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STABILITY OF NONLINEAR SYSTEMS WITH CHARACTERISTIC EQUATIONS HAVING COMPLEX COEFFICIENTS*

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Abstract—*Stability criteria for nonlinear systems with linearized characteristic equations having complex coefficients are presented. The presented criteria are more general and the proofs are more complete than that in the current literature.*

1. INTRODUCTION

Linearized characteristic equations of nonlinear systems have complex coefficients frequently. Han and Thaler (1966-1968) have introduced a method, called stability equation method, to predict the stability and the limit cycle of nonlinear systems having complex coefficients in linearized characteristic equations. The general approach is as follows:

Consider an Nth order system with characteristic equation as

$$F(s)=\sum_{i=0}^{N} a_i s^i=0 \tag{1}$$

where $a_n=1$, $a_i=\alpha_i+j\beta_i$. The stability equations are

$$F_e(s)=\alpha_0+j\beta_1 s+\alpha_2 s^2+j\beta_3 s^3+\ldots\ldots\ldots\ldots\ldots=0 \tag{2}$$

$$F_0(s)=j\beta_0+\alpha_1 s+j\beta_2 s^2+\alpha_3 s^3+\ldots\ldots\ldots\ldots\ldots=0 \tag{3}$$

or with the substitution of $s=j\omega$

$$F_R(\omega)=F_e(j\omega)=\sum_{i=0}^{N_e} c_i\omega^i=0 \tag{4}$$

$$F_I(\omega)=jF_0(j\omega)=\sum_{i=0}^{Nq} d_i\omega^i=0 \tag{5}$$

where α_i, β_i, c_i, d_i are all real, and

$$F(s)=F_e(s)+F_0(s)=0 \tag{6}$$

or $$F(j\omega)=F_R(\omega)+jF_I(\omega)=0 \tag{7}$$

where $F_e(s)$ and $F_0(s)$ are the even part and the odd part of $F(s)$ respectively. Equations (6) and (7) can be written, respectively, as

$$G(s)=\frac{F_e(s)}{F_0(s)}=-1 \tag{8}$$

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$$G(j\omega)=\frac{F_R(\omega)}{jF_I(\omega)}=-1 \tag{9}$$

The conditions for a stable system are that (1) all poles and zeros of eqn. (8) are on the imaginary axis of s-plane, and (2) poles and zeros must occur in alternating sequences:

$$......z_{-2}<p_{-1}<z_{-1}<p_0<z_1<p_1<z_2.............. \tag{10}$$

From root-locus point of view, these conditions indicate that all the root-loci of $m\times 180$ degrees (where m is an odd integer) are in the left half of s-plane (LHP), so the linearized system is stable (Han and Thaler 1966).

Note that these conditions are meaningful only if the signs of coefficients of the highest order terms of $F_e(s)$ and $F_0(s)$ are the same.

The purpose of this paper is to present general stability criteria that the assumption of $a_n=1$ is not necessary, the selection of $F_e(s)$ and $F_0(s)$ may not be as eqns. (2) and (3), and that the angle of root loci may not be of $m\times 180$ degrees.

2. A STABILITY CRITERION FOR THE CASE THAT G(s) HAS A GENERAL PHASE-ANGLE LOCI

Basing on the characteristics of 180°-loci, the stability equation method has been developed for the prediction of stability of high order linear systems (Han and Thaler 1966). If $F_e(s)$ and $F_0(s)$ are not chosen as eqns. (2) and (3), the root loci associated with $G(s)$ are not limited to 180°-loci. Following developments are based on a general phase-angle loci of $G(s)$.

Lemmr 1. If the poles and zeros are all on the imaginary axis of the s-plane, and occur in alternating sequences, the departure angles of the poles are all equal to a certain angle θ_p, and the arrival angles of the zeros are all equal to θ_z, and $\theta_z=-\theta_p$.

Proof:

Write the transfer function $G(s)$ in factored form:

$$G(s)=k_s\frac{\prod_{n=1}^{N_z}(s-jz_n)}{\prod_{n=1}^{N_p}(s-jp_n)}=-1 \tag{11}$$

where k_s is a complex gain of $G(s)$, *i.e.*, $k_s=|k_s|e^{jA_s}$.

Assume the phase-angle of the phase-angle loci of $G(s)$ is ϕ, then by definition the departure angle of p_i is

$$\theta_{p_i}=ARG\left(\frac{\prod_{n=1}^{N_z}(p_i-z_n)j}{\prod_{\substack{n=1\\n\neq i}}^{N_p}(p_i-p_n)j}\right)-\phi \tag{12}$$

$$\triangleq A_{p_i}-\phi$$

The departure angle of p_{i+1} is

$$\theta_{p_{i+1}}=A_{p_{i+1}}-\phi \tag{13}$$

From Fig. 1, it is evident that

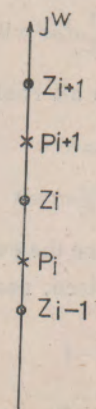

Fig. 1. A typical pole-zero distribution on the imaginary axis.

$$A_{p_{i+1}}=A_{p_i}-(\Delta\theta_{p_{i+1},\ p_i}+\Delta\theta_{z_i,\ p_i})+(\Delta\theta_{p_i,\ p_{i+1}}+\Delta\theta_{z_i,\ p_{i+1}}) \tag{14}$$

where

$$\Delta\theta_{p_{i+1},\ p_i}=90° \tag{15}$$

$$\Delta\theta_{p_i,\ p_{i+1}}=-90° \tag{16}$$

$$\Delta\theta_{z_i,\ p_i}=-90° \tag{17}$$

$$\Delta\theta_{z_i,\ p_{i+1}}=90° \tag{18}$$

Hence

$$A_{p_{i+1}}=A_{p_i} \tag{19}$$

Substituting eqn. (19) into eqns. (12) and (13), yields

$$\theta_{p_i}=\theta_{p_{i+1}} \tag{20}$$

Equation (20) is satisfied for all $i=1,2,\ldots\ldots N_p$, since there is no restriction on i; therefore eqn. (20) implies that

$$\theta_{p_i}=\theta_p, \quad \text{for} \quad i=1,2,\ldots\ldots\ldots\ldots N_p \tag{21}$$

By the same reasoning, it is easy to prove that all the arrival angles of the zeros are equal to some certain angle

$$\theta_{z_i}=\theta_z, \quad \text{for} \quad i=1,2,\ldots\ldots\ldots\ldots N_z \tag{22}$$

From Fig. 1, the relation between θ_p and θ_z can be derived.

$$\theta_{p_i}=A_{p_i}-\phi \tag{23}$$

$$\theta_{z_i}=-A_{z_i}+\phi \tag{24}$$

where

$$A_{p_i}=ARG\left(\frac{\prod_{n=1}^{N_z}(p_i-z_n)j}{\prod_{\substack{n=1\\n\neq i}}^{N_p}(p_i-p_n)j}\right) \tag{25}$$

$$A_{z_i}=ARG\left(\frac{\prod_{\substack{n=1\\n\neq i}}^{N_z}(z_i-z_n)j}{\prod_{n=1}^{N_p}(z_i-p_n)j}\right) \tag{26}$$

But

$$A_{p_i}=A_{z_i}-\Delta\theta_{p_i,\ z_i}+\Delta\theta_{z_i,\ p_i} \tag{27}$$

where

$$\Delta\theta_{p_i,\ z_i}=\Delta\theta_{z_i,\ p_i} \tag{28}$$

for either $p_i>z_i$ or $z_i>p_i$; therefore

$$A_{p_i}=A_{z_i} \tag{29}$$

Substituting eqn. (29) into eqns. (23) and (24), yields

$$\theta_{p_i}=-\theta_{z_i} \tag{30}$$

since

$$\theta_{p_i}=\theta_p=A_p-\phi, \quad \text{for} \quad i=1,\ldots\ldots\ldots N_p, \tag{30a}$$

and

$$\theta_{z_i}=\theta_z=-A_z+\phi, \quad \text{for} \quad i=1,\ldots\ldots\ldots N_z. \tag{30b}$$

Hence

$$\theta_p=-\theta_z \tag{31}$$

Lemma 2. As the condition of poles and zeros stated in Lemma 1 is true, then

$$\theta_p=(N_p-N_z-1)\frac{\pi}{2}-\phi \quad \text{when the smallest singularity is a pole,}$$

and

$$\theta_p=(N_p-N_z+1)\frac{\pi}{2}-\phi \quad \text{when the smallest singularity is a zero.}$$

Proof:

If the smallest singularity is a pole, then the alternating sequence of poles and zeros is as

$$p_1<z_1<\ldots\ldots\ldots p_{i-1}<z_{i-1}<p_i<z_i<p_{i+1}<\ldots\ldots\ldots\ldots\ldots \tag{32}$$

and

$$\begin{aligned} A_p=A_{p_i}&=\Big[(i-1)\frac{\pi}{2}-(N_z-i+1)\frac{\pi}{2}\Big]+\Big[-(i-1)\frac{\pi}{2}+(N_p-i)\frac{\pi}{2}\Big] \\ &=(N_p-N_z-1)\frac{\pi}{2} \end{aligned} \tag{33}$$

Substitufing eqn. (33) into (30a) gives

$$\theta_p=(N_p-N_z-1)\frac{\pi}{2}-\phi \tag{34}$$

If the smallest singularity is a zero, the sequence is as

$$z_1<p_1<\ldots\ldots\ldots z_{i-1}<p_{i-1}<z_i<p_i<z_{i+1}<\ldots\ldots\ldots\ldots\ldots \tag{35}$$

and

$$\begin{aligned} A_p=A_{p_i}&=\Big[\frac{\pi}{2}i-(N_z-i)\frac{\pi}{2}\Big]+\Big[-(i-1)\frac{\pi}{2}+(N_p-i)\frac{\pi}{2}\Big] \\ &=(N_p-N_z+1)\frac{\pi}{2} \end{aligned} \tag{36}$$

Substituting eqn. (36) into eqn. (30a) yields

$$\theta_p=(N_p-N_z+1)\frac{\pi}{2}-\phi \tag{37}$$

Lemma 3. The phase-angle of the phase-angle loci of $G(s)$, defined as ϕ, can be given as

$$\phi=\pi-A_z$$

or

$$\phi=(N_z-N_p-1)\frac{\pi}{2}-A_\omega$$

where A_ω is the argument of k_ω, and k_ω is the gain of $G(j\omega)$.
Proof:

Rearrnaging eqn. (11) yields

$$|k_s|e^{jA_s}\frac{\prod_{n=1}^{N_z}(s-jz_n)}{\prod_{n=1}^{N_z}(s-jp_n)}=-1 \tag{38}$$

or

$$|k_s|\frac{\prod_{n=1}^{N_z}(s-jz_n)}{\prod_{n=1}^{N_p}(s-jp_n)}=-e^{-jA_s} \tag{39}$$

By definition

$$\begin{aligned}\phi&=ARG[-e^{-jA_s}]=ARG[-1]+ARG[e^{-jA_s}]\\&=\pi-A_s\end{aligned} \tag{40}$$

Substituting $s=j\omega$ into eqn. (11) gets

$$G(j\omega)=k_s\frac{(j)^{N_z}}{(j)^{N_p}}\frac{\prod_{n=1}^{N_z}(\omega-z_n)}{\prod_{n=1}^{N_p}(\omega-p_n)}=k_s\ j^{(N_z-N_p+1)}\frac{\prod_{n=1}^{N_z}(\omega-z_n)}{j\prod_{n=1}^{N_p}(\omega-p_n)} \tag{41}$$

From eqns. (4), (5), (9) and (11)

$$G(j\omega)\overset{\Delta}{=}k_\omega\frac{\prod_{n=1}^{N_z}(\omega-z_n)}{j\prod_{n=1}^{N_p}(\omega-p_n)} \tag{42}$$

Comparing eqns. (41) and (42), gives

$$k_\omega=k_s\ j^{(N_z-N_p+1)} \tag{43}$$

The phase angles of both side of eqn. (43) are equal, *i.e.*

$$ARG(k_\omega)=ARG(k_s)+\frac{\pi}{2}(N_z-N_p+1) \tag{44}$$

or

$$A_s=A_\omega-\frac{\pi}{2}(N_z-N_p+1) \tag{45}$$

Substituting eqns. (45) into eqn. (40) yields

$$\phi=\frac{\pi}{2}(N_z-N_p-1)-A_\omega \tag{46}$$

Lemma 4. As the condition of poles and zeros stated in Lemma 1 is true, then

$$\left.\begin{aligned}&\theta_p=(N_p-N_z+1)\frac{\pi}{2}+A_s\\ \text{or}\quad&\\&\theta_p=(N_p-N_z)\pi+A_\omega\end{aligned}\right\}\quad\text{if the smallest singularity is a pole,}$$

and

$$\left.\begin{array}{l}\theta_p=(N_p-N_z-1)\dfrac{\pi}{2}+A_s \\ \text{or} \\ \theta_p=(N_p-N_z-1)\pi+A_\omega\end{array}\right\}\begin{array}{l}\text{if the smallest singularity} \\ \\ \text{is a zero.}\end{array}$$

Proof: From Lemma 2 and 3, the results are evident.

Theorem 1. The roots of the characteristic equation are all in the LHP if

(1) The poles and zeros of $G(s)$ are all on the imaginary axis of the s-plane and in alternating sequences.

(2) The departure angles of the poles are θ_p, and

$$\frac{\pi}{2}<\theta_p<\frac{3\pi}{2}$$

Proof: if $\dfrac{\pi}{2}<\theta_p<\dfrac{3\pi}{2}$ (47)

then from Lemma 1

$$\theta_z=-\theta_p \tag{48}$$

which is also in the region

$$\frac{\pi}{2}<\theta_z<\frac{3\pi}{2} \tag{49}$$

The imaginary axis is the loci of phase angle

$$\phi=\pm\left(\frac{1}{2}\pm n\right)\pi \tag{50}$$

where n is an integer: thus no root loci can cross the imaginary axis. The departure (arrival) directions of all poles (zeros) are toward (from) LHP, so that all root loci are confined to the left half of the s-plane.

Corollary 1. Let's define the following two quantities:

$$C_s=\frac{N_p-N_z}{2}+\frac{A_s}{\pi} \tag{51}$$

$$C_\omega=(N_p-N_z)+\frac{A_\omega}{\pi} \tag{52}$$

Then, the characteristic equation has all root in LHP if

$$\left.\begin{array}{l}2N<C_s<2N+1 \\ \text{or} \\ 2N+0.5<C_\omega<2N+1.5\end{array}\right\}\begin{array}{l}\text{for the case that the smallest} \\ \text{singularity is a pole,}\end{array}$$

and

$$\left.\begin{array}{l}2N+1<C_s<2N+2 \\ \text{or} \\ 2N-0.5<C_\omega<2N+0.5\end{array}\right\}\begin{array}{l}\text{for the smallest singularity is a} \\ \text{zero}\end{array}$$

To prove corollary 1, Theorem 1 and Lemma 4 must be used, and she result is not difficult to achieve.

Example 1. Determine the root distribution of the following equation

$$F(s)=(2+2j)s^2+(3+j)s+4+4j=0$$

If $F_e(s)$ and $F_0(s)$ are chosen as

$$F_e(s)=(2+2j)s^2+4+4j$$
$$F_0(s)=(3+j)s$$

which give

$$F_R(\omega)=(2+2j)\,[-\omega^2+1]$$
$$F_I(\omega)=(3+j)\omega$$

the zeros are at 1, -1, and the pole is at 0. The pole zero sequence is alternative and the smallest singularity is a zero. In addition,

$$K_s=\frac{2+2j}{3+3j}$$
$$A_s=\tan^{-1}\ 2=83.4°$$

and

$$C_s=\frac{N_p-N_z}{2}+\frac{A_s}{\pi}=-0.147$$
$$i.e.\quad -1<C_s<0$$

Thts all the roots of the considered equation are in LHP.

3. THE NECESSARY AND SUFFICIENT CONDITIONS OF STABILITY

In this section, the stability equations $F_e(s)$ and $F_0(s)$ are chosen as the even part and odd part of the characteristic equation as eqns. (2) and (3), respectively, but with $a_n=\alpha_n+j\beta_n$. The sufficient conditions for stability are those of Theorem 1, since in this case the phase angle loci are a special case of those of Theorem 1.

Lemma 5. If the stability equations $F_e(s)$ and $F_0(s)$ are chosen as eqns. (2) and (3), the phase-angle of the phase-angle loci of $G(s)=F_e(s)/F_0(s)=-1$ (defined as ϕ) can be expressed as

$$\phi=\frac{m}{2}\pi \qquad m=0,\ \pm1,\ \pm2,\ \ldots\ldots\ldots\ldots$$

Proof:

Using Lemma 3

$$\phi=(N_z-N_p-1)\frac{\pi}{2}-A_\omega$$

$A_\omega=ARG(k_\omega)$, from eqns. (4) and (5)

$$k_\omega=\frac{c_{N_z}}{d_{N_p}} \tag{53}$$

Since c_{N_z} and d_{N_p} are all real, therefore

$$A_\omega=n\pi \tag{54}$$

where n is either zero or one. Substituting eqn. (54) into eqn. (40), yields

$$\phi=(N_z-N_p-1)\frac{\pi}{2}-n\pi=(N_z-N_p-1-2n)\frac{\pi}{2}=m\frac{\pi}{2} \tag{55}$$

where

$$m=N_z-N_p-(2n+1) \tag{56}$$

Lemma 6. On the imaginary axis the phase angle of any point is

$$\theta=\frac{l}{2}\pi, \qquad l=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ldots\ldots\ldots\ldots$$

and θ is constant between any two singularities.

Proof:

If $F_e(s)$ and $F_0(s)$ are chosen as eqns. (2) and (3) then $F_R(\omega)$ and $F_I(\omega)$ are those of eqns. (4) and (5), which are all with real coefficients. In such a case, the roots of either $F_R(\omega)=0$ or $F_I(\omega)=0$ (zero or poles) can exist only in the following two classes:

(1) ω is real.

(2) ω is complex with that both ω and its conjugate $\bar{\omega}$ are all roots: *i.e.* the complex roots of $F_R(\omega)=0$ and $F_I(\omega)=0$ are symmetric with respect to the imaginary axis of s-plane.

Fig. 2 Contains both these two classes of poles and zeros. Consider the phase angle of the point on the imaginary axis, which is not a singularity. Each real singularity contributes an angle $\theta_R=\pm\frac{\pi}{2}$; each complex singularity pair contributes an angle $\theta_c=\pm\pi$ to any point on the imaginary axis. Thus the phase angle of any point which is not a singularity is

$$\theta=\sum_{R=1}^{N_R}\theta_R+\sum_{c=1}^{N_c}\theta_c=\frac{l}{2}\pi \qquad (57)$$

where $l=\pm1,\ \pm2,\ \pm3,\ \pm4,\ldots\ldots$; N_R and N_c are the numbers of real roots and pairs of complex roots respectively.

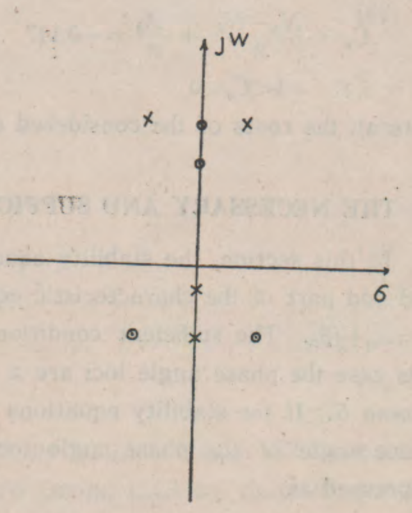

Fig. 2. A general pole-zero distribution in s-plane.

The phase angle of a singularity on the imaginary axis is

$$\theta=\sum_{\substack{R=1\\R\neq i}}^{N_R}\theta_R+\sum_{c=1}^{N_c}\theta_c=\frac{l}{2}\pi, \qquad (58)$$

where $l=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ldots\ldots\ldots$. From Fig. 2 it can be seen that, for a point between any two singularities on the imaginary axis, there is no change of the phase angle contributed from any real singularity and any pair of complex singularities; thus θ is constant between any two singularities on the imaginary axis.

Lemma 7. The root loci of $G(s)=F_e(s)=-1$ can't meet the imaginary axis except at the singularities.

Proof:

From eqns. (55) and (56), $\phi=m\frac{\pi}{2}$, where m is odd when N_z-N_p is even, and m is even when N_z-N_p is odd. In Eq. (57),

$$N_R=N_p+N_z-2N_c \qquad (59)$$

Let

$$\sum_{c=1}^{N_c} \theta_c = n_1\pi = 2n_1\left(\frac{\pi}{2}\right) \tag{60}$$

where n_1 is an integer, and

$$\sum_{R=1}^{N_R} \theta_R = p_1 \frac{\pi}{2} \tag{61}$$

where p_1 is odd when N_p+N_z or N_p-N_z is odd, and, and p_1 is even when N_p-N_z is even. Substituting eqns. (60), (61) into (57), one has

$$l = 2n_1 + p_1 \tag{62}$$

where l is odd when p_1 is odd, and even when p_1 is even. From the statements above, m is even and l is odd when N_z-N_p is odd; m is odd and l is even when N_z-N_p is even. Therefore

$$\theta \neq \phi \tag{63}$$

for any point on the imaginary axis, which is not a singularity; Thus the root loci of $G(s)$ can't meet the imaginary axis except at the singularities.

Remarks:

From Theorem 1 and Lemma 7, it is eviden that any root locus in one half of s-plane can not cross the imaginary axis and enter into another half plane. Therefore, for a stable system the roots of $F_R(\omega)$ and $F_I(\omega)$ can not be complex, since in this case there will be at least one complete branch of the root loci in the RHP.

Lemma 8. If the characteristic equation has all roots in LHP, the poles and zeros of the stability equation must be simple and all on the imaginary axis.

Proof:

Assume the i^{th} pole is an m_0-multiple pole, the departure angles of this multiple pole are

$$\theta_{p_i} = (A_{p_i} - \phi \pm 2n\pi)m_0 \qquad n=0,\ 1, \ldots\ldots m_0-1 \tag{64}$$

where ϕ is defined as Eq. (55) and

$$A_{p_i} = ARG\left(\frac{\prod_{n=1}^{N_z^*}(p_i - z_n)j}{\prod_{\substack{n=1\\ n\neq i}}^{N_p^*}(p_i - k_n)j}\right) \tag{65}$$

N_z^* and N_p^* are the number of distinct zeros and poles

The difference of any two departure angles of neighbor loci is

$$\delta = \frac{2\pi}{m_o} \tag{66}$$

For $m_o \geq 2$, $\delta \leq \pi$. This indicates that it is impossible to put all of the loci starting from an m_o-multiple pole ($m_o \geq 2$) in one half of s-plane; *i.e.* if the characteristic equation has all roots in LHP, the poles of the stability equation must be simple. By the same reasoning, the zeros must also be simple. Complex singularities are impossible (Remark), therefore the poles and zeros must be simple and all on the imaginary axis for a stable system.

Lemma 9. If the characteristic equation has all roots in LHP, the poles and zeros must be simple and on the imaginary axis in alternative sequence.

Proof:

Assume the pole-zero configuration is not alternative, then there is at least one occurrence that two of the same class of singularities appear sequently. Consider the case that two poles come sequently, the upper one has the departure angle

$$\theta_{p_{i+1}}=A_{p_{i+1}}-\phi \tag{67}$$

and the lower one has

$$\theta_{p_i}=A_{p_i}-\phi \tag{68}$$

where A_{p_i} is defined as Eq. (25). From Eq. (25) we can find that

$$A_{p_{i+1}}=A_{p_i}-\pi \tag{69}$$

and

$$\theta_{p_{i+1}}=\theta_{p_i}-\pi \tag{70}$$

From last equation, the loci starting from these two poles can't be in the same half of s-plane. The result is counter to the assumption that the characteristic roots are all in LHP. The same conclusion is true for the case that two zeros appear sequently. Hence the poles and zeros must be in alternative sequence.

Lemma 10. If the characteristic equation has all roots in LHP, the departure and arrival angles of the poles and zeros of the stability equations must be all equal to π.

Proof:

From eqn. (12), the departure angle of p_i is

$$\theta_{p_i}=A_{p_i}-\phi \tag{71}$$

where A_{p_i} is defined as eqn. (58), *i.e.*

$$A_{p_i}=\sum_{\substack{R=1\\R\neq i}}^{Np+N_z-1}\theta_R=\frac{l}{2}\pi \tag{72}$$

where $l=(N_r+N_z-1)$. From eqns. (55) and (56)

$$\phi=m\frac{\pi}{2}=(N_z-N_p\pm 1)\frac{\pi}{2} \tag{73}$$

Therefore

$$\theta_{p_i}=\left(\frac{l-m}{2}\right)\pi \tag{74}$$

But in this case l is odd and m is odd when N_z-N_p is even; l is even and m is even when N_z-N_p is odd; thus $l-m$ is always even, and eqn. (74) may be written as

$$\theta_{p_i}=k\pi \tag{75}$$

where K is an integer. If $K=2n$, $n=0,1,2...$from Teeorem 1, the characteristic equation has a root locus in RHP. This is a counter conclusion of the assumption that all roots of the characteristic equation are in LHP. Hence $K=2n+1$ and if the principal value is taken as $K=1$,

$$\theta_{p_i}=\pi \qquad \text{for } i=1,\ldots\ldots\ldots\ldots N_p \tag{76}$$

The same reasoning gives that the arrival angle of any zero is

$$\theta_{z_i}=\pi \qquad \text{for } i=1,\ldots\ldots\ldots\ldots N_z \tag{77}$$

Theorem 2. If the characteristic equation as eqn. (1) is given with $a_n=\alpha_n+j\beta_n$, and $F_e(s)$ and $F_o(s)$ are chosen as eqns. (2) and (3), and the associated stability equations in ω-domain are eqns. (4) and (5), eqn. (1) has all roots in the LHP if and only if the poles (roots of $F_I(\omega)$) and zeros (roots of $F_R(\omega)$) are all real and occur in alternating sequences; and the departure (arrival) angles of the singularities are all equal to π.
proof:

(1) Sufficient: From Theorem 1, the conditions are sufficient for all roots of eqn. (1) in LHP.

(2) Necessary: From Lemma 10, if eqn. (1) has all roots in LHP, then the conditions must be true.

Corollary 2. The characteristic equation has all root in LHP, if and only if

(1) The poles and zeros of the stability equations are on the imaginary axis and occur in alternating sequences.

(2a) When $N_z=N_p$, coefficients of highest terms of $F_e(s)$ and $F_o(s)$ have the same (opposite) signs if the smallest singularity is a zero (pole).

(2b) If $N_z=N_p\pm 1$, coefficients of highest tenms of $F_e(s)$ and $F_o(s)$ have the same signs.

Note that corollary (2) is just an alternative statement of the conditions for having the departure or arrival angles of the singularities equal to π.

Corollary 3. The limit cycle of a nonlinear system can occur only when there is one pole and one zero coincide, and all the other poles and zeros construct a pattern which corresponding to the case that the other roots are all in the LHP (Han and Thaler 1968).

Corollary 4. The limit cycle predicted by corollary 3 is stable, if the pole-zero sequences of the system get to a stable configuration when the magnitude of the input signal to the nonlinear element is increased (Han and Thaler 1967).

Example 2. Consider the characteristic equation in Example 1. The stability equations are chosen as the real part and the imaginary part of $F(s)$, such as

$$F_e(s)=2s_1+js+4=0$$

$$F_o(s)=2js^2+3s+4j=0$$

which give

$$F_R(\omega)=-\omega^2-0.5\omega+2=0$$

$$F_I(\omega)=-\omega^2+1.5\omega+2=0$$

The zeros are at -1.69 and $+1.19$, and the poles are at 2.35 and -0.85. The pole-zero sequence is alternative; the smallest singularity is a zero and $N_z=N_p$; the signs of $F_e(s)$ and $F_o(s)$ are the same; thus all the characteristic roots are in LHP. The root loci are sketched in Fig. 3, in which the characteristic roots are found approximately at $-0.4-j1.1$ and $-0.59+j1.6$.

Example 3. Consider the following characteristic equation.

$$F(s)=(-2-4j)s^2+(-2+4j)s+1=0$$

As in Example 2,

$$F_P(\omega)=-2\omega^2-4\omega+1$$

$$F_I(\omega)=4\omega^2-2\omega$$

The zeros are at 1.707 and 0.293; the poles ane at 0.5 and 0. The pole-zero sequence is alternative, and $N_p=N_z$. The smallest singularity is a pole but the coefficients of highest terms of $F_e(s)$ and $F_o(s)$ are the same: thus all roots are in RHP. The root loci are sketched in Fig. 4, and the characteristic roots are found approximately at $0.1+j0.3$ and $0.5+j0.5$.

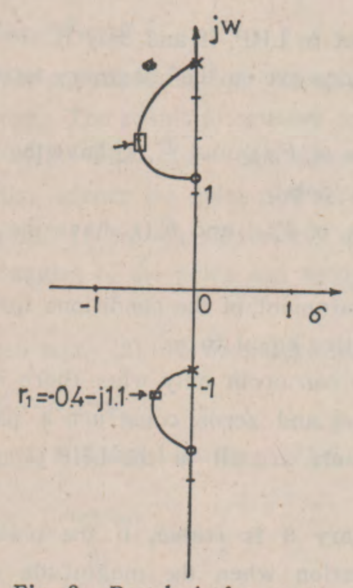

Fig. 3. Root loci of a second order equation.

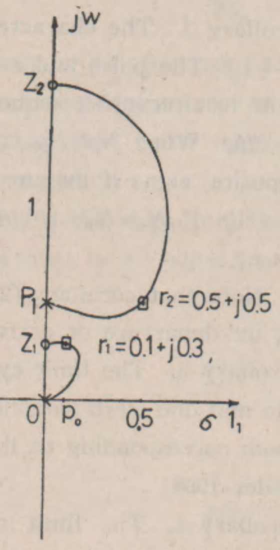

Fig. 4. Root loci of a second order equation.

4. CONCLUSIONS

The stability criteria presented in this paper are more general than that given in the references, and the proofs are more complete. The presented criteria are simple and only need to deal with real numbers. Although the main objective of this paper is to apply the criteria to test stability of nonlinear control systems with linearized characteristic equations having complex coefficients, the presented criteria may have applications to may engineering problems.

LIST OF SYMBOLS

a_i Coefficients of characteristic equation
A_s Phase angle of the complex gain K_s
A_ω Phase angle of the complex gain K_ω
$F(s)$ Characteristic equation
$F_e(s)$ Even part of characteristic equation
$F_o(s)$ Odd part of characteristic equation
K_ω Complex gain of $G(j\omega)$, $K_\omega=|K_\omega|e^{jA\omega}$
K_s Complex gain of $G(s)$, $K_s=|K_s|e^{jAs}$
N_R Number of real roots
N_σ Number of complex roots
N_p Number of poles
N_s Number of zeros
P_i Poles of $G(s)$
Z_i Zeros of $G(s)$
θp_i Departure angle of p_i
ϕ Phase angle of phase-angle loci

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