The Journal of National Chiao Tung University
Vol. I, April 1976
pp.167-174

哥氏空間之準補餘子空間

On Quasi-Complemented Subspace of Grothendieck Spaces

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(Received January 16, 1976)

ABSTRACT —Let Z be a quasi-complemented subspace of a Grothendieck space X with quasi-complement Y such that the unit ball of Z* is weak*-sequentially compact, then Z is quasi-complemented in X** and Y**/Y is isomorphic to X**/X. If a Grothendieck space has a fundamental biorthogonal system, then it possesses a separable quasi-complemented subspace.

Introduction

Let Z be a subspace of a Banach space X. Z is said to ge quasi-complemented in X if there exists a subspace $Y \subset X$ such that $Z \cap Y = \{0\}$ and Z+Y is dense in X. Y is then called a quasi-complement of Z. For a detailed discussion of quasi-complemented subspaces in a weakly compactly generated (WCG) Banach space, we refer to Mackey [11], Murray[12] and Lindenstrauss [7,8]. Conclusively, it has been shown that every subspace of a WCG space is quasi-complemented.

In [13, p.186], Rosenthal established a quasi-complementation criterion for a subspace A whose dual space is weak*-separable. More precisely, for such a subspace A of a Banach space, A is quasi-complemented if its annihilator contains a reflexive subspace. From this criterion, an excellent exposition of quasi-complementation theorems in Grothendieck spaces has been developed. When a quasi-complemented subspace Z of a Grothendieck space X has the property that the unit ball of Z* is weak*-sequentially compact, the situation owns its particular interest. As a result of this paper, Z is then quasi-complemented in X**. If Y is a quasi-complement of Z, we derive that Y**/Y~X**/X. It is unknown whether X** is a Grothendieck space if X is. We remark here that if in addition X**/X is also a Grothendieck space, then the answer is affirmative[6]. In this case, Y** is a Grothendieck space.

In [1], Amir and Lindenstrauss proved that a Banach space X is

WCG if and if the unit ball B_{X^*} of X^* is affine homeomorphic to a weakly compact set (and hence by Eberlein's Theorem, B_{X^*} is weak*-sequentially compact). Occationally, for the purpose of application, one only needs to know that B_{X^*} is weak*-sequentially compact. Thus we shall provide a rather simple proof for this fact. As an observation from the proof, a Banach space has a separable quasi-complemented subspace if it has a WCG quotient space.

Finally, in connection with the open problems that whether every Banach space has a fundamental biorthogonal system and whether every Grothendieck space has a separable quasi-complemented subspace, we have the following implication: If a Grothendieck space has a fundamental biorthogonal system, then it possesses a separable quasi-complemented subspace.

1. It is an open question whether every Banach space has a separable quasi-complemented subspace. A Banach space X possesses this property if and only if X has a WCG quotient space. This can be shown by using the fact that every separable subspace of a WCG space is quasi-complemented. However, we shall give an elementary proof.

LEMMA 1.1. Let X be a WCG Banach space. Then the dimension of X is equal to the smallest cardinality m for which there is a total subset of X^* of cardinality m.

PROOF. [9, Prop. 2.2].

PROPOSITION 1.2. Let X be a WCG Banach space. Then:

(i) There exists a separable quotient space of X.

(ii) The unit ball B_{Y^*} of Y^* is weak*-sequentially compact for any subspace $Y \subset X$.

PROOF. Let (x_n^*) be a sequence in X^* and $\tilde{Y}$ be the weak* closed subspace spanned by (x_n^*). Then

$$\tilde{Y} = ((x_n^*)^{\top})^{\perp} \sim (X/(x_n^*)^{\top})^*.$$

Note that since $(X/(x_n^*)^{\top})^*$ is weak*-isomorphic to $\tilde{Y}$, $(X/(x_n^*)^{\top})^*$ is weak*-separable. By Lemma 1.1, $X/(x_n^*)^{\top}$ is then separable; which proves (i).

If $Y \subset X$, then B_{Y^*} is a weak* continuous image of B_{X^*}. Thus, to prove (ii), it suffices to show that B_{X^*} is weak*-sequentially compact. Let (x_n^*) be a bounded sequence in X^*. From the proof of (i), we know that $X/(x_n^*)^{\top}$ is separable. Thus the unit ball of $(X/(x_n^*)^{\top})^*$ is metrizable in its weak* topology, and hence it is weak*-

sequentially compact. By the weak*-isomorphism between $(X/(x_n^*)^T$ and $\tilde{Y}$, we conclude that the bounded sequence $(x_n^*) \subset \tilde{Y}$ has a weak* convergent subsequence. Our argument shows that every bounded sequence in X* has a weak* convergent subsequence. This is equivalent to saying that $B_X{}^*$ is weak*-sequentially compact.

COROLLARY 1.3. A Banach space X has a separable quasi-complemented subspace if and only if it has a WCG quotient space.

PROOF. Assume X has a WCG quotient space. Then by Proposition 1.2 (i), a WCG space has a separable quotient space, hence X itself has a separable quotient space, say X/Z. It follows that there exists a separable subspace Y such that Y + Z is dense in X. By a result of Mackey [11], $Y \cap Z$ is quasi-complemented in Y. Denote by W its quasi-complement in Y. We have that $(Y \cap Z)+W+Z = W + Z$ is dense in X and $W \cap Z=\{0\}$. Thus W is a separable quasi-complemented subspace of X.

REMARK. A recent result of Davis,Figiel,Johson and Pelczynski [3] shows that a Banach space X is WCG if and only if there exists a reflexive space R and a one-to-one operator $T:R \to X$ with T(R) dense in X. Using this theorem,Lemma 1.1 follows readily. Indeed,$T^*:X^* \to R^*$ has a weak*-dense image, which is the same as weakly dense image in a reflexive space. Thus the weak* density character of X* is no less than the density character of R. But since the density character of R is no less than that of X, the assertion of Lemma 1.1 is verified. (ii) of Proposition 1.2 also follows from above theorem. For if X admits a continuous linear dense image of a Banach space X_1 such that $B_{X_1}{}^*$ is weak*-sequentially compact then $B_X{}^*$ has the same property.

2. In this section, we denote X a Grothendieck space and Z a subspace of X such that B_{Z^*} is weak*-sequentially compact. Recall that when X is a non-reflexive continuous linear image of a C(S) space, where S is an F-space, then l^∞ is a continuous linear image of X. Thus by Rosenthal's theorem [THEOREM 2.1, 13], every subspace of X whose dual space is weak*-separable is quasi-complemented. In particular, every subspace of l^∞ and $L^\infty[0,1]$ is quasi-complemented. Many of the quasi-complementation theorems rely on the way in which a subspace is imbedded [Sec. 2, 13]. However, we remark here that in the specific Grothendieck space $l^\infty(\Gamma)$, the quasi-complementation of subspaces is "essentially" a linear isomorphic property in the following sense.

PROPOSITION 2.1. Let Y be a quasi-complemented subspace of $l^\infty(\Gamma)$ such that $\dim Y < \dim l^\infty(\Gamma) = 2^{\text{card}\,\Gamma}$. Suppose Y_1 is a subspace isomorphic to Y; then Y_1 is quasi-complemented.

PROOF. Let $T: Y \to Y_1$ be a isomorphism from Y onto Y_1. Then since $\dim Y < \dim l^\infty(\Gamma)$, by a result of [10, p.238], there exists an automorphism $\tilde{T}: l^\infty(\Gamma) \to l^\infty(\Gamma)$ which extends T. Thus if W is a quasi-complement of Y in $l^\infty(\Gamma)$, $\tilde{T}(W)$ is a quasi-complement of Y_1

The following corollary is known at present as we indicated at the beginning of this section. While as an application of above proposition, one can deduce it without employing the rather strong result of Rosenthal that l^∞ is a continuous linear image of every non-reflexive quotient space of a C(S) space with S an F-space.

COROLLARY 2.2. If Y is a subspace of $l^\infty(\Gamma)$, card $\Gamma \geq c$, such that Y* is weak*-separable; then Y is quasi-complemented. The assertion also holds in case Y is a separable subspace of l^∞.

PROOF. By Corollary 2.3 of [13], Y is isomorphic to a quasi-complemented subspace of $l^\infty(\Gamma)$. But since Y* is weak*-separable, $\dim Y \leq c$. Thus it follows from Proposition 2.1 that Y is itself quasi-complemented.

As an application to Banach space geometry, we observe that.

COROLLARY 2.3. Suppose that Z is a Banach space that Z* is weak*-separable and that the unit ball of Z* is weak*-sequentially compact. Then Z has an equivalent strictly convex norm.

PROOF. Embed Z into a suitable Banach space $l^\infty(\Gamma)$. Since Z* is weak*-separable, Z is quasi-complemented in $l^\infty(\Gamma)$ with quasi-complement Y. Let $\pi : Z \to l^\infty(T)/Y$ be the restriction to Z of the quotient map $\phi: l^\infty(\Gamma) \to l^\infty(\Gamma)/Y$. Observe that since π has a dense image and the unit ball of Z* is assumed to be weak*-sequentially compact, the unit ball of $(l^\infty(\Gamma)/Y)^*$ has the same property. But $l^\infty(\Gamma)/Y$ is a Grothendieck space, it then must be reflexive. The map π is also one-to-one, hence the assertion follows from a theorem by Day [4].

Before we proceed our main result, recall that if Z is a quasi-complemented subspace of a Grothendieck space X such that the unit ball of X* is weak*-sequentially compact (In particular, when Z is WCG) then every quasi-complement of Z in X is a Grothendieck space [6].

THEOREM 2.4. Let X be a Grothendieck space and Z be a quasi-complemented subspace of X with quasi-complement Y such that the unit ball of Z* is weak*-sequentially compact. Then Z is quasi-complemented in X**. If in addition, X** is also a Grothendieck space, then Y** is a Grothendieck space.

PROOF. Let $\phi|_Z : Z \to X/Y$ be the restriction of the quotient map ϕ of X into X/Y. $\phi|_Z$ is then injective and has dense image. Thus the adjoint map $\phi|_Z{}^* : (X/Y)^* \to Z^*$ has weak* dense image and is injective also. Let B be the unit ball of $(X/Y)^*$, B is then weak* compact. Now since $\phi|_Z{}^*$ is injective and weak* continous, B and $\phi|_Z{}^*(B)$ are homeomorphic. By the hypothesis that the unit ball of Z* is weak* sequentially compact, we conclude that B has the same property. But X/Y is a Grothendieck space, It then follows that X/Y must be reflexive.

Consider now the diagram:

$$\begin{tikzcd}
 & X^{**}/Y^{\perp\perp} \arrow[dr, "\widetilde{\phi}^{**}"] & \\
X^{**} \arrow[ur, "\psi"] \arrow[rr, "\phi^{**}"] & & (X/Y)^{**} \\
X \arrow[u, "\kappa"] \arrow[rr, "\phi"] & & X/Y \arrow[u, "J"']
\end{tikzcd}$$

where ϕ, ψ are quotient maps; κ, J are the canonical imbeddings and $\widetilde{\phi}^{**}$ is the induced isometric isomorphism of $X^{**}/Y^{\perp\perp}$ onto $(X/Y)^{**}$.

Note that ϕ^{**} is a linear extension of ϕ, hence $\psi\kappa = \widetilde{\phi}^{**-1} J \phi$. Now since $\phi(Z)$ is dense in the reflexive space X/Y, $\psi\kappa(Z) = \widetilde{\phi}^{**-1} J \phi(Z)$ is dense in $X^{**}/Y^{\perp\perp}$, from which in turn it follows that $\kappa Z + Y^{\perp\perp}$ is dense in X**.

Also note that κZ and $Y^{\perp\perp}$ are disjoint. Indeed if $\kappa z \in \kappa Z \cap Y^{\perp\perp}$ for some $z \in Z$, then $\kappa z(x^*) = x^*(z) = 0$ for all $x^* \in Y^{\perp}$, i.e., $z \in (Y^{\perp})^{T} = Y$. It follows that $z=0$ for Z and Y are disjoint. Thus κZ is quasi-complemented in X** with quasi-complement $Y^{\perp\perp}$.

The assertion that Y** is a Grothendieck space follows easily from the fact that Y** is isometrically isomorphic to $Y^{\perp\perp}$.

COROLLARY 2.5. Let X,Y,Z be as in Theorem. Then $Y^{**}/Y \sim X^{**}/X$.

PROOF. From the proof of above Theorem, we know that Z is quasi-complemented in X** with quasi-complement $Y^{\perp\perp}$. Hence in particular, $Z + Y^{\perp\perp}$ is dense in X**. But a result of civin and Yood[2] shows that $Y^{\perp\perp} + X$ is closed in X** and $Y^{**}/Z \sim (Y^{\perp\perp}+X)/X$. Thus $Y^{\perp\perp}+X$

$=X^{**}$ and $Y^{**}/Y \sim X^{**}/X$.

REMARK. It is unknown whether X^{**} is a Grothendieck space if X is. However, as proved in [6], if both X and X^{**}/X are Grothendieck spaces then the answer is affirmative.

3. In connection with the open questions whether every Banach space possesses a fundamental biorthorgonal system and whether every Grothendieck space has a separable quasi-complemented subspace, we have the following implication. For the proof, we use the technique of Johson in [5].

THEOREM 3.1. Let X be a Grothendieck space which possesses a fundamental biorthogonal system. Then X has a separable quasi-complemented subspace.

PROOF. Let (x_α, x_α^*) be a fundamental biorthogonal system of X, i.e., the linear span of (x_α) is dense in X and $x_\alpha^*(x_\beta) = 0$ if $\alpha \neq \beta$. Let $\tilde{Y}$ be the closed subspace generated by (x_α^*). We shall prove that $\tilde{Y}$ is a reflexive subspace of X^*. Let (f_n) be a bounded sequence in $\tilde{Y}$ and $A_n = \{x_\alpha : f_n(x_\alpha) = 0\}$. Note that since each f_n is in the closure of the linear span of (x_α^*), A_n is at most countable. Let $A = \bigcup_{n=1}^{\infty} A_n$. A is countable. Thus by a standard diagonal process, there exists a subsequence (f_{n_k}) such that, $(f_{n_k}(x_\alpha))$ converges for each x_α. Moreover, since (f_{n_k}) is bounded and the linear span of (x_α) is dense in X. $(f_{n_k}(x))$ converges for each $x \varepsilon X$. But X is a Grothendieck space, (f_{n_k}) is then convergent to some element in $\tilde{Y}$ in the weak topology of X^*. Our argument shows that every bounded sequence in $\tilde{Y}$ has a weak convergent subsequence in $\tilde{Y}$. This is equivalent to saying that $\tilde{Y}$ is reflexive. It follows then that $X/\tilde{Y}^{\top}$ is reflexive. Hence by Proposition 1.2 (i), X has a separable quasi-complemented subspace.

COROLLARY 3.2. Let X be a Grothendieck space such that every quotient space of X has a fundamental biorthogonal system. Then a subspace Y is quasi-complemented if Y^* is weak*-separable.

PROOF. Let Y be a subspace of X such that Y^* is weak*-separable. Consider the quotient space X/Y. By hypothesis, X/Y has a fundamental biorthogonal system. Hence there exists a separable quasi-complemented subspace of X/Y with quasi-complement, say W, in X/Y. Note that this implies that (X/Y)/W is a separable Grothendieck space. (X/Y)/W is thus necessarily reflexive. We have then that $W^{\perp}$

is a reflexive subspace of $(X/Y)^* \sim Y^{\perp}$. The assertion now follows from Theorem 2.1 of [13].

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