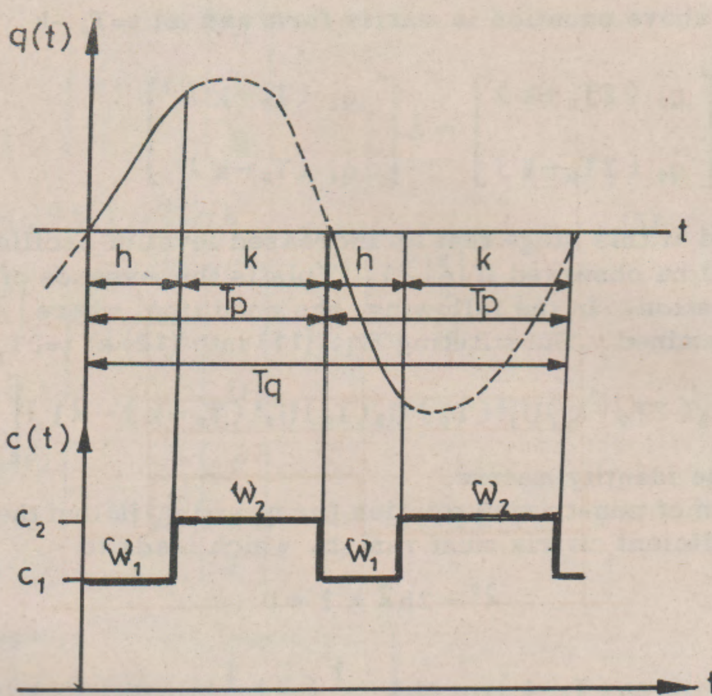

Fig. 3 High-mood - plates restared always at zero force (regular pumping).

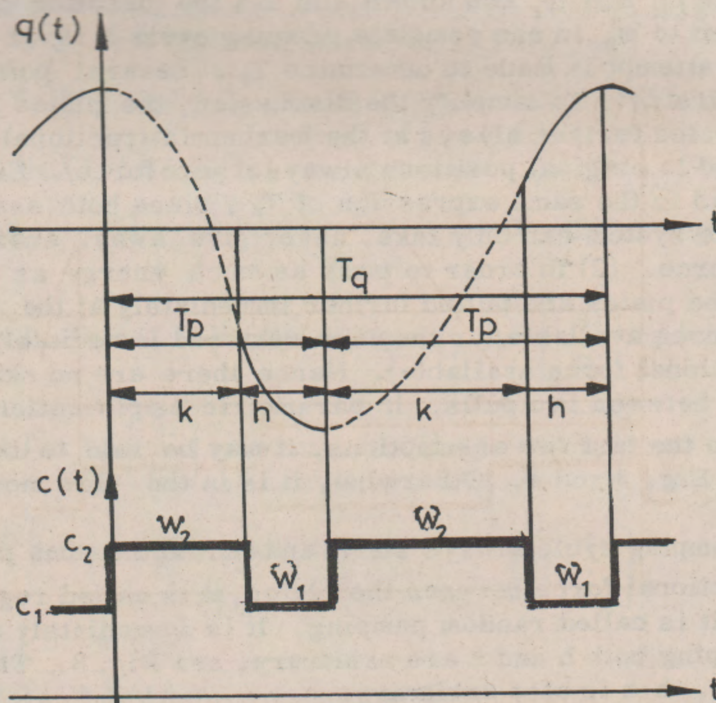

Fig. 4 High-mood - plates separated farther always at maximum force (regular pumping).

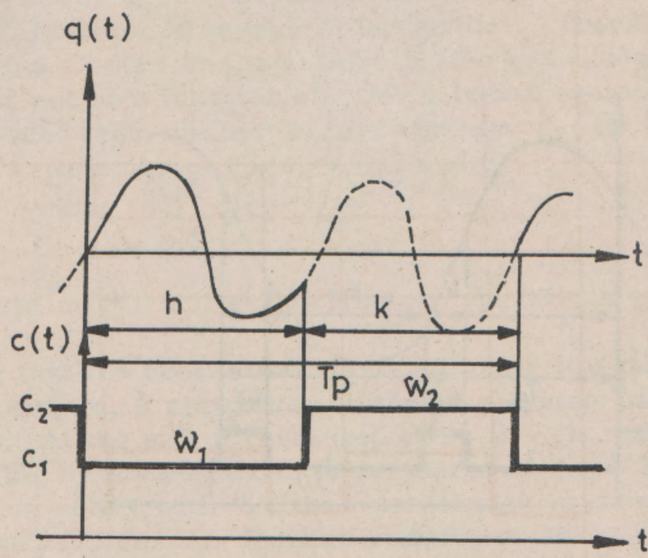

Fig. 5 Low-mood - plates always restored at zero force. (regular pumping).

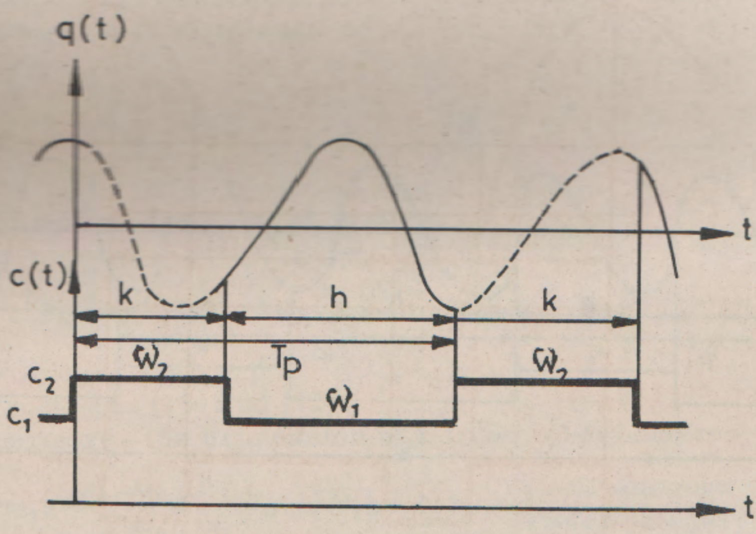

Fig. 6 Low-mood - plates separated farther always at maximum force. (regular pumping).

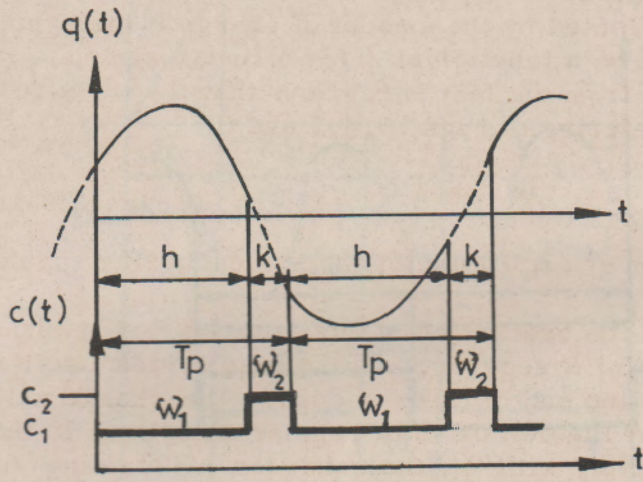

Fig. 7 Low-mood - (regular pumping).

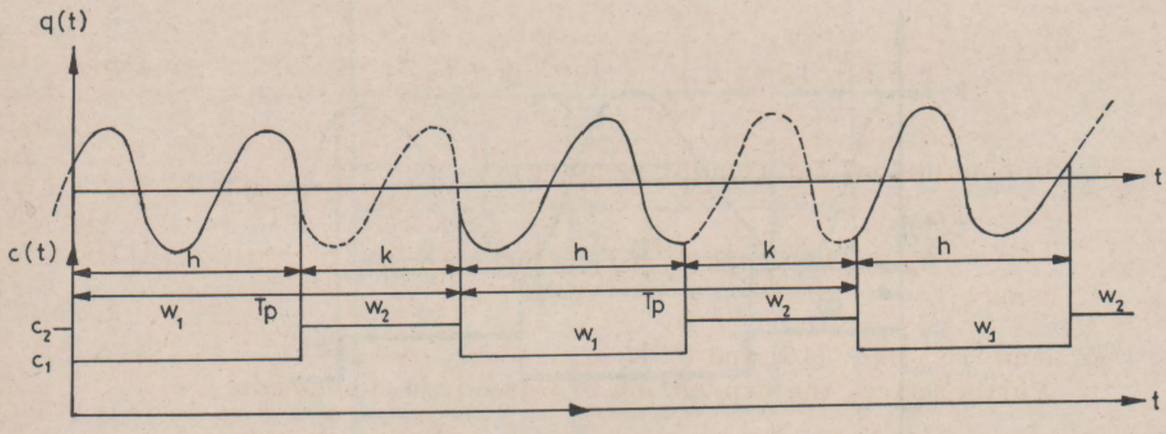

Fig. 8 Low-mood - The most general case (random pumping).

which is excluded from the following consideration. For regular pumping, the pumping period must be a function of ω_1, ω_2 and k, since h can be expressed in terms of ω_1 , ω_2 and k. Notice that since the oscillation frequency of a linear system depends solely on the values of L and C, it is independent of the level of charge in the circuit. Therefore, the pumping period is also unaffected by the amount of charge in the system. In other words, T_p can not be a function of λ for a linear system.

It is concluded from the last paragraph that T_p is a function of ω_1 , ω_2 and k only. Refering to Figs. 3,4,5 and 6.

$$T_p = k + h \tag{15}$$

where h is duration of oscillation of the system having angular frequency ω_1.

Supposing that the oscillation starts at q(t)=0 (or maximum) with frequency ω_2 lasting for k seconds, and then the system oscillates with ω_1 for h seconds. At the end of (k+h) seconds, the charge will be zero (or maximum) again, by the definition of regular pumping. If the oscillation were carried on solely with ω_2, the condition of starting and ending on the same phase requires the oscillation duration being $\frac{m}{2}T_2$ (m=1, 2, . . .), (where m may be interpreatated as the average number of times of q(t) passing through the zero value). Since the actual oscillation with ω_2 only lasts for k seconds, hence there are $\omega_2(\frac{m}{2}T_2-k)$radians to be succeeded by oscillation with the frequency ω_1 in order to be able to return to the original starting phase, corresponding to zero (or maximum) charge. Recalling that every phase change of 2π radians requires T_1 seconds when the system oscillates with frequency ω_1, thus $\omega_2(\frac{m}{2}T_2-k)$ radians requires $\omega_2(\frac{m}{2}T_2-k)T_1/2\pi$ seconds to complete the change.

Therefore, h is expressed as

$$h = (\frac{m}{2}T_2 - k)\frac{T_1}{T_2} \qquad \text{for} \quad m = 1, 2, \cdots\cdots \tag{16}$$

In conclusion, for regular pumping,

$$T_p = (1 - \frac{T_1}{T_2})k + \frac{m}{2}T_1 \qquad \text{for} \quad m = 1, 2, \cdots\cdots \tag{17}$$

as seen from eqs. (15) and (16).

Furthermore, the expression of b in eq. (14) becomes

$$b = \mp\cos^2\omega_2 k \mp \frac{1}{2}(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})\sin_2^2\omega_2 k \quad \text{for m=} \begin{cases} \text{odd integers} \\ \text{even integers} \end{cases} \tag{18}$$

where the minus sign and the plus sign correspond to m=odd integers and m=even integers, respectively. Adopting the following definitions.

$$\frac{\omega_2}{\omega_1} = \left(\frac{C_1}{C_2}\right)^{1\ 2} = y \qquad 0 < y \tag{19}$$

and

$$k = x\left(\frac{T_2}{2}\right) \qquad 0 < x \tag{20}$$

eq. (18) becomes,

$$b = \mp\cos^2 x \mp \left(\frac{y^2+1}{2y}\right)\sin^2 x \quad \text{for } m = \begin{cases} \text{odd integers} \\ \text{even integers} \end{cases} \tag{21}$$

As the final step of the analysis, λ is solved from eq. (13),

$$\lambda_{1,2} = +b \pm \sqrt{b^2 - 1} \tag{22}$$

and λ_1, λ_2 v.s. b are plotted in Fig. 9.

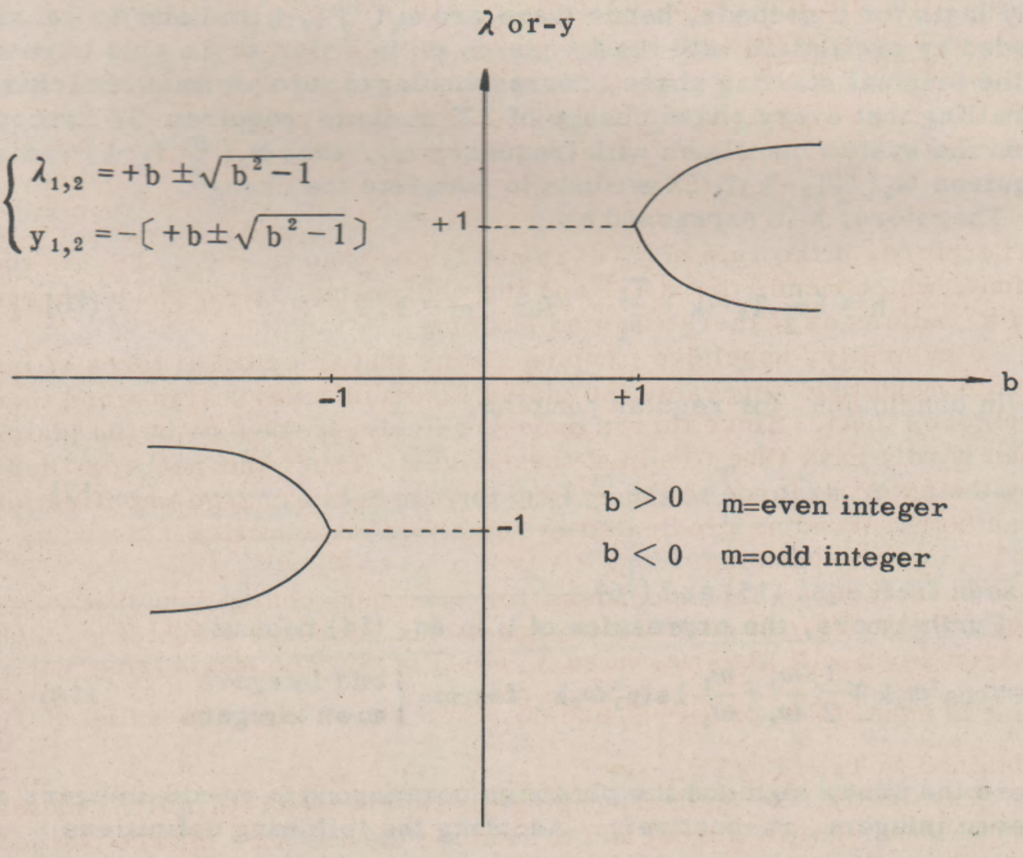

Fig. 9 The plotting of λ v.s. b. and-y v.s. b.

Discussions

For simplicity, only the high-mood case is considered (m=1), which means that $1>x>0$ and the upper (minus) signs in eqs. (18) and (21) will be employed.

Case 1 $\omega_1=\omega_2$, or y=1 no pumping

It is seen from eqs. (17) and (18) that $T_p=\frac{1}{2}T_1$ and b=-1, which leads to $\lambda=-1$, see Fig. 9. The charge changes only by a phase ($=\pi$ radians). The mathematical analysis certainly holds for this case.

Case 2 k= 0, or x= 0. no pumping

From eqs. (17) and (18), $T_p=\frac{1}{2}T_1$ and b=1. Therefore, $\lambda=-1$, obtained from Fig. 9, which again is physical permissible.

Case 3 $k=\frac{1}{2}T_2$ or $x=\frac{1}{2}$ no pumping

From eqs. (17) and (18), $T_p=\frac{1}{2}T_2$ and b=-1. Therefore, $\lambda=-1$ which is anticipated.

Case 4 Impulsive pumping $k\to 0$ and $\omega_i\to 0$, ($x\to 0$, $y\to 0$)

From eq. (21)

$$\lim_{\substack{x\to 0\\ y\to 0}} b = \lim_{\substack{x\to 0\\ y\to 0}} \left[-\cos^2\pi x - \frac{y^2+1}{2y}\sin^2\pi x\right]$$

$$= 1-\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{1}{2y}\left(\pi x-\frac{(\pi x)^3}{3!}+\cdots\right)^2\right] \tag{23}$$

Therefore, if the rate of $y\to 0$ is not faster than $x^2\to 0$, b=-1 is the limit, which means that $\lambda=-1$ and the charge has only a phase change of π radians as if there were no pumping.

Physically, impulsive pumping means that an external force of infinite magnitude separates the plates for an infinitesimal time and then restores them. Since this is done so quickly, the change on the plates can hardly have time to adjest themselves. Thus, the net work done by the external force to the system is zero and no charge amplification can be observed as predicated by the above mathematics.

Case 5 Optimal pumping

To determine the conditions for maximum charge amplification, eq. (19) is differentiated with respect to x, since the value of k which maximizer b will also maximize λ, see Fig.9. The maximizing value of x is equal to $\frac{1}{2}$. or $k=\frac{1}{4}T_2$ and eq. (17) shows that the value of T_p obtained is $T_p=(T_1+T_2)/4$.

Physically, the optimal pumping case corresponds to pull the plates at maximum charge (force) and to restore them at zero charge (force). This is wellknow, and the idea has been used repeatedly by authors decribing the phenomenon of parametric amplification.

Case 6 General case

Solving for x from eq. (21), yield

$$x = \pm \frac{1}{\pi} \sin^{-1} \sqrt{\frac{-b-1}{A-1}} \qquad (24)$$

where

$$A = \frac{(y^2+1)}{2y} \qquad (25)$$

and the plus sign is to be adopted since $x>0$ due to the definition of x in eq.(20). In order to have real solution for x,

$$\frac{-b-1}{A-1} \leq 1 \qquad (26)$$

or

$$y^2 + 2by + 1 \geq 0 \qquad (27)$$

The solution of eq. (27) is

$$y > y_1, \quad y < y_2 \quad \text{for} \ (y_1 > y_2)$$

where

$$y_{1,2} = -b \mp \sqrt{b^2-1} \qquad (28)$$

The permissible values of y for real x are plotted in Fig. 9 and the corresponding solution region of x can be seen from eq. (24) to be $0<x<1$ for $m=1$. [and $(m-1)<x<m$ for $m<1$]. Since $\sin\pi x=\sin\pi(1-x)$ for $0<x<1$ it is seen from eq. (18) that $k=\pi x$ and $k=\pi(1-x)$ will yield the same value of b and hence the same value of λ, which implies that the plotting of v.s. b for m=1 is symmetrical with respect to $x=\frac{1}{2}$. This is physically permissible, since the attractional force between the plates while restoring them with $k=\pi x$ is the same as that with $k=\pi(1-x)$.

Applications of the above analysis

1. It is possible to predict the charge gain λ by knowing the values of $x=2k/T_1$ and $y=(C_1/C_2)^{1/2}$; simply by substiting the values of x and y into eq. (21) obtaining the value of b and the charge gain λ can be read directly from Fig. 9.
   This is illustrated in example 1.
   Example 1:
   Substituting $x=0.25$, $y=1.01$ into eq. (21) obtaining $b=-1.000025$, it is obtained that $\lambda=-1.007$ from eq. (22) or longitudinal axis of Fig. 9.

Note that if $x=0.5$, the obtained value is $\lambda=-1.01=\lambda_{max}$, which is just the negative value of y, namely $\lambda_{max}=-y$ for $x=0.5$.

2. It is also possible to determine the set of values of x and y to attain the desired value of charge gain λ. This is done by determining the corresponding value of b for the desired value of λ from Fig. 9, then choosing a values of x between 0 and 1, and calculating the corres-

ponding value of y from eq. (21) for the known values of b and x. Repeat this process, one can obtain as many pairs of (x,y) as one wishes to achieve the given charge gain
This is illustrated in example 2.
Example 2:

Let $\lambda=-1.0140$, then $b=-1.0001$ from fig. 9. If we chose various value of x from 0 to 1 as shown in the table. Substituting b and x into eq. (21), corresponding value of y obtained accordingly are shown to the same table.

x	0.30	0.35	0.40	0.45	0.50
y	1.0176	1.0166	1.0150	1.0144	1.0140

3. When the value of pumping ratio y is known, the maximum charge gain is determined accordingly and has a value $\lambda_{max}=-y$ corresponding to $x=0.5$. This concluded from equations. (22), (24) and (28), see Example 1.

Comments on Equation (14)

It is seen that eq. (14) has the same functional form as the characteristic equation obtained from the Kronig-penney potential in quantum mechanics [3] . The same functional form also appeared in Meissner's work in the analysis of the vibrations of the driving rods of locomotives [4] . In the analysis of periodic optical resonant reflectors, Mahlein and Schollmeier [5] have obtained an expression identical to eq. (14), using a matrix method developed by Abeles, and Born and Wolf.

Energy Considerations:

1. Let the maximum energy stored in the capacitor be

$$E_I = \frac{1}{2}\frac{q_I^2}{c}$$

and the maximum energy stored in the capacitor after n complete cycles of pumping is

$$E_f = \frac{1}{2}\frac{q_f^2}{c} \qquad \text{where } q_f = \lambda^n q_I$$

Therefore, the energy absorbed by the system in n pumpings is

$$E_f - E_I = \frac{q_I^2}{2c}\left(\lambda^{2n} - 1\right)$$

and the energy is absorbed at the rate,

$$\text{power absorbed} = \frac{E_f - E_I}{nT_p}$$

The energy gain in db is

$$\text{gain (db)} = 10 \log_{10} \frac{E_f}{E_I} = 20 \text{ nlog}$$

2. The maximum work done to a capacitor is to pull of the plates to infinity, and by doing so it can be verified that the work done by the external force is equal to the value of original energy stored in the capacitor. Hence the total energy stored after the plate being brought to infinity is twice its original stored energy. This argument leads to the conclusion that the maximum possible charge gain in one pumping cycle is at most $|\lambda| = \sqrt{2}$, since the charge squared is proportional to the energy. This result has placed an upper bound on the value of λ. This viewpoint has not been revealed by the above analysis, which is due to the lack of such an energy constraint at the beginning of the analysis. This short coming may be overcome without too much efforts.

Conclusion

1. To the best knowledge of the authors, the charge generation mechanism, the oscillation criteria and the classification of amplification have not been seen elsewhere.
2. It has been shown previously that the charge gain can be obtained by knowing the value of pumping hight, specified by the ratio y, and the value of pumping duration, denoted by x. Also, it is shown that in order to obtain a certain charge gain the required set of values of pumping height and pumping duration at a certain frequency can be obtained, as illustrated by the examples.
3. It was concluded previously that the maximum charge gain occurs at $x=\pi-\frac{1}{2}$ and has a magnitude equal to the value of y with a phase charge of m.
4. Sudden change of the capacitance levels requires an infinite power source, since work must be done in an infinitesimal time whenever the capacitance is changed. This is physically impossible, but this assumption has been adopted because it simplifies the investigation greatly. To analyze the more general cases such as linear pumping or sinusoidal pumping, namely the capacitance level changing with time linearly or sinusoidally, the techniques and basic ideas presented in this paper would be valuable.
5. Since parametric amplification may be achieved with sinusoidal pumping, it is sometimes concluded from intuition that it may also be obtained with impulsive pumping, since an impulse train contains a

sequence of sinusoids. Nevertheless, this is definitely not the case as the paper has shown from both physical and mathematical viewpoints.

6. Although only the high-mood pumping has been considered in the section of discussion. The more general cases such as random pumping and various forms of low-mood regular pumping may be investigated in a similar way.
7. Although only the charge as a function of time has been considered, other quantities such as voltage or energy may be simply by dividing the charge curve by the capacitance occure or by dividing the squared charge curve by twice the capacitance curve respectively.
8. Although the considerations of this work is limited to an ideal tank circuit, the results may be applied to the general class of problems pertaining to wave propagation in time-variant medium and to light reflection in periodic optical resonators.
9. This paper only considered lossless case. If losses are allowed, such as ohmic loss or radiation loss, the Lagrange's equation for nonconservative system must be used.

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