SCIENCE BULLETIN
CHIAO TUNG UNIVERSITY
Vol. 5, No. 1, APRIL 1971

A Study on the Range and Accuracy of Long-Range Ballistic Missilcs

by

Hsu Chen
College of Engineering
National Chiao Tung University

I. Introduction

The flight of a ballistic missile is idealized by the assumptions that the speed is acquired impulsively at sea level and that drag is absent. With practical long-range rockets the distance traversed under power is usually 5-10% of the total range, and the length of re-entry path in which drag is significant is even smaller. Consequently the impulsive assumption gives a good first approximation both for the range and for the impact errors of long-range rockets.

II. Maximum Ballistic Height

We first find the maximum height h which is reached by a missile leaving the surface of the Earth vertically with speed u. Equating the initial kinetic energy to the integral of force times distance, we have

$$(1)\quad \frac{1}{2} mu^2 = -\int_R^{R+h} mG\,(R/r)^2 dr$$

where m is the mass of the missile, G is the accerlation of gravity at sea level, R is the Earth's radius and $-G(R/r)^2$ is the accerlation of gravity at radius r. Equating the integral and solving for h yield.

$$(2)\quad \frac{h}{R} = \frac{u^2/2GR}{1-(u^2/2GR)}$$

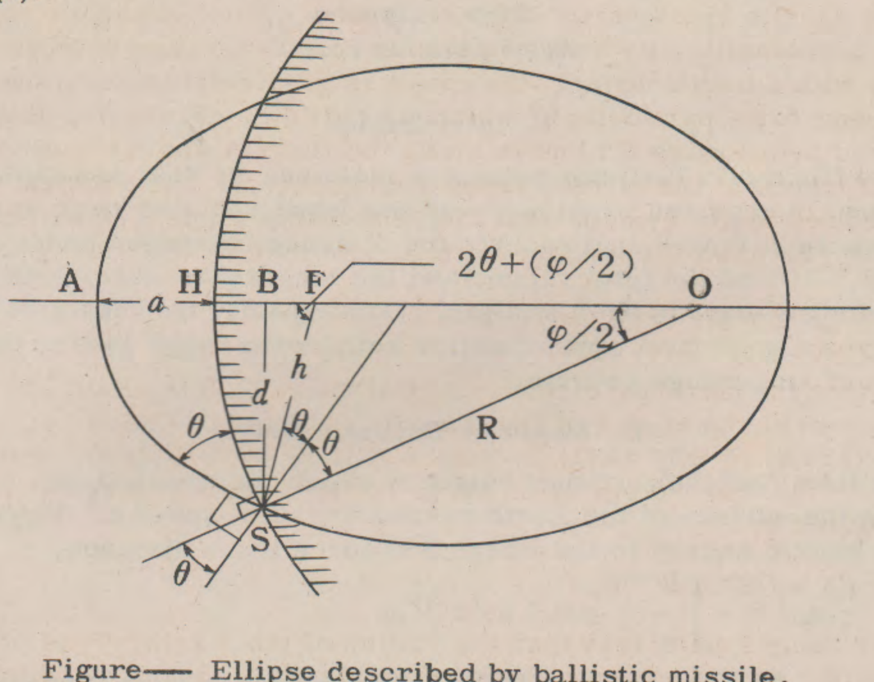
Now consider a satellite in a circular orbit at sea level with speed V. The centripetal acceration is just that of gravity. Hence $GR=V^2$. It is also casily shown that escape speed is $\sqrt{2}V$. Substitution for GR in (2) gives for the maximum vertical ballistic high h

$$(3)\quad \frac{h}{R} = \frac{u^2/2V^2}{1-(u^2/2V^2)}$$

Note that missile moving vertically at sea level with speed u equal to satellite speed V will reach a height equal to one Earth radius.

III. The Elliptic Orbit

The vertical trajectory we have just considered is a degencrate elliptic orbit with one focus at the center of the Earth and the other at the apex of the flight. Thus the major axis of the degenerate ellipse is R+h. It is shown in many standard references[1,2] that that major axis of an elliptic orbit depends only on the total energy of the missile. Consequently the major axis of the ellipse described by a missile, having speed u at sea level, depends only on u and not at all on the elevation angle θ from the horizontal; in fact the major axis is just R+h(u), where h(u) is given by Eq. (3).



Figure— Ellipse described by ballistic missile.

In Figure an elliptic orbit is depicted with one focus at F. At the starting point S the ellipse makes angle θ with the surface of the Earth. Note that by a well-known property of ellipses OS+SF=major axis=R+h. But OS=R so that SF=h. A second property of ellipses requires the bisector of $\widehat{OSF}$ to make a right angle with the tangent to the ellipse at S. It follows immediately that the angle $\widehat{OSF}=2\theta$. Let the range semi-angle $\widehat{BOS}$ be termed. $\varphi/2$.

We can now write two equations for the length d of the semichord SB. Equating these two equations for d yields

(4) $\sin 2\theta + (\varphi/2) = (R/h)\sin(\varphi/2)$

and dividing through by $\cos(\varphi/2)$, we find

$$\text{(5)} \quad \tan\frac{\varphi}{2} = \frac{\sin 2\theta}{(R/h) - \cos 2\theta}$$

Eliminating (R/h) with Eq.(3), we obtain finally

$$\text{(6)} \quad \tan(\varphi/2) = \frac{\sin\theta\cos\theta}{(V/u)^2 - \cos^2\theta}$$

which gives the range angle φ as a function of u and θ. Some practical applications of Eq.(6) are made by Allen and Eggers.[3]

IV. The Trajectory of Maximum Range

If a missile rises vertically with speed u, its range is zero; if it moves horizontally, its range is also zero. For values of θ between $\pi/2$ and 0, with u held constant, the range is greater than zero and reaches a maximum for a particular θ which we call θ_m. Following the figure θ is varied by allowing F to move along the line OA and S to move along the circumference of the circle, remembering that SF has constant length h. It is geometrically evident that the semi-range angle $\varphi/2$ will be a maximum, $\varphi_m/2$, when F coincides with B, and then

$$\text{(7)} \quad \varphi_m/2 = \sin^{-1}(h/R), \quad h \leq R$$

so that the maximum circumferential range $\varphi_m R$ is $2R\sin^{-1}(h/R)$. Now the sum of the angles of the triangle OSF is π. When $\varphi = \varphi_m$, the angle $\widehat{SFO}$ is $\pi/2$, $\widehat{SOF}$ is $\varphi_m/2$, and so $\widehat{OSF}$ is $(\pi/2)-(\varphi_m/2)$. But $\widehat{OSF}$ is also $2\theta_m$. So the elevation angle θ_m for maximum range with speed u is given by

$$\text{(8)} \quad \theta_m = (\pi/4) - (\varphi_m/4)$$

For ranges much less than the radius of the Earth, φ is much less than π, so the optimum elevation angle θ_m is just $\pi/4$, which is a familiar result of "flat-earth" dynmaics. When h is much less than R, the maximum range $\varphi_m R$ as given by Eq.(7) becomes very much 2h, a second familiar result. We can substitute Eq.(3) into Eq.(7) to eliminate h/R; this connects maximum range with sea level speed.

$$\text{(9)} \quad \varphi_m/2 = \sin^{-1}\frac{u^2/2V^2}{1-(u^2/2V^2)}$$

Eq.(8) has shown that at high speeds the maximum range trajectories become very flat, if u is held constant. To a first approximation rockets can be characterized by a constant value of u, independent of the burnout elevation angle, as has been verified by Culler and Fried.[4]

In fact the burnout speed increases slightly with decreasing θ, so that the maximum range trajectories are even flatter than indicated by Eq. (8).

Eqs. (7) and (9) have no solution for $h>R$, i.e., for $u>V$. There is no maximum range for speeds greater than satellite speed.

V. Apex Hight of the Trajectory

The apex height a can be found in terms of h and φ from the relation

$$(10)\quad \begin{aligned} 2a &= (OA + AF)-OH-HB-BF \\ &= (R+h) - R - R[1-\cos(\varphi/2)] - h[\cos(2\theta+(\varphi/2)] \\ &= h\{1-\cos\ 2\theta+(\varphi/2)]\} - R[1-\cos(\varphi/2)] \end{aligned}$$

or, on using Eq. (4) to elliminate

$$(11)\quad 2a = h - R[(h/R)^2 - \sin^2(\varphi/2)]^{\frac{1}{2}} - R[1-\cos(\varphi/2)]$$

When θ is chosen to give maximum range, $\theta=\theta_m$ in Eq. (8) $\cos[2\theta_m+(\varphi_m/2)] = \cos(\pi/2) = 0$ and the apex height is given simply by substituting from Eq. (7)

$$(12)\quad a_m = (R/2)[\sin(\varphi_m/2) + \cos(\varphi_m/2) - 1]$$

Thus the apexes a_m of the max. range trajectories are never very great, at most $0.207R$ when the range φ_m is $\pi/2$.

VI. Range Errors

From Eq. (6) we have

$$(13)\quad \varphi(u,\theta) = 2\tan^{-1}\frac{\sin\theta\cos\theta}{(V/u)^2-\cos^2\theta}$$

The range error $\Delta\varphi$ due to errors Δu and $\Delta\theta$ is obtained most easily by expanding φ in Taylor scries

$$(14)\quad \Delta\varphi = \frac{\partial\varphi}{\partial u}\Delta u + \frac{\partial\varphi}{\partial\theta}\Delta\theta + \frac{1}{2!}\left[\frac{\partial^2\varphi}{\partial u^2}(\Delta u)^2 + 2\frac{\partial^2\varphi}{\partial u\partial\theta}\Delta\theta\Delta u + \frac{\partial^2\varphi}{\partial\theta^2}(\Delta\theta)^2\right] + \cdots\cdots$$

The values of Δu and $\Delta\theta$ are determined by the accuracy of the guiding equipment[5], while the derivatives are purely a function of the chosen trajectory, now

$$(15)\quad \frac{\partial\varphi}{\partial\theta} = \frac{2}{1+\tan^2(\varphi/2)}\;\frac{[(V/u)^2-\cos^2\theta]\cos 2\theta - 2\sin^2\theta\cos^2\theta}{[(V/u)^2-\cos^2\theta]^2}$$

We eliminate V/u with Eq. (13) and obtain $\frac{\partial\varphi}{\partial\theta}$ as a function of φ and θ.

$$(16)\ \frac{\partial\varphi}{\partial\theta} = 2\sin\varphi \left[\frac{1}{\tan 2\theta} - \tan\frac{\varphi}{2} \right]$$

This elimination of (V/u) simplifies subsequent differentiations. Similar operations yield

$$(17)\ \frac{\partial\varphi}{\partial u} = \frac{8\sin^2(\varphi/2)}{\sin 2\theta} \left(\frac{V}{u}\right)^2 \frac{1}{u}$$

High derivatives are of complicated appearance. They are interesting only when lower derivatives are zero. This never occurs to $\partial\varphi/\partial u$, but it does occur to $\partial\varphi/\partial\theta$ when the departure angle is selected to give max. range, i.e., when $\theta = \theta_m$. For this value of θ, the higher derivatives are simpler, and we have

$$(18)\ \left(\frac{\partial\varphi}{\partial u}\right)_m = \frac{8\sin^2(\varphi_m/2)}{\cos(\varphi_m/2)} \left(\frac{V}{u}\right)^2 \frac{1}{u}$$

$$(19)\ \left(\frac{\partial\varphi}{\partial\theta}\right)_m = 0$$

$$(20)\ \left(\frac{\partial^2\varphi}{\partial u\,\partial\theta}\right)_m = -\frac{16\sin^3(\varphi_m/2)}{\cos^2(\varphi_m/2)} \left(\frac{V}{u}\right)^2 \frac{1}{u}$$

$$(21)\ \left(\frac{\partial^2\varphi}{\partial\theta^2}\right)_m = -8\tan(\varphi_m/2)$$

Table I ——— Ronge Errors Resulting from Errors in Speed and Elevation Angle

Range, R, statute miles	3000		5000		7000	
Elevation angle, , degrees	34.1 optimum	45	26.9 optimum	45	19.7 optimum	45
Speed, u, ft./sec.	19100	19650	22300	23900	24200	27500
$R\frac{\partial\varphi}{\partial u}$, miles/(ft./sec.)	0.445	0.380	0.830	0.542	1.41	0.609
$R\frac{\partial\varphi}{\partial\theta}$, miles/millirad.	0	-2.11	0	-5.51	0	-9.32
$R\frac{\partial^2\varphi}{\partial u\partial\theta}$, miles/(ft./sec.)-millirad.	-0.352×10^{-3}	—	-1.21×10^{-3}	—	-3.38×10^{-3}	—
$\frac{\partial^2\varphi}{\partial\theta^2}$, miles/(millirad.)	-0.0126	—	-0.0232	—	-0.0380	—

In Table I we have shown some typical values of $\partial\varphi/\partial\theta$ and of $\partial\varphi/\partial u$, computed from Eqs. (16) through (21), and it is noteworthy that $\partial\varphi/\partial u$ becomes very large at long ranges and small elevation angle θ. In fact this dependence of φ on u is inconveniently strong in terms of present-day guidance technology. By using larger elevation angle θ, one can considerably reduce $\partial\varphi/\partial u$ and hence the range error; however, greater speed is required and therefore a large propulsion system. The second derivative in Table I appear almost negligible. The values of $R(\partial\varphi/\partial\theta)$ should not be compared directly with those of $R(\partial\varphi/\partial u)$, since each has to be multiplied by the probable value of $\Delta\theta$ and Δu to estimate the resulting error.

VII. Errors due to Earth Rotation

In the preceding analysis the effect of terrestrial rotation was ignored. The range, -angle φ; was considered in a stationary system. If the vector u, θ incorporates the velocity of the launch point, the preceding results can be applied to the rotating Earth. However, the range is still computed in the fixed system. It is a simple matter to calculate the motion of the target during the time of flight and hence to convert the range on the Earth to the range in the non-rotating system.

An error in u or θ will lead to an error in the flight time t, so that the target will not be at the predicted impact point in the fixed system at the time of impact. Thus $\partial t/\partial\theta$ and $\partial t/\partial u$ must be found; when multiplied by the target speed u_t in the fixed system, they will give an additional miss-distance due to time error. The expresions[6] for time of flight $t(u, \theta)$, for $\partial t/\partial u$ and for $\partial t/\partial\theta$ are very unwiedly.

$$(22)\quad t(u,\theta) = (P/\pi)(\sin^{-1} f - cf)$$

where f, the period P, and the eccentricity e are given by

$$(23)\quad P(u) = (2\pi R/V)[1 + (h/R)]^{3/2}$$

$$(24)\quad h(u) = R(u^2/2V^2)[1 - (u^2/2V^2)]^{-1}$$

$$(25)\quad e^2(u,\theta) = 1 - [2 - (u^2/V^2)](u^2/V^2)\cos^2\theta$$

$$(26)\quad f(u,\theta) = (1 - e^2)^{1/2}[1 - \cos(\varphi/2)]^{-1}\sin(\varphi/2)$$

Table II—Miss Distances Caused by Earth Rotation

Range, φR, statue miles	3000		5000		7000	
Elevation angle, θ, degrees	34.1	45	26.9	45	19.7	45
Assumed typical torget speed due to Eorth rotation U_t, miles/sec.	0.14	0.14	0.14	0.14	0.14	0.14
Flight time t, sec.	1050	1520	1680	2660	1950	4790
$U_t(\partial t/\partial u)$, miles/(ft./sec.)	0.021	0.038	0.076	0.13		
$U_t(\partial t/\partial\theta)$, miles/millirad.	0.16	0.15	0.27	0.16		

In Table II we show some typical values of miss-distance. There are to be compared with entries in Table I; in particular, $u_t(\partial t/\partial u)$ is to be compared with $R(\partial \varphi/\partial u)$ and $u_t(\partial t/\partial \theta)$ with $R(\partial \varphi/\partial \theta)$. Usually the miss-distance caused by Earth rotation will be less than the fixed system miss-distance of Table I. Thus the Earth rotation errors are usually small but not unnoticeable. But when the speed of the rocket is very near escape speed and when lofting trajectories are used, the earthe rotation miss-distances can become predominant. In extreme cases the Earth rotates many times before the rocket lands.

VIII. Conclusion

The range and accuracy of long-range ballistic missiles have been approximately computed in terms of the initial velocity voctor and errors in the initial velocity vector. The calculations are too crude for the compilation of range tables, but they will suffice for overall system comparisions. Some important perturbing factors excluded from consideration were oblateness [7,8] of the Earth, local irregularities of gravity near the launch site, and the precise shape of the powered trajectory [4] and re-entry path [9] .

References

1. Goldstein, H., Classical Mechanics, Addison-Wesley, 1951.

2. Corben, H. C. and Stehlc, P., Classical Mechanics. John Wiley 1950

3. Allen, H. J. and Eggers, A. J., A Comparative Analysis of the Performance of Long-Range Hypervelocity Vehicles, Tech. Notes Nat. Adv. Comm. Aero., 4046, 1957.

4. Culler, G. J. and Fried, B. D., J. A. P. Vol. 28, No. 672, 1957.

5. Russell, W. T., Jet Propulsion, Vol. 28, No. 17, 1958.

6. Whittaker, E. T., Analytical Dynamics, Dover Publications, 1944.

7. Blitzer, L., Weisfeld, M. and Wheelon, A. D., J. A. P. Vol. 27, No. 1141, 1956.

8. Blitzer, L. and Wheelon, A. D., J. A. P. Vol. 28, No. 279, 1957.

9. Allen, H. J. and Eggers, A. J., A Study of the Motion and Aerodynamic Heating of Missiles Entering the Earth's Atmosphere at High Supersonic Speeds, Tech. Notes, Nat. Adv. Comm. Aero., No. 4047, 1957.