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OPTIMAL SINGULAR CONTROL COMPUTATION BY EPSILON TECHNIQUE

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Abstract: *This paper extends the ε-technique which was proposed by A. V. Balakrishnan in 1968 to optimal singular control problems and shows that the extended technique can solve the singular control problem in a straight-forward way. A set of necessary conditions for optimality by using ε-technique for singular odtimal control problems is derived and a computational algorithm is presented.*

1. INTRODUCTION

In recent years the singular control problems have received considerable attention[1~7]. It is found that the singular arc is encountered in many practical problem[1,2]. However, when singular control exists, the maximum principle does not provide enough information for the solution of the problem and additional conditions have to be derived.

Most indirect methods for computing optimal control problems are based on iterative solutions of two-point boundary value differential or difference equations[6~8]. In 1968, a new computing technique was proposed by A. V. Blakrishnan called epsilon technique[9,10]. This technique bypasses the difficulties associated with the classical interative algorithms for two-point boundary value problems. It uses a penalty function approach to include the dyanmical constraints in the cost function. Under some minor assumptions, it was shown in [10], that a solution for this continuous time problem with penalty functions converged to a solution of the original control problem. Later, on, the epsilon technique is modified and improved[11] to satisfy problems with inequality constraints and disurete time system problems. Also the modified technique can be efficiently employed on digital computers. The purpose of this paper will state briefly the epsilon technique and extend it to the singular control problem. Also it shows how the epsilon technique can be applied to determine the optimal in the singular. For couvenience, only the discrete time system will be considered. And two illustrative numerical examples are presented.

2. THE STATEMENT OF THE SINGULAR CONTROL PROBLEM

Consider the discrete-timd optimal control system described by

$$\underline{x}(k+1)=\underline{x}(k)+\underline{f}_1(x(k),k)+\underline{f}_2(\underline{x}(k),k)\underline{u}(k),\quad 0\leq k\leq N-1 \qquad (1)$$

with fixed initial conditions

$$\underline{x}(0)=\underline{x}_0 \tag{2a}$$

fixed final conditions

$$\underline{x}(N)=\underline{x}_N \tag{2b}$$

and the additional control inequality constraints

$$|u_j(k)|\leq 1,\ k=0,1,\ldots\ldots,N-1.\quad j=1,2,\ldots\ldots,m. \tag{2c}$$

where N is finite, and

$$\underline{x}(k)=[x_1(k),x_2(k),\ldots\ldots,x_n(k)]^T\in R^n$$

is the state vector. T denotes transpose.

$$\underline{u}(k)=[u_1(k),u_2(k),\ldots\ldots,u_m(k)]^T\in R^m$$

is the control vector. $\underline{f}_1$ is a $n\times 1$ column vector, and $\underline{f}_2$ is an $n\times m$ matrix. The objective of the control problem is to choose the control sequence $\{\underline{u}(0),\underline{u}(1),\ldots..,\underline{u}(N-1)\}$ to satisfy the condition of equations (1) and (2), such that the cost functional

$$J[\underline{x}(.),\underline{u}(.)]=\sum_{k=0}^{N-1}J_k(\underline{x}(k),\underline{u}(k)) \tag{3}$$

is minimized.

The classical approach to this class of problems is that according to the discrete maximum principle, the Hamiltonian for the proposed problem must be a maximum:

$$H[\underline{x}(k),\underline{\lambda}(k+1),\underline{u}(k);k]=-J_k[x(k)]+\underline{\lambda}^T(k+1)[\underline{x}(k)+\underline{f}_1(.)+\underline{f}_2(.)\underline{u}(k)] \tag{4}$$

For nonzero vector of $\underline{\lambda}^T(k+1)\underline{f}_2(.)$, it is clear that the optimal control to maximize the Hamiltonian is given by

$$u_j^*(k)=\begin{cases}+1, \text{ if } \underline{A}_j(k)>0\\ -1, \text{ if } \underline{A}_j(k)<0\end{cases} \tag{5}$$

where the switching function $\underline{A}_j(k)$ is defined by

$$\underline{A}_i(k)=[\underline{\lambda}^{*T}(k+1)\,\underline{f}_2(x^*(k),k)]_j \tag{6}$$

and $\underline{A}_j(k)$ denote the j-th component of $\underline{A}(k)$.

However, it is possible that the switching function $\underline{A}_j(k)$ may be identically zero over some nonzero time intervals. During these intervals the Hamiltonian function ceases to be an explict function of the control variables. And the maximum principle yields no information about the desired optimal control. Now, since $H(.)$ does not depend upon $u(k)$ explicity, the usual procedure of selecting the optimal control as to maximize $H(.)$ with respect to $u(k)$ breaks down. The problems in which $\underline{A}_j(k)$ becomes identically zero over some finite time intervals (usually called as singular) have been referred to as singular control problem.

3. DISCRETE EPSILON PROBLEM

To modify the proposed original problem stated in equation (1) to equation (3) by using a penalty function approach to include the dynamical constraints in the cots function becomes the new problem called discrete ε-problem as:

Choose $\{\underline{u}(0), \underline{u}(1), \ldots\ldots, \underline{u}(N-1)\}$ and $\{\underline{x}(1), \underline{x}(2), \ldots\ldots, \underline{x}(N)\}$ such that the new cost functional

$$J_\varepsilon[\underline{x}(.), \underline{u}(.)] = J[\underline{x}(.), \underline{u}(.)] + \frac{1}{\epsilon}\sum_{k=0}^{N-1} ||\underline{x}(k+1) - \underline{x}(k) - \underline{f}(\underline{x}(k), \underline{u}(k), k||^2 \tag{7}$$

is minimized, where $\underline{f}(.) = \underline{f}_1(.) + \underline{f}_2(.)\underline{u}(k)$, ϵ is a positive number, and $||.||$ denotes the assigned norm.

In [11], it was shown that under some mild conditions the solution of ε-problem approaches the solution of the original problem as ε approaches zero. For convenience, we refer to the procedure which converts the original problem with dynamical constraints to one without dynamical constraints as the epsilon technique.

4. THE EPSILON MAXIMUM PRINCIPLE FOR DISCRETE EPSILON PROBLEM

The epsilon maximum principle for continuous system had been derived in [10] and for discrete-time system was derived in [11]. Here, for the purpose of convenience only the results in [11] will be given.

Let $\underline{z}(k) = \underline{x}(k+1) - \underline{x}(k) - \underline{f}(\underline{x}(k), \underline{u}(k), k]$ (8)

and $\{\underline{x}^*(k)\}$, $\{\underline{u}^*(k)\}$ and $\{\underline{z}^*(k)\}$

be the optimal vector sequences for discrete ϵ-problem for some fixed ϵ. We assume that the systems are not abnormal and that there exists a nonzero vector sequence $\{\underline{\lambda}_\varepsilon(k)\}$, then the epsilon maximum principle may be stated as:

(i) Maximization of the Hamiltonian

$$<\underline{F}_k[\underline{x}^*(k), \underline{u}(k)], \underline{\lambda}_\varepsilon^*(k+1)> \geq <\underline{F}_k[\underline{x}^*(k), \underline{u}(k)], \underline{\lambda}_\varepsilon^*(k+1)> \tag{9}$$

for $|u_j(k)| \leq 1$, $j = 1, 2, \ldots\ldots, m$.

where

$$\underline{F}_k = \begin{bmatrix} \underline{f} \\ -J_k \end{bmatrix}$$

and the $<.,.>$ is defined as the inner product in the Euclidean space.

(ii) Adjoint equation

$$\underline{\lambda}_\varepsilon^*(k) - \underline{\lambda}_\varepsilon^*(k+1) = \left[\frac{\partial \underline{F}_k(\underline{x}(k), \underline{u}^*(k))}{\partial \underline{x}(k)}\bigg|_{\underline{x}(k) = \underline{x}^*(k)}\right]^T \underline{\lambda}_\varepsilon^*(k+1) \tag{10}$$

(iii) Traversality conditions

$$\underline{\lambda}_\varepsilon^*(k+1) = \rho\begin{bmatrix} \underline{z}^*(k) \\ \varepsilon/2 \end{bmatrix}, \text{ where } \rho > 0.$$

For all $k = 0, 1, 2, \ldots\ldots, N-1$.

5. DETERMINATION OF OPTIMAL CONTROL IN THE SINGULAR STAGES

The ε-maximum principle has been stated in the last section, for the singular problem in equations (1) to (3), it can be stated as follows:

Maximize the Hamiltonian for $\varepsilon>0$ defined by (let $\rho=1$)

$$H[\varepsilon, \underline{z}_\varepsilon(k), \underline{x}_\varepsilon(k), \underline{u}_\varepsilon(k), k]=-J_k(.)+\frac{2}{\varepsilon}\underline{z}_\varepsilon^T(k)[\underline{f}_1(\underline{x}_\varepsilon(k), k+\underline{f}_2(k), k)\underline{u}_\varepsilon(k)] \quad (11)$$

where $\underline{z}_\varepsilon(k)=\underline{x}_\varepsilon(k+1)-\underline{f}_1(\underline{x}_\varepsilon(k), k)-\underline{f}_2(\underline{x}_\varepsilon(k), k)$, $k=0, 1, 2, \ldots\ldots, N-1.$ (12)

It is clear that for nonzero vector of switching function defined by

$$G_j(\varepsilon, k)=(2/\varepsilon)[\underline{z}_\varepsilon^{*T}(k)\cdot \underline{f}_2(\underline{x}_\varepsilon^*(k), k)]_j \quad (13)$$

the optimal control is

$$[u_\varepsilon^*(k)]_j=\begin{cases}+1, \text{ if } G_j(\varepsilon, k)>0\\ -1, \text{ if } G_j(\varepsilon, k)<0\end{cases} \quad (14)$$

where $G_j(.)$ denotes the j-th component of vector G. And $G=(2/\varepsilon)[\underline{z}_\varepsilon^{*T}, \underline{f}_2(\underline{x}_\varepsilon^*(k), k)]$ is a $1\times m$ row vector. But in the case $G_j(\varepsilon, k)=0$, from the definitions of $G_j(.)$ and $\underline{z}_\varepsilon(k)$, we can obtain that:

$$(\underline{u}_\varepsilon^*(k)]_j=U_j[\underline{x}_\varepsilon(k+1), \underline{x}_\varepsilon(k), u_{\varepsilon,1}, \ldots\ldots, u_{\varepsilon,j-1}, u_{\varepsilon,j+1}, \ldots\ldots, u_{\varepsilon,m}] \quad (15)$$

for $j=1, 2, \ldots\ldots, m.$ $k=0, 1, 2, \ldots\ldots, N-1.$

For the scalar control case, the optimal control in the singular stages takes a simple from:

$$u_\varepsilon^*(k)=[\underline{f}_2^T(x_\varepsilon^*(k), k)\underline{f}_2(.)]^{-1}\underline{f}_2^T(.)[\underline{x}_\varepsilon^*(k+1)-\underline{x}_\varepsilon^*(k+1)-\underline{x}_\varepsilon^*(k)-\underline{f}_1(\underline{x}_\varepsilon(k), k)] \quad (16)$$

here, $[\underline{f}_2^T\cdot \underline{f}_2]^{-1}$ is always a nonzero scalar.

Hence, we can determine the optimal control in the singular stage. And as ε approaches zero, $\underline{u}_\varepsilon^*(k)$ approaches the solution of original problem.

6. THE NECESSARY CONDITION FOR OPTIMALITY OBTAINED FROM EPSILON TECHNIQUE

The necessary conditions for optimality for the proposed problem can easily be derived by the following two procedures:

(a) Change the original problem, to ε-problem, that is we will minimize the cost functional

$$J_\varepsilon[\underline{x}(k), \underline{u}(k)]=\sum_{k=0}^{N-1}J_k[\underline{x}(k)]+\frac{1}{\varepsilon}\sum_{k=0}^{N-1}||\underline{x}(k+1)-\underline{x}(k)-\underline{f}_1(\underline{x}(k), k)-\underline{f}_2(\underline{x}(k), k)\underline{u}(k)||^2 \quad (17)$$

(b) Maximize the Hamiltonian defined in equation (11). The constraints still remain the same. So the necessary conditions for optimality are:

$$\frac{\partial \underline{x}_\varepsilon(k)}{\partial J_\varepsilon(.)}=0 \quad k=1, 2, \ldots\ldots, N-1; \quad (18)$$

and
$$[\underline{u}_\varepsilon(k)]_j=\begin{cases}+1, \text{ if } G_j(\varepsilon, k)>0 & (19)\\ -1, \text{ if } G_j(\varepsilon, k)<0 & (20)\\ U_j[\underline{x}_\varepsilon(k+1), \underline{x}_\varepsilon(k), u_{\varepsilon,j-1}, u_{\varepsilon,j+1}, \ldots\ldots, u_{\varepsilon,m}], \text{ if } G_j(\varepsilon, k)=0 & (21)\end{cases}$$

for $j=1,2,\ldots\ldots,m$ $k=0,1,2,\ldots\ldots,N-1$.

Equation (18) is a set of simultaneous equations, this implies that we are required to solve the simultaneous equations instead of solving the boundary value equations. From equation (21) we see that the indetermination is eliminated if the ε-technique is employed to solve the singular control problem.

7. COMPUTATIONAL ALGORITHM

The computational algorithm for solving the set of equations (18) to (21) can be stated as follows:

(i) Choose a starting value $\varepsilon_1>0$ and the coresponding nominal initial control sequence:

$|u_j(k)|<1$ $k=0,1,2,\ldots\ldots,N-1;$ $j=1,2,\ldots\ldots,m.$

(ii) Solve the simultaneous equations derived from equation (18).

(iii) For every ε_i compute the quantity $\underline{G}_j(\varepsilon_i,k)=(2/\varepsilon_i)[\underline{z}^T\cdot(k)\cdot\underline{f}_2(\underline{x}(k),k)]_j$ then compute the new control sequences by the following way:

$$[\underline{u}^{new}(k)]_j=\begin{cases}+1, \text{ if } G_j(\varepsilon_i,k)>0\\ -1, \text{ if } G_j(\varepsilon_i,k)<0\\ \text{Sat }[U_j(\underline{x}(k+1),\underline{x}(k),u_1,u_2,\ldots\ldots,u_{j-1},u_{j+1},\ldots\ldots,u_m)], \text{ if } G_j(\varepsilon_i,k)=0^*\\ k=0,1,\ldots\ldots,N-1;\\ j=1,2,\ldots\ldots,m.\end{cases} \quad (22)$$

where we define

$$\text{Sat }[x]=x \text{ if } |x|<1$$
$$=x/|x| \text{ if } |x|>1$$

(iv) Adjust the control sequences ($n=1$ initially) for ε_i by the following way:

$$u_j^{n+1}(k)=u_j^n(k)+[u_j^{new}(k)-u_j^n(k)]\cdot p \quad (23)$$

where $0<p<1$

and n denotes n-th iteration.

(v) For every ε_i, the procedure can be terminated when $|u_j^{n+1}(.)-u_j^n|<r$, r is a small predetermined positive quantity, then we set

$$[\underline{u}_{\varepsilon_i}^*(k)]_j=[u_{\varepsilon_i}^{n+1}(k)]_j, \quad k=0,1,2,\ldots\ldots,N-1;\ j=1,2,\ldots\ldots,m \quad (24)$$

(vi) Choose $\varepsilon_{i+1}<\varepsilon_i$ and set the initial nominal control sequences for ε_{i+1} equal to $\underline{u}_{\varepsilon_i}^*(k)$, $i=i+1$, and go to (ii).

(vii) Terminate the computation when either

a) ε_i is so small that numerical instability occurs, or

b) $\varepsilon_i<a$, a is a small predetermined positive quantity.

* From the computational point of view, $\underline{G}(\varepsilon_i,k)=0$ is not suitable for the determination of the existence of singular control since the accumulation of round-off errors may render the results inaccurate. So we must estimate adequately a small quantity $d>0$, such that if $\underline{G}_j|(\varepsilon_i,k)|<d$, the $\underline{G}_j(\varepsilon_i,k)$ seems to be equal to zero

8. COMPUTED EXAMPLES

In this section, two numerical examples are solved by using the ε-technique. For convenience, the norm in equation (17) is assigned as Euclidean norm, and only the cases of cost functional of quadratic form and linear system equations will be discussed.

Example 1.

Minimize

$$J=0.1\sum_{k=0}^{20} x^2(k) \tag{25}$$

Subject to

$$x(k+1)=x(k)+0.1\,u(k) \tag{26}$$

$$x(0)=1 \tag{27}$$

$$|u(k)|\leq 1 \text{ for all } k=0.1,\ldots\ldots,19. \tag{28}$$

One may think this problem is euqivalent to the continuous problem:

$$J=\int_0^2 x^2 dt$$

$$x=u \quad x(0)=1$$

$$|u(t)|\leq 1$$

we define $z(k)=x(k+1)-x(k)-0.1\,u(k)$

hence the ε-problem is to minimize

$$J_\varepsilon=(\cdot)=0.1\sum_{k=0}^{20} x^2(k)+\frac{1}{\varepsilon}\sum_{k=0}^{19} z^2(k) \tag{29}$$

subject to equations (27) and (28). The necessary conditions for optimality according to equations (18)-(21) are

$$x(k-1)-(0.1\varepsilon+2)x(k)+x(k+1)=0.1[u(k)-u(k-1)]$$

for $k=1,2,\ldots\ldots,19$;

$$x(19)-(1+0.1\varepsilon)x(20)=-0.1\,u(19)$$

and

$$u_\varepsilon(k)=\begin{cases}+1, \text{ if } (2/\varepsilon)[0.1\,z_\varepsilon(k)]>0\\ -1, \text{ if } (2/\varepsilon)[0.1\,z_\varepsilon(k)]<0\\ \text{sat } [10(x_\varepsilon(k+1)-x_\varepsilon(k))] \text{ if } (2/\varepsilon)[0.1\,z_\varepsilon(k)]=0, \; k=0,1,\ldots\ldots,19.\end{cases}$$

Startidg with $\varepsilon_1=5$, choose $r=10^{-5}$, and $p=0.02$, then to solve this problem by using the algorithm in the last section. The resulting ε-problem is solved for a monotonically decreasing sequence $\{\varepsilon_1>\varepsilon_2\ldots\ldots>\varepsilon_l>0\}$, where $\varepsilon_{l+1}=\varepsilon_l/5$. When $\varepsilon_l=0.1280\times10^{-3}$, the solutions of the ε-problem tends to the solution of the original problem. The resulting value of $J^*(.)$ that was produced by the ε-technique was 0.3851. For every ε, the values of $J_\varepsilon^*(.)$, $J^*(.)$ and $\sum_{k=0}^{3}||z^*(k)||^2$ were presented in Table 1. In

Table 1.

$\varepsilon>0$	$J_\varepsilon(x^*(.),z^*(.))$	$J(x^*(.))$	$\Sigma\|z^*(.)\|^2$
$\varepsilon_1=5.0$	0.1692	0.1304	0.19382948
$\varepsilon_2=1.0$	0.2451	0,1926	0.05252104
$\varepsilon_3=0.2$	0.3238	0.2837	0.00802029
$\varepsilon_4=0.04$	0.3677	0.3510	0.00066695
$\varepsilon_5=0.008$	0.3814	0.3770	0.00003509
$\varepsilon_6=0.0016$	0.3843	0.3835	0.00000126
$\varepsilon_7=0.00032$	0.3850	0.3848	0.00000005
$\varepsilon_8=0.000064$	0.3851	0.3851	0.00000000
$\varepsilon_9=0.0000128$	0.3851	0.3851	0.00000000

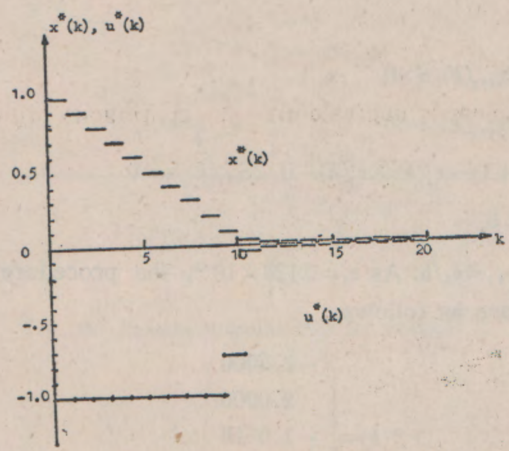

Fig. 1. Trajectory, and computed optimal control for example 1.

Fig. 1, we display the results of optimal control and trajectory.

Example 2.

This example is a second order linear discrete time singular control problem discussed in [5]. Here, we will use the ε-technique to compute the optimal control.

The system equations are described by

$$x_1(k+1)=x_2(k)$$
$$x_2(k+1)=x_1(k)+x_2(k)+u(k) \tag{30}$$

find the $u(k)$ to minimize

$$J=\sum_{k=0}^{3}[x_1^2(k)+1.8x_1(k)x_2(k)+x_2^2(k)] \tag{31}$$

subject to (30) and

$$x_1(0)=-2.5 \quad x_1(4)=0$$
$$x_2(0)=2.0 \quad x_2(4)=0$$
$$|u(k)|\leq 1 \quad 0\leq k\leq N-1 \tag{32}$$

here we let the initial condition be located in the region of singular isochrone.** Let us define

$$z_1(k)=x_1(k+1)-x_2(k)$$
$$z_2(k)=x_2(k+1)-x_1(k)-x_2(k)-u(k)$$

then the ε-problem is to minimize

$$J_\varepsilon(.)=\sum_{k=0}^{3}[x_1^2(k)+1.8x_1(k)x_2(k)+x_2^2(k)]+\frac{1}{\varepsilon}\sum_{k=0}^{3}[z_1^2(k)+z_2^2(k)] \quad (33)$$

subject to (32).

The necessary conditions for optimality are

$$(\varepsilon+2)x_1(k)-x_2(k-1)+(0.9\varepsilon+1)x_2(k)-x_2(k+1)=-u(k),$$
$$-x_1(k-1)+(0.9\varepsilon+1)x_1(k)-x_1(k-1)-x_2(k-1)+(\varepsilon+2)x_2(k)-x_2(k+1)=u(k-1)-u(k) \quad (34)$$

for $k=1,2,3$

and

$$u_\varepsilon(k)=\begin{cases}+1 & \text{if } 2z_{\varepsilon,2}(k)/\varepsilon>0\\ -1 & \text{if } 2z_{\varepsilon,2}(k)/\varepsilon<0\\ \text{sat } x_2(k+1)-x_1(k)-x_2(k) & \text{if } 2z_{\varepsilon,2}(k)/\varepsilon=0\\ k=0,1,2,3 & \end{cases} \quad (35)$$

Starting $\varepsilon_1=0.5$. Let $\varepsilon_{i+1}=\varepsilon_i/5$. As $\varepsilon_i=0.128\times10^{-5}$, the procedure was terminated and the optimal solutions are as follows:

$$u^*(k)=\begin{pmatrix}-.55\\-.39\\.49\\-.55\end{pmatrix} \qquad x_1^*(k)=\begin{pmatrix}-2.5000\\2.0000\\-1.0518\\.5553\\.0000\end{pmatrix}$$

$$z_2^*(k)=\begin{pmatrix}2.0000\\-1.0518\\.5553\\.0001\\.0000\end{pmatrix} \qquad z_1^*(k)=\begin{pmatrix}.000011\\-.000027\\.000078\\.000125\end{pmatrix}$$

$$z_2^*(k)=\begin{pmatrix}-.000005\\.000045\\-.000045\\.000096\end{pmatrix}$$

The cost functional $J^*(.)$ is equal to 3.2417. We present the values of $J_\varepsilon^*(.)$, $J^*(.)$ and $\sum_{k=0}^{3}[z_1^{*2}(k)+z_2^{*2}(k)]$ in Table 2. And in Fig. 2, we also display the optimal controls and the corresponding singular arcs in the phase plane.

** The singular isochrone is a convex set in the state space contains all the initial states which are transferred to the final states by singular controls in N singular stages. (For this problem, N=4).

Table 2.

$\varepsilon>0$	$J_\varepsilon(x^*(.).z^*(.))$	$J(x^*(.))$	$\Sigma=z^*(.)\|^2$
$\varepsilon_1=0.02$	3.21852	3.2043	0.000070
$\varepsilon_2=0.004$	3.22206	3.2115	0.000041
$\varepsilon_3=0.0008$	3.23188	3.2237	0.000006
$\varepsilon_4=000016$	3.24185	3.2414	0.000000
$\varepsilon_5=0.000032$	3.24287	3.2417	0.000000
$\varepsilon_6=0.0000064$	3.24725	3.2417	0.000000
$\varepsilon_7=0.00000128$	3.26852	3.2418	0.000000

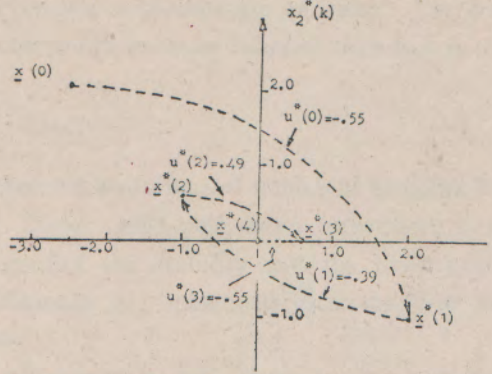

Fig. 2. System singular arcs for example 2.

9. CONCLUSION

We have shown the singular control problem can be solved by using the ɘ-technique. In the singular stages, the optimal controls are found in a straightforward way.

The advantage of the ε-technique, from both practical and theoretical point of view, is its simplicity since it will bypass the complexity on solving the boundary value problem. The examples, show that the ε-technique can be applied singular problems and the difficulties caused by systems with abnormal solution is then avoided.

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