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廣義特徵值問題之研究
Localization of Eigenvalues of Generalized Eigenproblem

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Abstract — The localization of eigenvalues of the generalized eigenproblem Ax= λ Bx, where A and B are M matrices, is studied in this paper. We show that all eigenvalues of the aboue problem lie in an annulus centered at the origin in the complex plane, and the inner and outer radii of the annulus depend only on the moduli of the entries of the matrices A and B. Moreover, a detailed study of eigenvalues appearing on the boundary of the annulus is also being made.

I. Introduction

The generalized eigenproblem [2] can be formulated as

$$Ax = \lambda Bx \tag{1}$$

where A and B are n x n complex matrices. This eigenproblem generally occurs in the study of physical problems such as the vibration mode superposition analysis, the linearized buckling analysis and the heat transfer analysis etc. In most cases, it is impractical to solve the eigenvalues of the generalized eigenproblem by directly solving the equation

$$\det(\lambda B - A) = 0 \tag{2}$$

The objective of this paper is to study lower and upper bounds for eigenvalues of the generalized eigenproblem with A and B being M-matrices. The case where eigenvalues appear on the boundary is discussed in detail. At the end, we shall extend the theory of eigenvalues in §2 for single matrices to a class $\hat{\Omega}(A,B)$ of matrices.

II. Lower and Upper Bounds of the Eigenvalues

We shall adopt the notations $C^{n,n}$ $(R^{n,n})$ and $\mathbb{C}^n$ (R^n) to denote the set of all n x n matrices with complex (real) entries, and the complex (real) n-dimensional vector space of all column vectors $x=(x_1, x_2,..., x_n)^T$ with each component x_i a complex (real) number respectively.

Definition 1 The spectrum $\sigma(A, B)$ of the generalized eigenproblem (1) is the collection of all eigenvalues of (1) i.e.

$$\sigma(A,B) \equiv \{\lambda \mid \lambda \in \mathbb{C}, \det(\lambda B-A) = 0\} \tag{3}$$

It is convenient to set

$$\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} = \sum_j{}' a_{ij}$$

Definition 2 A matrix $A=(a_{ij})\in \mathbb{C}^{n,n}$ is said to be diagonally dominant if

$$|a_{ii}| \geq \sum_j{}'|a_{ij}| \tag{4}$$

for all $1\leq i \leq n$. An nxn matrix A is strictly diagonally dominant if strict inequality in (4) is valid for all $1\leq i\leq n$.

In 1931, Gerschgorin proposed a localization theorem for the eigenvalues of any nxn complex matrix. The following theorem is known as Gerschgorin's theorem [5].

Theorem 1 Let $A=(a_{ij})$ be an nxn complex matrix and $G_i=\{z \mid |z-a_{ii}|\leq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$, then the eigenvalues of A lie in $\bigcup_{i=1}^{n} G_i$. Moreover, if $J=\{1,2,\ldots n\}$ and $J_k\subset J$ consists of k elements, $S=\bigcup_{i\in J_k} G_i$ and $T=\bigcup_{i\in J-J_k} G_i$. If $S\cap T=\phi$, then S is isolated and contains exactly k eigenvalues of A. G_i, in the above theorem, is called the i-th Gerschgorin disk of A.

Using a method similar to the one in the proof of Gerschgorin's theorem, we obtain the following:

Theorem 2 Let A, $B\in\mathbb{C}^{n,n}$ with A arbitrary and B strictly diagonally dominant, then all the eigenvalues of the generalized eigenproblem (1) lie in the disk $G_1(z)=\{z \mid |z|\leq\lambda, z\in\mathbb{C}\}$ where $\tilde\lambda=\max_i \tilde\lambda_i$ and $\tilde\lambda_i=(|a_{ii}|+\sum_j{}'|a_{ij}|)/(|b_{ii}|-\sum_j{}'|b_{ij}|)$.

Proof: Let λ be any eigenvalue of the generalized eigenproblem (1), and x be an eigenvector of (1) corresponding to λ. We may normalize the vector x so that its largest component has modulus one. By definition,

$$a_{ii}x_i+\sum_j{}' a_{ij}x_j = \lambda b_{ii}x_i + \lambda \sum_j{}'b_{ij}x_j$$

$$(\lambda b_{ii}-a_{ii})x_i=\sum_j{}'(a_{ij}-\lambda b_{ij})x_j \tag{5}$$

In particular, if $|x_r|=1$, then

$$|\lambda b_{rr}-a_{rr}| \leq \sum_j{}'|a_{rj}-b_{rj}|\ |x_j|$$

which implies that

$$|\lambda b_{rr}-a_{rr}| \leq \sum_j{}'|a_{rj}-\lambda b_{rj}|$$

Applying the triangle inequality, we get

$$|\lambda|\,|b_{rr}|-|a_{rr}| \leq \sum_j{}'|a_{rj}|+|\lambda|\sum_j{}'|b_{rj}|$$

which is

$$|\lambda|\,(|b_{rr}|-\sum_j{}'|b_{rj}|)\leq |a_{rr}|+\sum_j{}'|a_{rj}|$$

Since B is strictly diagonally dominant, i.e. $|b_{rr}|-\sum_j{}'|b_{rj}| > 0 \ \forall\ r$, we have

$$|\lambda| \le \frac{|a_{rr}| + \sum_j{}' |a_{rj}|}{|b_{rr}| - \sum_j{}' |b_{rj}|} \equiv \tilde{\lambda}_r \tag{6}$$

Thus λ lies in the disk $\{z \mid |z| \le \tilde{\lambda}_r\}$. Since λ is arbitrary, it follows that all the eigenvalues of (1) lie in the union of the disks $\{z \mid |z| \le \tilde{\lambda}_i\}$, $1 \le i \le n$. And this union is $G_1(z)$.

We have seen in Theorem 2 that the eigenvalues of (1) are bounded above. A similar result for lower bounds of the eigenvalues of (1) also holds.

Theorem 3 Let $A, B \in C^{n,n}$ with A strictly diagonally dominant, then all the eigenvalues of the generalized eigenproblem (1) lie outside the disk $G_2(z) = \{z \mid z \in \mathbb{C}, |z| < \underset{\sim}{\lambda}\}$ where $\underset{\sim}{\lambda} = \min_i \underset{\sim}{\lambda}_i$ and $\underset{\sim}{\lambda}_i = (|a_{ii}| - \sum_j{}' |a_{ij}|)/(|b_{ii}| + \sum_j{}' |b_{ij}|)$.

Proof: Replacing (5) by

$$(a_{ii} - \lambda b_{ii})x_i = \sum_j{}' (\lambda b_{ij} - a_{ij})x_j ,$$

the proof of this theorem follows that of Thm. 1 similarly and we obtain

$$|\lambda| \ge \frac{|a_{rr} - \sum_j{}' |a_{rj}|}{|b_{rr} + \sum_j{}' |b_{rj}|} \equiv \underset{\sim}{\lambda}_r > 0 ,$$

i.e. any eigenvalue of (1) lie outside the disk $\{z \mid |z| < \underset{\sim}{\lambda}_r\}$. Thus all the eigenvalues of (1) lie outside $G_2(z)$.

The following corollary is obvious in view of the above two theorems.

Corollary 1 Let $A, B \in \mathbb{C}^{n,n}$, with A and B strictly diagonally dominant, then all the eigenvalues of $Ax = \lambda Bx$ lie in the annulus $G(z) = \{z \mid z \in \mathbb{C}, \underset{\sim}{\lambda} \le |z| \le \tilde{\lambda}\}$, where $\underset{\sim}{\lambda}$ and $\tilde{\lambda}$ are $\min_i \underset{\sim}{\lambda}_i$ and $\max_i \tilde{\lambda}_i$, respectively, $\underset{\sim}{\lambda}_i = (|a_{ii}| - \sum_j{}' |a_{ij}|)/(|b_{ii}| + \sum_j{}' |b_{ij}|)$ and $\tilde{\lambda}_i = (|a_{ii}| + \sum_j{}' |a_{ij}|)/(|b_{ii}| - \sum_j{}' |b_{ij}|)$.

We know that the eigenvalues of the simple eigenvalue problem are not changed by similarity transformation. The same result holds for the generalized eigenproblem, i.e.

$$D^{-1}ADx = \lambda D^{-1}BDx$$

has the same eigenvalues as that of

$$Ax = \lambda Bx$$

since

$$\det(\lambda D^{-1}BD - D^{-1}AD) = \det\{D^{-1}(\lambda B - A)D\} = \det(\lambda B - A). \tag{7}$$

The matrix B in Theorem 2 and the matrix A in Theorem 3 are both assumed to be strictly diagonally dominant. This assumption can be weakened and after a diagonal similarity transformation the conclusions of the theorems still hold. In order to show this, the following definition is necessary.

Definition 3 A matrix with all its entries equal to zero is called a null matrix and is denoted by $\mathbf{0}$. A matrix A is said to be nonnegative (positive), denoted by $A \geq \mathbf{0}$ ($>\mathbf{0}$), if each entry of A is nonnegative (positive).

Definition 4 A real nxn matrix $A=(a_{ij})$ with $a_{ij} \leq 0$ for all $i \neq j$ is an M-matrix, if A is nonsingular, and $A^{-1} \geq \mathbf{0}$.

Using the concept of M-matrix, one may relax the condition of 'strictly diagonally dominant'.

Theorem 4 Let $A \in \mathbb{C}^{n,n}$. If $B=(b_{ij})$ with $b_{ij} = -|a_{ij}|$ for $i \neq j$ and $b_{ii} = |a_{ii}|$ is an M-matrix, then there exists a positive diagonal matrix X, such that $X^{-1}AX$ is strictly diagonally dominant.

Proof: Let $X=\mathrm{diag}(x_1, x_2, \ldots, x_n)$ be a diagonal matrix with $x_i > 0$ for all $1 \leq i \leq n$.

After similarity transformation, we find

$$X^{-1}AX = (a_{ij}x_jx_i^{-1})_{nxn}$$

Now we want to find an X which is diagonal and positive such that $X^{-1}AX$ has the property that

$$|a_{ii}| > \sum_j{}' |a_{ij}| x_jx_i^{-1} \qquad 1 \leq i \leq n \tag{8}$$

then A is transformed to a matrix $X^{-1}AX$ which is strictly diagonally dominant.

Equation (8) is equivalent to

$$|a_{ii}| x_i > \sum_j{}' |a_{ij}| x_j \qquad 1 \leq i \leq n$$

which is the same as

$$|a_{ii}| x_i - \sum_j{}' |a_{ij}| x_j > 0$$

Let

$$|a_{ii}| x_i - \sum_j{}' |a_{ij}| x_j \equiv y_i > 0 \qquad 1 \leq i \leq n$$

The system of equation (9) can be written in the following form:

$$\begin{bmatrix} |a_{11}| & -|a_{12}| \cdots\cdots -|a_{1n}| \\ -|a_{21}| & |a_{22}| \cdots\cdots -|a_{2n}| \\ \vdots & \ddots \\ -|a_{n1}| & -|a_{n2}| \qquad |a_{nn}| \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} > 0$$

i. e. $BX=Y$

$$X = B^{-1}Y > 0$$

since B is an M-matrix, $B^{-1} \geq 0$. Thus we obtain the positive diagonal matrix X such that $X^{-1}AX$ is strictly diagonally dominant.

In the rest of this paper, A^c denotes the matrix B which is constructed from A in Theorem 4 i.e.

$$A^c = B = (b_{ij}) \text{ with } b_{ii} = |a_{ii}| \text{ and } b_{ij} = -|a_{ij}| \quad i \neq j .$$

III. Eigenvalues on the Boundary

All the eigenvalues of the generalized eigenproblem described in Theorem 2 and Theorem 3 lie inside some disks or outside some circles which have the origin as their centers, respectively. The problem which now arises is 'when will the eigenvalues be on the boundary?'

Definition 5 A matrix is said to be irreducible if there is no permutation matrix P such that

$$PAT^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

The concept of irreducibility may be interpreted by means of graph theory. We first consider some elementary notions. Consider any n points $P_1, P_2, \ldots P_n$ in the plane, we shall call them nodes. Let $A = (a_{ij}) \in \mathbb{C}^{n,n}$, for every nonzero entry a_{ij} of the matrix A, we say tht there is a path between node P_i and node P_j, directed from P_i to P_j, denoted by $\overrightarrow{P_i\ P_j}$. In this way, every $A \in \mathbb{C}^{n,n}$ can be associated with a finite directed graph G(A).

Definition 6 A directed graph is strongly connected if for any ordered pair of nodes P_i and P_j, there exists a direct path

$$\overrightarrow{P_iP_{\ell_1}}, \overrightarrow{P_{\ell_1}\ P_{\ell_2}}, \ldots\ldots \overrightarrow{P_{\ell_m}\ P_j}$$

connecting P_i and P_j.

The following Lemma 1 shows that the matrix property of irreducibility is equivalent to the condition that the graph associated with the matrix is strongly connected.

Lemma 1 A matrix $A \in \mathbb{C}^{n,n}$ is irreducible if and only if its directed graph is strongly connected.

With the concept of irreducibility, we can sharpen the result of Theorem 1 as follows.

Theorem 5 Let A, B $\in \mathbb{C}^{n,n}$ with A irreducible and B strictly diagonally dominant. If λ , an eigenvalue of the generalized eigenproblem $Ax = \lambda Bx$, is a boundary point on the set $\{ z \mid |z| = \tilde{\lambda} \}$, where $\tilde{\lambda} = \max_i \tilde{\lambda}_i$ and $\tilde{\lambda}_i = (|a_{ii}| + \sum_j{}' |a_{ij}|) / (|b_{ii}| - \sum_j{}' |b_{ij}|)$, then $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \ldots\ldots = \tilde{\lambda}_n$.

Proof: Suppose λ is an eigenvalue of the generalized eignproblem

$$Ax = \lambda Bx$$

and x is an eigenvector corresponding to λ. Normalize x such that $|x_r| = 1 \geq x_i$ for all $i \neq r$.

As in the proof of Theorem 2 we have

$$(\lambda b_{rr} - a_{rr}) x_r = \sum_j{}' (a_{rj} - \lambda b_{rj}) x_j$$

$$|\lambda| |b_{rr}| - |a_{rr}| \leq \sum_j{}' |a_{rj}| + |\lambda| \sum_j{}' |b_{rj}|$$

$$|\lambda| \leq \frac{|a_{rr}| + \sum_j{}' |a_{rj}|}{|b_{rr}| - \sum_j{}' |b_{rj}|} = \tilde{\lambda}_r \tag{10}$$

However, since λ is assumed to be a boundary point on the upper bound, equality in (10) must hold, and we have

$$|\lambda| \; |b_{rr}| - |a_{rr}| = \sum_j{}' (|a_{rj}| + |\lambda| |b_{rj}|) |x_j|$$

Consequently for all $a_{r\ell} \neq 0$ and $b_{rm} \neq 0$, $|x_\ell| = |x_m| = 1$.

Since A is irreducible, there exists at least one ℓ for which $a_{r\ell} \neq 0$, then

$$|\lambda b_{\ell\ell} - a_{\ell\ell}| \leq \sum_j{}' |a_{\ell j} - \lambda b_{\ell j}|$$

$$|\lambda| \; |b_{\ell\ell}| - |a_{\ell\ell}| \leq \sum_j{}' |a_{\ell j}| + |\lambda| \sum_j{}' |b_{\ell j}|$$

$$|\lambda| \leq \frac{|a_{\ell\ell}| + \sum_j{}' |a_{\ell j}|}{|b_{\ell\ell}| - \sum_j{}' |b_{\ell j}|} = \tilde{\lambda}_\ell \tag{11}$$

Since λ is a boundary point on the upper bound, the inequality of (11) is again an equality.

Combining (10) and (11), we have

$$|\lambda| = \tilde{\lambda}_r = \tilde{\lambda}_\ell$$

By repeating the above arguments and using the equivalent condition for the irreducibility of A given in Lemma 1, one can find a sequence of nonzero entries of matrix A of the form $a_{r\ell_1}, a_{\ell_1 \ell_2}, \ldots, a_{\ell_{m-1}, j}$ for any integer j, $1 \leq j \leq n$. Thus we conclude that all $\tilde{\lambda}_i$'s have the same value.

A similar result holds for lower bounds of eigenvalues of the generalized eigenproblem.

Theorem 6 Let A, B ε $\mathbf{C}^{n,n}$ with A strictly diagonally dominant, and B irreducible. If λ, an eigenvalue of (1), is a boundary point of lower bound $\{ z \mid |z| = \underset{\sim}{\lambda} \}$, then all $\underset{\sim}{\lambda}_i$ are equal.

The proof for Theorem 6 is similar to that of Theorem 5, we omit it.

The converses of Theorem 5 and Theorem 6 are not valid, we have the following example which shows this.

Example 1

$$\begin{pmatrix} 8 & 4 & 0 \\ 1 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} X = \lambda \begin{pmatrix} 14 & 2 & 0 \\ 2 & 12 & 2 \\ 1 & 0 & 7 \end{pmatrix} X$$

We may calculate the lower and upper bounds directly, and find

$$\tilde{\lambda} = \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = 1$$

$$\underset{\sim}{\lambda} = \underset{\sim}{\lambda}_1 = \underset{\sim}{\lambda}_2 = \underset{\sim}{\lambda}_3 = 1/4$$

but the eigenvalues of the above eigenproblem are

$$\lambda_1 = 1/2, \qquad \lambda_2 = \frac{1}{576}(314 + \sqrt{476}\ i)\ ,\ \lambda_3 = \frac{1}{576}(314 - \sqrt{476}\ i)$$

which are calculated from the equation

$$\det(\lambda B - A) = \begin{vmatrix} 14\lambda - 8 & 2\lambda - 4 & 0 \\ 2\lambda - 1 & 12\lambda - 6 & 2\lambda - 1 \\ \lambda - 1 & -1 & 7\lambda - 4 \end{vmatrix} = 0$$

We have seen that all $\tilde{\lambda}_i$ are equal to 1, but there is no eigenvalue lying on the outer boundary. We also have all $\underset{\sim}{\lambda}_i$ equal to 1/4, but there is no eigenvalue lying on the circle with center 0 and radius 1/4.

Even though the converses of Theorem 5 and Theorem 6 are not valid, in general, they are true if one puts some restrictions on the matrices in the eigenproblem.

IV. Nonnegative Matrices and Improvement on Bounds

In this section, we shall investigate conditions for which the converses of Theorem 5 and Theorem 6 are valid. i.e. For some pairs of matrices if $\tilde{\lambda}_i$ ($\underset{\sim}{\lambda}_i$) are equal for all i, then the maximal (minimal) eigenvalue lie on the outer (inner) boundary. We denote the maximal (minimal) eigenvalue of the generalized eigenproblem (1) by ρ (A, B) (γ (A,B)).

$$\begin{aligned} \rho(A,B) &= \max\{|\lambda| \mid \lambda \in \sigma(A,B)\} \\ \gamma(A,B) &= \min\{|\lambda| \mid \lambda \in \sigma(A,B)\} \end{aligned} \tag{12}$$

We develope the theory by means of Wielandt's lemma [1].

Definition 7 Let $A = (a_{ij})$ and $B = (b_{ij})$ be two nxn matrices. Then $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all $1 \leq i \leq n$, $1 \leq j \leq n$.

Let $C = (c_{ij}) \in \mathbb{C}^{n,n}$, then $|C|$ denotes the matrix with entries $|c_{ij}|$.

Lemma 2 [Wielandt] [1] Let $A \in \mathbb{C}^{n,n}$ be nonnegative and irreducible, and let $B \in \mathbb{C}^{n,n}$ with $|B| \leq A$. If β is any eigenvalue of B, then

$$|\beta| \leq \rho(A). \tag{13}$$

Moreover, equality in (13) is valid, i.e. $\beta = \rho(A) e^{i\phi}$, if and only if $|B| = A$, where B has the form

$$B = e^{i\phi} DAD^{-1} \tag{14}$$

and with D a diagonal matrix whose diagonal entries have modulus unity.

Lemma 3 shows that irreducibility of a nonnegative matrix is preserved when it is multiplied by the inverse of an M-matrix.

Lemma 3 Let $A, B \in \mathbb{R}^{n,n}$, if $A \geq \mathbb{O}$ is irreducible and B is an M-matrix, then $B^{-1}A \geq \mathbb{O}$ and irreducible.

Proof: Since B is an M-matrix, B is nonsingular and $B^{-1} \geq \mathbb{O}$ so $B^{-1}A \geq \mathbb{O}$.

Now, suppose $B^{-1}A$ is reducible, and for convenience we denote $D \equiv B^{-1}$ and $C \equiv B^{-1}A = DA$.

By the definition of reducible matrix, there exists a permutation matrix P, such that

$$PCP^T = \begin{pmatrix} C_{11} & C_{12} \\ \mathbb{O} & C_{22} \end{pmatrix}$$

where C_{11} is a rxr block of PCP^T, C_{22} a (n-r)x(n-r) block of PCP^T and C_{12} a (n-r)xr block.

$$PCT^T = PB^{-1}AP^T = PDAP^T = PDP^TPAT^T$$

Write PDP^T and PAP^T in the same block form as PCP^T

$$PDP^TPAP^T = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} C_{11} & C_{12} \\ \mathbb{O} & C_{22} \end{pmatrix}$$

Thus D_{11} and A_{11} are rxr blocks of PDP^T and PAP^T respectively, D_{22} and A_{22} are (n-r)x(n-r) blocks, D_{12} and A_{12} are (n-r)xr blocks.

We must have

$$D_{21}A_{11}+D_{22}A_{21} = \mathbb{O},$$

That is,

$$(D_{21} \;\; D_{22})_{(n-r)xn} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}_{nxr} = \mathbb{O}$$

Since A is irreducible, there is at least one nonzero entry in each row and each column. Equality (15) holds only when $(D_{21} \;\; D_{22})_{(n-r)xn} = \mathbb{O}_{(n-r)xn}$, however this contradicts with the nonsingularity of D. We conclude that $B^{-1}A$ is irreducible.

Basing on the above two lemmas, we have

Lemma 4 Let $A \geq \mathbf{0}$ be an irreducible nxn matrix, $B \in \mathbb{R}^{n,n}$ is an M-matrix, and let $C \in \mathbb{C}^{n,n}$ with $|C| \leq A$. If is any eigenvalue of the eigenproblem $Cx=Bx$,then

$$|\nu| \leq \rho(A, B). \tag{16}$$

Moreover, equality is valid in (16), if and only if C is of the form

$$C = e^{i\phi} DAD^{-1}$$

where $\quad |D| = I.$

Proof: Since $A \geq \mathbf{0}$ is irreducible and B is an M-matrix, multiplying equation

$$Ax = \lambda Bx$$

through by B^{-1} gives

$$B^{-1}Ax = \lambda x$$

That is

$$\rho(A, B) = \rho(B^{-1} A) \tag{17}$$

Moreover, $B^{-1}A \geq \mathbf{0}$ is irreducible by Lemma 3, and by assumption

$$|C| \leq A, \text{ thus}$$

$$B^{-1} |C| \leq B^{-1}A$$

Since $B^{-1}A \geq 0$ is irreducible and $B^{-1} |C| \leq B^{-1}A$, we may apply Wielandt's lemma to conclude that

$$|\nu| \leq \rho(B^{-1}A) = \rho(A, B)$$

and equality holds if and only if

$$C = e^{i\phi} DAD^{-1}$$

where $\quad |D| = I.$

Lemma 4 describes the lower bound for $\rho(A, B)$, where A is a nonnegative and irreducible matrix, and B is an M-matrix. From Theorem 4, we know that an M-matrix can be transformed to a strictly diagonally dominant matrix by positive diagonal similarity transformation. For convenience we now use a stronger condition 'strictly diagonally dominant M-matrix'. Combining Lemma 4 with Theorem 2, we get

Lemma 5 Let A, B $\varepsilon\ \mathbb{R}^{n,n}$ with A nonnegative and irreducible and B a strictly diagonally dominant M-matrix.

(1) If $\tilde{\lambda}_i$ are all equal, then $\rho\,(A, B) = \tilde{\lambda}_i = \tilde{\lambda}$

(2) otherwise $\min_i \tilde{\lambda}_i < \rho\,(A, B) < \max_i \tilde{\lambda}_i$

Proof: First, suppose that all $\tilde{\lambda}_i$ are equal to $\tilde{\lambda}$.

If ξ is the column vector with all components equal to unity, then it is obvious that

$$A\ \xi = \lambda B \xi$$

Hence we have

$$\tilde{\lambda} \leq \rho(A, B)$$

Theorem 1 showed that $\rho(A, B) \leq \tilde{\lambda}$, thus we conclude that

$$\rho\,(A, B) = \tilde{\lambda}$$

which proves part (1).

Secondly, if $\tilde{\lambda}_i$ are not all equal, we can construct a nonnegative irreducible matrix A' by decreasing certain positive entries of A, so that

$$\tilde{\lambda}_i' = \frac{a_{ii} + \sum_j' a'_{ij}}{|b_{ii}| - \sum_j' |b_{ij}|} = \min_k \tilde{\lambda}_k \text{ for all i.}$$

Then $A \geq A' \geq 0$ and $A \neq A'$. As all $\tilde{\lambda}_i$ are equal to $\min_k \tilde{\lambda}_k$, we may apply the result of part (1) of this lemma to $\rho\ (A', B)$.

Thus $\qquad \rho\,(A', B) = \min_i \tilde{\lambda}_i$

By the result of Lemma 4, we must have

$$\rho\,(A', B) < \rho(A, B)$$

So that $\qquad \min_i \tilde{\lambda}_i < \rho\,(A,B)$

On the other hand, we can construct a nonegative irreducible matrix A'' by increasing certain positive entries of A. So that $A'' \geq A$, and $A'' \neq A$. Thus all $\tilde{\lambda}_i''$ are equal to $\max_k \tilde{\lambda}_k$ and therefore

$$\rho\,(A, B) < \rho\,(A'', B) = \max_i \tilde{\lambda}_i$$

These inequalities are combined and give

$$\min_i \tilde{\lambda}_i < \rho\,(A, B) < \max_i \tilde{\lambda}_i$$

which completes the proof.

Lemma 5 answered the question on eigenvalues on the boundary which was raised in the last paragraph of § 3. There exists a similar result for minimal eigenvalues.

Lemma 6 Let A, B $\in \mathbb{R}^{n,n}$ with A a strictly diagonally dominant M-matrix and B nonnegative and irreducible. If $\underset{\sim}{\lambda}_i$ are all equal, then $\gamma(A, B)= \underset{\sim}{\lambda}_i$, otherwise $\min_i \underset{\sim}{\lambda}_i < \gamma(A, B) < \max_i \underset{\sim}{\lambda}_i$.

The proof of Lemma 6 is the same as that of Lemma 5 and is omitted.

As a consequence of Lemma 6, we have the following corollaries by which one may test whether $\lambda = 0$ is an eigenvlaue of the generalized eigenproblem or not.

Corollary 2 Let A, B $\in \mathbb{R}^{n,n}$ with A an M-matrix and B irreducible and nonnegative, then $\lambda = 0$ is an eigenvlue of the generalized eigenproblem (1) if and only if $\sum_{k=1}^{n} a_{ik}=0$ for all i = 1, 2, ...n.

Definition 8 A matrix $A=(a_{ij}) \in \mathbb{C}^{n,n}$ is said to be irreducibly diagonally dominant if A is irreducible and diagonally dominant with strict inequality in (4) for at least one i.

Corollary 3 Let A, B $\in \mathbb{R}^{n,n}$ with B nonnegative and irreducible. If A is strictly diagonally dominant or irreducibly diagonally dominant, then $\lambda=0$ is not an eigenvalue of the generalized eigenproblem (1).

Proof: The case where A is strictly diagonally dominant is obvious.

Thus, suppose A is irreducibly diagonally dominant, and suppose $\lambda=0$ is an eigenvlue, then $\underset{\sim}{\lambda}_i=0$ for some i, and thus $\underset{\sim}{\lambda}_i=0$ for all i=1,2,...,n, but this contradicts with the assumption that A is irreducibly diagonally dominant. Thus $\lambda=0$ is not an eigenvalue of (1).

From (7), we know that the eigenvalues of the generalized eigenproblems are not changed by similarity transformation. We improve the bounds for the eigenvalues of (1) as follows.

Theorem 7 Let A, B $\in \mathbb{R}^{n,n}$ with A nonnegative and irreducible and B an M-matrix, and let $P_1^* = \{ x \in \mathbb{R}^n | x > 0$ and $X^{-1}(x)BX(x)$ is strictly diagonally dominant where $X(x)=\mathrm{diag}(x_1, x_2,...,x_n)$ is a diagonal matrix $\}$. Then for any $x \in P_1^*$, either

$$\min_i \tilde{\lambda}_i(x) < \rho(A, B) < \max_i \tilde{\lambda}_i(x) \tag{18}$$

or

$$\tilde{\lambda}_i(x) = \rho(A,B) \quad \text{for all } 1 \le i \le n, \text{ where} \tag{19}$$

$$\tilde{\lambda}_i(x) = \frac{|a_{ii}| + \sum_j' |a_{ij}| x_j x_i^{-1}}{|b_{ii}| - \sum_j' |b_{ij}| x_j x_i^{-1}} .$$

Moreover, $\sup_{x \in P_1^*} \{ \min_i \lambda_i(x) \} = \rho(A, B) = \inf_{x \in P_1^*} \{ \max_i \tilde{\lambda}_i(x) \}$.

Proof: For any positive vector $x \in P_1^*$, consider the equation

$$X^{-1}(x)AX(x)z = \lambda X^{-1}(x)BX(x)z \tag{20}$$

From (7) we know that (20) has the same eigenvalues as $Az = \lambda Bz$.

It is clear that $X^{-1}(x)AX(x)$ is nonnegative and irreducible and $X^{-1}(x)BX(x)$ is a strictly diagonally dominant matrix. Thus (18) and (19) follow directly form Lemma 5.

We certainly have from (18) and (19) that

$$\sup_{x \in P^*}\{\min_i \lambda_i(x)\} \leq \rho(A, B) \leq \inf_{x \in P_1^*}\{\max_i \tilde{\lambda}_i(x)\}, \tag{21}$$

From (7) and Perron-Frobenius theorem [3] we know that if y is the eigenvector corresponding to the spectral radius $\rho(A, B)$ then $y > 0$, and by Lemma 5 we know $y \in P_1^*$. Choosing the positive vector $y \in P_1^*$ corresponding to the eigenvalue $\rho(A, B)$ shows that equality is valid in (21), which completes the poof.

A similar result for improving the lower bounds of the eigenvalues is as follows.

Theorem 8 Let $A, B \in \mathbb{R}^{n,n}$ with A an M-matrix and B nonnegative and irreducible, and let $P_2^* = \{x \in \mathbb{R}^n \mid x > 0,$ $X^{-1}(x)AX(x)$ be diagonally dominant where $X(x) = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ is a diagonal matrix$\}$. Then for any $x \in P_2^*$, either

$$\min_i \underset{\sim}{\lambda}_i(x) < \gamma(A, B) < \max_i \underset{\sim}{\lambda}_i(x) \tag{22}$$

or $\underset{\sim}{\lambda}_i(x) = \gamma(A, B)$ for all $i = 1, 2, \ldots, n$, where

$$\underset{\sim}{\lambda}_i(x) = \frac{|a_{ii}| - \sum_j{}' |a_{ij}| x_j x_i^{-1}}{|b_{ii}| + \sum_j{}' |b_{ij}| x_j x_i^{-1}}. \tag{23}$$

Moreover, $\sup_{x \in P_2^*}\{\min_i \underset{\sim}{\lambda}_i(x)\} = \gamma(A, B) = \inf_{x \in P_2^*}\{\max_i \underset{\sim}{\lambda}_i(x)\}$.

V. The Extended Set Ω (A, B)

Now, we make some extensions of the previous results. We observe that the upper bounds λ_i and lower bounds $\underset{\sim}{\lambda}_i$ in Theorem 2 and Theorem 3, respectively, depend only on the modulii of the entries of A and B. Consequently, all pairs of equimodular matrices (A, B) have the same bounds for the eigenvalues of the associated generalized eigenproblem $Ax = \lambda Bx$. We consider the generalized eigenproblem (1) under the assumption that A and B are strictly diagonally dominant matrices. The above arguments imply that the eigenvalues of the class of eigenproblems

$$Cx = \lambda Dx \text{ with } |C| = |A| \text{ and } |D| = |B|$$

all lie in the annulus with inner and outer radii given by Corollary 1. It is logical to make the following definition [4].

Definition 9 Let $C, D \in \mathbb{C}^{n,n}$, we say $(C, D) \in \Omega(A, B)$ if $|C| = |A|$ and $|D| = |B|$. i.e.

$$\Omega(A, B) = \{(C, D) \mid C, D \in \mathbb{C}^{n,n} \text{ with } |C| = |A| \text{ and } |B| = |D|\}.$$

The assumption that both A and B are strictly diagonally dominant matrix can be relaxed to both A^c and B^c are M-matrices by considering Theroem 4. Denote the set of positive column vectors x, such that $X^{-1}(x)AX(x)$ or $X^{-1}(x)BX(x)$ is strictly diagonally dominant by P^*, i.e.

$$P^* = \{x \in \mathbb{R}^n \mid x > 0, \text{ and } X^{-1}(x)AX(x) \text{ or } X^{-1}(x)BX(x) \text{ is strictly diagonally dominant}\}$$

We now restrict our attention to the eigenproblem $CZ = \lambda\ DZ$ where $(C, D) \in \Omega\ (A, B)$. For any $x \in P_2^*$, consider the equation $X^{-1}(x)CZ(x)Z = \lambda\ X^{-1}(x)DX(x)Z$ and denote $\min_i\ \underset{\sim}{\lambda}_i(x)$ by $\underset{\sim}{\lambda}_x(C, D)$.

Let $G(x_1, x_2)$ denote the annulus in which the eigenvalues lie. After similarity transformation by $X(x_1)$ and $X(x_2)$, $G(x_1, x_2)$ can be written as follows:

$$G(x_1, x_2) = \{z \mid \underset{\sim}{\lambda}_{x_2}(\Omega(A, B)) \leq |z| \leq \tilde{\lambda}_{x_1}(\Omega(A, B))\} \tag{24}$$

where

$$\underset{\sim}{\lambda}_{x_2}(\Omega(A, B)) = \min_{\substack{x \in P_2^* \\ (C, D) \in \Omega(A,B)}} \underset{\sim}{\lambda}_x(C, D) \tag{25}$$

and

$$\tilde{\lambda}_{x_1}(\Omega(A, B)) = \max_{\substack{x \in P_1^* \\ (C,D) \in \Omega(A, B)}} \tilde{\lambda}_i(x) \tag{26}$$

where P_1^* is defined in Théorem 7 and P_2^* is defined in Theorem 8.

It is clear that

$$\sigma(\Omega(A, B)) \subseteq \bigcap_{\substack{x_1 \in P_1^* \\ \text{and } x_2 \in P_2^*}} G(x_1, x_2) \subseteq \bigcap_{x_1, x_2 \in P^*} G(x_1, x_2) \tag{27}$$

We make further extension by defining

$$\hat{\Omega}(A, B) = \{ (C, D) \mid C, D \in \mathbb{C}^{n,n},\ |c_{ij}| \leq |a_{ij}|,\ |b_{ij}| \geq |d_{ij}| \ i \neq j \text{ and } |c_{ii}| = |a_{ii}|,\ |b_{ii}| = |d_{ii}|\ \forall_i \}$$

and let $\tilde{\lambda}_x(\hat{\Omega}(A, B))$ and $\underset{\sim}{\lambda}_x(\hat{\Omega}(A, B))$ denote the upper and lower bounds of eigenvalues of generalized eigenproblem of class $\hat{\Omega}(A, B)$ after similarity transformation by $X(x)$.

It is easy to verify taht

$$\tilde{\lambda}_x(\hat{\Omega}(A, B)) \leq \tilde{\lambda}_x(\Omega(A, B)) \text{ for all } x \in P_1^*$$

and

$$\underset{\sim}{\lambda}_x(\hat{\Omega}(A, B)) \geq \underset{\sim}{\lambda}_x(\Omega(A, B)) \text{ for all } x \in P_2^*$$

Hence, we have

Theorem 9

$$\sigma(\hat{\Omega}(A,B)) \subseteq \bigcap_{\substack{x_1 \in P_1^* \\ x_2 \in P_2^*}} G(x_1, x_2) \subseteq \bigcap_{x_1, x_2 \in P^*} G(x_1, x_2)$$

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