The Journal of National Chiao Tung University
Vol. I, April 1976
pp. 247-262

聯絡及規範場論
Connection & Gauge Field Theory

邵錦昌 Jiin-Chang Shaw
Department of Applied Mathematics

(Received January 14, 1976)

ABSTRACT — The purpose of this report is to show, in an explicit manner, that the theory of what the physicists call "gauge field theory" is identical to the theory of "connection in a vector bundle".

I. Introduction

In this paper, we show, in an explicit manner, that the gauge field theory is identical to the theory of "connection on a vector bundles". It has long been realized that electromagnetic field can be formulated in terms of abelian gauge field. This ideal was extended in 1954 by Yang and Mills [1] to the gauge field for isotopic spin rotation. Under the action of this transformation field variables are subjected to transformations isomorphic with rotation in three dimensions. The angle of rotation, however, is allowed to vary from point to point in space time. This is consistent with the localized field concept that underlies the usual physical theories. This implies that every space-time point has its own isotopic spin space so that the relative orientation of the isotopic spin at two space-time points becomes a physically meaningless quantity and in that case a triplet of vector fields has to be introduced in order to maintain the general isotopic spin rotation invariance. Utiyama [2] then generalized this idea to arbitrary Lie group and identify the gravitational field as gauge field associating with the Lorentz group [3].

The theory of vector bundles with connection, however, provides a convenient framework for discussing the local gauge transformation. With this mathematical tool, the gauge field is related to the connection form of the vector bundles by

$$\omega = -A^a_\mu T_{(a)} dx^\mu ,$$

Where $T_{(a)}$ is the generator of the Lie Algebra of the internal symmetric group and the local gauge transformation is interpreted as the changing of moving frame on the space-time manifold. In this way, the conneption of parallel displacement along a curve on the space-time manifold can be defined and then the space-time manifold is no longer a simple Minkowski space. After recognize this, we must generalize the usual field theory on the Minkowski space to the field theory on a general connection space when the gauge field exists. This is the new feature of gauge field we present here.

II. Frame Bundle

Let M be the 4-dimensional space-time and G be the internal symmetry group we interest. Let F be the representation space of G with dimension N which we call standard fible. Now at every $x \in M$ we attach a vector space F_x which are all isomorphic to F. A frame at x is an order basis of vector space F_x. A moving frame on an open set $U \subset M$ is an ordered N-tuple of vector fields on U, whose value at every point $x \in M$ form a frame.

Let B denote the set of all frames at all points of M. We have projection π: $B \rightarrow M$ defined by $\pi(u)=x$ if u is a frame at x. Let $\{(U_\alpha, \iota^\alpha)\}_{\alpha\in\Omega}$ denote all pairs where U_α are open sets of M which cover M and $\iota^\alpha=\{\iota^\alpha_1, \ldots, \iota^\alpha_N\}$ is a moving frame on U. For each α, we can assign a coordinate on $\pi^{-1}(U_\alpha)$ defined by

$$u \longrightarrow (\pi(u), f_\alpha(u)) = (x^\mu, u^A{}_B), \tag{1}$$

where

$$x = \pi(u)$$

$$u = \{ \iota^\alpha_A(x)\, u^A_1, \ldots, \iota^\alpha_A(x)\, u^A_N \} . \tag{2}$$

G is defined to act on B, $a=(a^A{}_B)$, the representation of $a \in G$ on F, sending a frame $u=\{u_1, \ldots, u_N\}$ at a point $x \in M$ to the frame $ua=\{u_A a^A{}_1, \ldots, u_A a^A{}_N\}$ at the same point x: we have $(ua)b=u(ab)$ for all $u \in B$ and $a, b \in G$. So as a manifold, B has dimension 4+n, where n is dimension of Lie group G. In coordinate notation, if $u=(x^\mu, u^A{}_B)$, then $ua=(x^\mu, u^A{}_C a^C{}_B)$, i.e.

$$f_\alpha(ua) = f_\alpha(u)\, a . \tag{3}$$

Now consider the function $f_\beta(u)\, f_\alpha(u)^{-1}$ from $\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)$ into G. Any frame at the same point $x=\pi(u)$ as u must be of the form ua and (3) says

$$f_\beta(u)\,f_\alpha(u)^{-1}=f_\beta(ua)\,f_\alpha(ua)^{-1}\ .$$

Now we have function $f_{\alpha\beta}: U_\alpha\cap U_\beta \longrightarrow G$ defined by

$$f_{\beta\alpha}(\pi(u))=f_\beta(u)\,f_\alpha(u)^{-1} \qquad \pi(u)\,\varepsilon\, U_\alpha\cap U_\beta\ . \tag{4}$$

They are called transition functions because

$$f_\beta(u)=f_{\beta\alpha}(\pi(u))\,f_\alpha(u)\ . \tag{5}$$

Notice that

$$f_{\gamma\alpha}(x)=f_{\gamma\beta}(x)\,f_{\beta\alpha}(x) \qquad x\,\varepsilon\, U_\alpha\cap U_\beta\cap U_\gamma\ . \tag{6}$$

Let $x\varepsilon M$. The fibre over x is the set $\pi^{-1}(x)$. Each fibre is a closed n-dimensional submanifold on B. A tangent vector $\bar{X}_u \varepsilon B_u$ is vertical if it is tangent to the fibre through u. A vector field on a subset of B is vertical if its value is vertical at every point of the subset. Given $u\varepsilon B$, V_u will denote the tangent subspace at u to the fibre through u. V_u is called the vertical space at u which has dimension n.

Consider the action of G on B. Every $A\varepsilon\mathcal{G}$(the Lie algebra of G) assigns a vector field A^* on B. It can be seen as follow: for every $A\varepsilon\mathcal{G}$, the one parameter subgroup exp tA induce a family of curves $u(\exp tA)\equiv u_t$ on B. Because of $u_{t+s}=u_t(\exp sA)$, their tangents form a vector field, which we denote by A^*, called fundamental vector field. They are vertical because G preserves each fibre. As G acts freely on B, A^* never vanishes on B if $A\neq 0$. The dimension of each fibre being equal to that of $\mathcal{G}$, the mapping $A\longmapsto A^*$ of $\mathcal{G}$ into B_u is a linear isomorphism of $\mathcal{G}$ onto vertical space V_u.

III. Connection, Gauge Field and Curvature

A connection $H=\{H_u\}_{u\varepsilon B}$ on B is a choice of subspace $H_u \subset B_u$ for every u B such that

$$B_u = H_u + V_u, \tag{i}$$

$$H_{ua}=(R_a)_* H_u \text{ for every } u\varepsilon B \text{ and } a\varepsilon G, \tag{ii}$$

where $R_a u = ua$ and $(Ra)_*$ is the transformation

$B_u \longrightarrow B_{ua}$ induced by R_a.

We call H_u the horizontal space at u. A tangent vector $\overline{X}_u \varepsilon B_u$ is horizontal if $\overline{X}_u \varepsilon H_u$, and a vector field on a subset of B is horizontal if its values are horizontal. Let $\overline{X}$ be a vector field on an open subset of B. Condition (i) for a connection says that we have a unique decomposition X=hX+vX where hX and vX are horizontal and vertical vector fields respectively. Condition (ii) says that G preserves the decompositions $B_u = H_u + V_u$; this will be used when we discuss parallel translation. Obviously H_u has dimension 4 as M.

The following fundamental lemma is stated without proof [4].

LEMMA 1. There is a one to one correspondence between connection H on B and $\mathcal{G}$ valued linear 1- form $\overline{\omega}$ on B such that

(i) $\overline{\omega}(A^*) = A$ for all $A \varepsilon \mathcal{G}$,

(ii) $(R_a)^* \omega = ad(a^{-1})\overline{\omega}$,

that is

$$\overline{\omega}((R_a)_* \overline{X})_{ua} = ad(a^{-1})\overline{\omega}(\overline{X})_u,$$

for every $a \varepsilon G$ and every vector field $\overline{X}$ on B, where ad denote the adjoint representation of G in $\mathcal{G}$.

Given H, we define $\overline{\omega}$ by $\overline{\omega}_u(H_u)=0$, $\overline{\omega}_u(A^*_u)=A$

Given $\overline{\omega}$, we define H by $H_u = \{\overline{X}_u\ B_u : \overline{\omega}_u(\overline{X}_u) = 0\}$

The form $\overline{\omega}$ is the connection form of H.

Let X_x be a tangent vector at point $x \varepsilon M$. Given a point $u \varepsilon \pi^{-1}(x)$ the projection $\pi_* : B_u \longrightarrow M_x$ has kernel V_u, so $\pi_* : H_u \simeq M_x$ where $\simeq$ means isomorphism. Thus there is unique horizontal vector $X^*_u \varepsilon B_u$ which π_* sends to $X_{\pi(u)}$. The horizontal vector field X^* along π^{-1} is called the horizontal lift of X_x.

In terms of coordinate notation, we have the following expression.

LEMMA 2. In the coordinate system (U_α, ι^α). $\overline{\omega}$ has the form $\overline{\omega} = (\overline{\omega}^E{}_B)$,

$$\overline{\omega}^E{}_B = (u^{-1})^E{}_A (du^A{}_B - A^a_\mu(\pi(u)) T^A_{(a)D} u^D{}_B dx^\mu), \qquad (7)$$

where $T_{(a)}$ is the generators of Lie algebra.

PROOF: We must prove the $\overline{\omega}$ satisfied two conditions of lemma 2.

(i) For any $A\varepsilon\mathcal{G}$, it is easy to see that the tangents of curves u(exp tA) have components

$$A^*_u = 0\cdot\frac{\partial}{\partial x^\mu} + u^A{}_C A^C{}_B \frac{\partial}{\partial u^A{}_B} \tag{8}$$

So we have

$$\overline{\omega}^E{}_B(A^*) = A^E{}_B\ ,$$

that is,

$$\overline{\omega}(A^*) = A\ .$$

(ii) For every vector field

$$\overline{X}_u = X^\mu\frac{\partial}{\partial x^\mu} + U^A{}_B\frac{\partial}{\partial u^A{}_B}\ ,$$

we have

$$((R_a)_*\overline{X})_{ua} = X^\mu\frac{\partial}{\partial x^\mu} + U^A{}_C a^C{}_B \frac{\partial}{\partial (ua)^A{}_B}$$

Hence

$$\overline{\omega}^E{}_B(R_{a*}\overline{X})_{ua}$$

$$= ((ua)^{-1})^E{}_A U^A{}_C a^C{}_B - ((ua)^{-1})^E{}_A A^a{}_\mu T^A{}_{(a)D}\ (ua)^D{}_B X^\mu$$

$$= (a^{-1})^E{}_F[(u^{-1})^F{}_A U^A{}_C - (u^{-1})^F{}_A A^a{}_\mu T^A{}_{(a)D} u^D{}_C X^\mu] a^C{}_B$$

$$= (a^{-1}\overline{\omega}(\overline{X})_u a)^E{}_B$$

$$= ((ada^{-1})\overline{\omega}(\overline{X})_u)^E{}_B \qquad \textbf{q.e.d.}$$

The $A^a{}_\mu(\pi(u)) = A^a{}_\mu(x)$ is called the gauge field corresponding to the connection form $\overline{\omega}$ and the coordinate system (U_α, ι^α).

LEMMA 3. The horizontal lift of vector field $\frac{\partial}{\partial x^\mu}$ is

$$X^*_u = \frac{\partial}{\partial x^\mu} + A^a{}_\mu T^A{}_{(a)E}\, u^E{}_B \frac{\partial}{\partial u^A{}_B} \tag{9}$$

PROOF: Because $\pi_*(X^*_u) = \frac{\partial}{\partial x^\mu}$, we only have to prove X^*_u is horizontal with respect to $\overline{\omega}$:

$$\bar{\omega}^{E}{}_{B}(X^{*}{}_{u}) = (u^{-1})^{E}{}_{A}\, du^{A}{}_{B}(A^{a}_{\mu} T_{(a)E}{}^{D} u^{E}{}_{F} \frac{\partial}{\partial u^{D}{}_{F}})$$

$$-(u^{-1})^{E}{}_{A} A^{a}_{\mu} T^{A}_{(a)D}\, u^{D}{}_{B}\, dx^{\mu}(\frac{\partial}{\partial x^{\mu}})$$

$$= 0 \qquad \text{q.e.d.}$$

A moving frame u can be looked as a mapping from M to B, that is, every $x\varepsilon M$ maps to $(x, u_x)\varepsilon B$. So u^* maps the differential form $\bar{\omega}$ on B to a differential form $u^*\bar{\omega}$ on M. Now we try to find $u^*\bar{\omega}$. If $X = X^{\mu}\frac{\partial}{\partial x^{\mu}}$ is a tangent field on M, then,

$$(u_* X) = X^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{\partial u^{A}{}_{B}}{\partial x^{\mu}} X^{\mu} \frac{\partial}{\partial u^{A}{}_{B}} .$$

Hence

$$(u^*\bar{\omega})^{E}{}_{B}(X) = \bar{\omega}^{E}{}_{B}(u_* X)_{u}$$

$$= (u^{-1})^{E}{}_{A} \frac{\partial u^{A}{}_{B}}{\partial x^{\mu}} X^{\mu} - (u^{-1})^{E}{}_{A} A^{a}_{\mu} T^{A}_{(a)D}\, u^{D}{}_{B}\, X^{\mu}.$$

So we get

$$u^*\bar{\omega} = (u^{-1}\frac{\partial u}{\partial x^{\mu}} - u^{-1} A^{a}_{\mu} T_{(a)}\, u)\, dx^{\mu} \quad . \tag{10}$$

If two moving frames u and v are related by the matrix function a(x) by v=ua, then by (10) we find

$$v^*\bar{\omega} = a^{-1}da + a^{-1}(u^*\bar{\omega})a \tag{11}$$

This is the gauge transformation of the gauge field. If u is taken as the standard moving frames ι, then we have

$$\omega \equiv \iota^*\omega = -A^{a}_{\mu} T_{(a)}\, dx. \tag{12}$$

THEOREM 4. (Structure Equations)

$$d\,\bar{\omega}^{E}{}_{B} + \bar{\omega}^{E}{}_{C} \wedge \bar{\omega}^{C}{}_{B} = \frac{1}{2}(u^{-1})^{F}{}_{A} R^{A}{}_{G\mu\nu}\, u^{G}{}_{B}\, dx^{\mu} \wedge dx^{\nu}$$

$$\equiv \bar{\Omega}^{E}{}_{B} \tag{13}$$

$$d\bar{\Omega}^{E}{}_{B} = \bar{\Omega}^{E}{}_{C} \wedge \bar{\omega}^{C}{}_{B} - \bar{\omega}^{E}{}_{C} \wedge \bar{\Omega}^{C}{}_{B} \tag{14}$$

where

$$R^{A}{}_{G\mu\nu} = -F^{a}_{\mu\nu}\, T^{A}_{(a)G} \tag{15}$$

$$F^a_{\mu\nu} = \frac{\partial A^a_\nu}{\partial x^\mu} - \frac{\partial A^a_\mu}{\partial x^\nu} - f^a_{bc} A^b_\mu A^c_\nu \tag{16}$$

and f^a_{bc} is the structure constant of Lie Algebra

$$[T_{(b)}, T_{(c)}] = f_b{}^a{}_c T_{(a)} . \tag{17}$$

PROOF:

Proof of (13)

$$d\bar{\omega}^E{}_B = d(u^{-1})^E{}_A \wedge du^A{}_B - A^a{}_\mu T^A{}_{(a)D} u^D{}_B \, d(u^{-1})^E{}_A \wedge dx^\mu$$

$$-(u^{-1})^E{}_A \frac{\partial A^a_\mu}{\partial x^\nu} T^A{}_{(a)D} u^D{}_B \, dx^\nu \wedge dx^\mu$$

$$-(u^{-1})^E{}_A A^a_\mu T^A{}_{(a)D} \, du^D{}_B \wedge dx^\mu$$

$$= -(u^{-1})^E{}_C (u^{-1})^F{}_A \, du^C{}_F \wedge du^A{}_B$$

$$+ A^a_\mu T^A{}_{(a)D} u^D{}_B (u^{-1})^E{}_C (u^{-1})^F{}_A \, du^C{}_F \wedge dx^\mu$$

$$-(u^{-1})^E{}_A A^a_\mu T^A{}_{(a)D} \, du^D{}_B \wedge dx^\mu$$

$$-(u^{-1})^E{}_A \frac{\partial A^a_\mu}{\partial x^\nu} T^A{}_{(a)D} u^D{}_B \, dx^\nu \wedge dx^\mu ,$$

$$\bar{\omega}^E{}_C \wedge \bar{\omega}^C{}_B = ((u^{-1})^E{}_A \, du^A{}_C$$

$$-(u^{-1})^E{}_A A^a_\mu T^A{}_{(a)D} u^D{}_C \, dx^\mu)$$

$$\wedge ((u^{-1})^C{}_F \, du^F{}_B$$

$$-(u^{-1})^C{}_F A^a_\nu T^F{}_{(a)G} u^G{}_B \, dx^\nu)$$

$$= (u^{-1})^E{}_A (u^{-1})^C{}_F \, du^A{}_C \wedge du^F{}_B$$

$$-(u^{-1})^E{}_A A^a_\mu T^A{}_{(a)D} u^D{}_C (u^{-1})^C{}_F \, dx^\mu \wedge du^F{}_B$$

$$-(u^{-1})^E{}_A (u^{-1})^C{}_F A^a_\nu T^F{}_{(a)G} u^G{}_B \, du^A{}_C \wedge dx^\nu$$

$$+(u^{-1})^E{}_A A^a_\mu T^A{}_{(a)D} u^D{}_C (u^{-1})^C{}_F A^b_\nu T^F{}_{(b)G} .$$

$$u^G{}_B\, dx^\mu \wedge dx^\nu$$

Hence

$$d\bar{\omega}^E{}_B + \bar{\omega}^E{}_C \wedge \bar{\omega}^C{}_B$$

$$= -\frac{1}{2}\Big(\frac{\partial A^a_\nu}{\partial x^\mu} T^A_{(a)G} - \frac{\partial A^a_\mu}{\partial x^\nu} T^A_{(a)G} - A^b_\mu T^A_{(a)D} A^C_\nu T^D_{(c)G}$$

$$+ A^C_\nu T^A_{(c)D} A^b_\mu T^D_{(b)G}\Big)(u^{-1})^E{}_A\, u^G{}_B\, dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} R^A{}_{G\mu\nu} (u^{-1})^E{}_A\, u^G{}_B\, dx^\mu \wedge dx^\nu \, .$$

Proof of (14). By (13)

$$d\bar{\Omega}^E{}_B = d\omega^E{}_C \wedge \bar{\omega}^C{}_B - \bar{\omega}^E{}_C \wedge d\bar{\omega}^C{}_B$$

$$= -\bar{\omega}^E{}_C \wedge \bar{\omega}^D{}_C \wedge \bar{\omega}^C{}_B + \bar{\Omega}^E{}_C \wedge \bar{\omega}^C{}_B + \bar{\omega}^E{}_C \wedge \bar{\omega}^C{}_D \wedge \bar{\omega}^D{}_B - \bar{\omega}^E{}_C \wedge \bar{\Omega}^C{}_B$$

$$= \bar{\Omega}^E{}_C \wedge \bar{\omega}^C{}_B - \bar{\omega}^E{}_C \wedge \bar{\Omega}^C{}_B \, .$$

Note: (i) (13) can be written by the form ω on M.

$d\omega^E{}_B + \omega^E{}_C \wedge \omega^C{}_B = \frac{1}{2} R^E{}_{B\mu\nu} dx^\mu \wedge dx^\nu$, or $d\omega + \omega \wedge \omega = \frac{1}{2} R_{\mu\nu} dx^\mu \wedge dx^\nu$.

(ii) (14) is called the Bianchi identity.

IV. Parallelism and Covariant Derivative

Now we are going to define the concept of parallel displacement of frames along any given curve τ in the base manifold M. After we define this we get a unique vector space isomorphism from $F_{x(t_o)}$ to $F_{x(t_1)}$ along τ.

Let $\tau = x_t$, $t_o \le t \le t_1$, be a curve in M. A horizontal lift of τ is a horizontal curve $\tau^* = u_t$, $t_o \le t \le t_1$ in B such that $\pi(u_t) = x_t$. Here a horizontal curve in B means a curve whose tangent vectors are all horizontal. The notion of lift of a curve corresponds to notion of lift of a vector field. Indeed, if X^* is the lift of a vector field X on M, then the integral curve of X^* through a point $u_o \in B$ is a lift of the integral curve of X through the point $x_o = \pi(u_o)$. The following theorem is also referring to Kobayasi & Nomizu's book p. 69 [4].

THEOREM 5. Let $\tau = x_t$ be a curve in M. For an arbitrary point

u_o of B with $\pi(u_o)=x_{t_o}$, there exists a unique lift $\tau^*=u_t$ of τ which starts from u_o.

Now using this theorem, we define the parallel displacement of frames as follows. Let $\tau=x_t$ be a curve on M. Let u_o be a arbitrary frame at x_{t_o}. The unique lift τ^* of τ through u_o has the end point u_1 which is a frame on x_{t_1}. We then call the mapping from u to u_1 be the parallel displacement along the curve τ. We again denote this mapping be the same letter τ.

THEOREM 6. The parallel displacement along any curve τ commute with action of G on B: $\tau(ua)=\tau(u)a$ for every $a\varepsilon G$.

PROOF: This follows from the fact that every horizontal curve is mapped into a horizontal curve by R_a. q.e.d.

So from this theorem we know parallel displacement indeed define a isomorphism from $F_{x(t_o)}$ to $F_{x(t_1)}$. That is, an element $q^A\tau(\iota Ax(t_o)A^\varepsilon F_{x(t_o)}$ is mapped to an element $q^A(\iota_{x(t_o)})A^\varepsilon F_{x(t_1)}$.

A single F valued function q is called a field in physical sense. In coordinate system (U_α, ι^α), q can be expressed out by components $q \rightarrow (q^1(x), \ldots, q^N(x))$ where $q=q^A(x)\iota_A(x)$. Now we define the covariant derivative. If X is a tangent field on M, x_t be its integral curve and q is a field, the covariant derivative of field q with respect to X is defined as

$$\nabla_X q= \lim_{h\to 0} \frac{1}{h}[\tau_t^{t+h}(q(x_{t+h}))-q(x_t)] \tag{18}$$

where $X=\dot{x}_t$, and $\tau_t^{t+h}: F_{x(t+h)} \rightarrow F_{x(t)}$ denotes the parallel displacement along τ from x_{t+h} to x_t. The covariant derivative has following properties:

THEOREM 7. Let X,Y be vector fields on M, q and q' be fields and f a function on M, then

(1). $\nabla_{X+Y}\, q = \nabla_X q+\nabla_Y q$

(2). $\nabla_X(q+q') = \nabla_X q+\nabla_X q'$

(3). $\nabla_{fX} = f\nabla_X q$

(4). $\nabla_X(fq) = f\nabla_X q+(Xf)q$

In the coordinate system (U,ι), by using lemma 3, we can show

$$\nabla_{\frac{\partial}{\partial x^{\mu}}} \iota_A = -A^a_{\mu} T^B_{(a)A} \iota_B . \tag{19}$$

Hence, by theorem 7 (4)

$$\nabla_{\frac{\partial}{\partial x^{\mu}}} q = \nabla_{\frac{\partial}{\partial x^{\mu}}} q^A \iota_A$$

$$= \left(\frac{\partial q^A}{\partial x^{\mu}} - A^a_{\mu} T^A_{(a)B} q^B\right)\iota_A$$

$$\equiv q^A{}_{|\mu} \iota_A . \tag{20}$$

The curvature tensor R defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} , \tag{21}$$

has been already gotten in theorem 4 (13). Since it is easy to compute from (19), that

$$R\left(\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}}\right) \iota_B = -F^a_{\mu\nu} T^A_{(a)B} \iota_A \equiv R^A{}_{B\mu\nu} \iota_A , \tag{22}$$

and if we choose another moving frames $u = \iota a$, the new curvature tensor

$$R\left(\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}}\right) u_B = -F'^a_{\mu\nu} T^A_{(a)B} u_A \equiv R'^A{}_{B\mu\nu} u_A ,$$

is related to the old one by

$$R^E{}_{D\mu\nu} = a^E{}_A R'^A{}_{B\mu\nu} (a^{-1})^B{}_D \tag{23}$$

Up to now, we have shown in an explicit manner that gauge field theory is actually the same as vector bundle theory with connection.

V. Riemanian Connection and Gravitation

One of an important special case of above consideration is that when F_x concide with the tangent space M_x, i.e. if F_x is the Minkowski space, and G chosen to be Lorentz group SO(3.1). Consider a Riemanian connection in this bundle of Lorentz frames with metric g. Let $\iota = \{\iota_a\} = \{\iota_o, \iota_1, \iota_2, \iota_3\}$ be a moving Lorentz frame on M, that is

$$\iota_a{}^{\mu}\iota_{b\mu} = \eta_{ab} \; , \qquad (24)$$
$$\iota_a{}^{\mu}\iota^{a\nu} = g^{\mu\nu} \; ,$$

where $_a{}^{\mu}$ are the components of vector ι_a. The generators of Lie algebra of Lorentz group are set of ten matrices $E_{(ab)}$ with elements

$$E_{(ab)}{}^{c}{}_{d} = \delta^{c}_{a}\eta_{bd} - \delta^{c}_{b}\,\eta_{ad} \; . \qquad (25)$$

From (19) we know, the gauge fields of this Riemanian connection are

$$\nabla_{\frac{\partial}{\partial x^{\mu}}}\iota_d = -\frac{1}{2}A^{ab}{}_{\mu}\; E_{(ab)}{}^{c}{}_{d}\,\iota_c \; . \qquad (26)$$

But it is well known that the usual Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ are related to the covariant derivative by

$$\nabla_{\frac{\partial}{\partial x^{\mu}}}\frac{\partial}{\partial x^{\nu}} = \Gamma^{\lambda}_{\mu\nu}\frac{\partial}{\partial x^{\lambda}} \; . \qquad (27)$$

By inserting the relation $\iota_a = \iota_a{}^{\mu}\frac{\partial}{\partial x^{\mu}}$ into (26) and using the property of covariant derivative, theorem 7(4), we can get

$$A^{ab}{}_{\mu} = \iota^{a\nu}\iota^{b}{}_{\rho}\Gamma^{\rho}_{\mu\nu} + \iota^{b}{}_{\rho}\frac{\partial\iota^{a\rho}}{\partial x^{\mu}} \; , \qquad (28)$$

where $g^{\mu\nu}$ are used for rising or lowing the indices μ, ν, ρ, etc. and η^{ab} are used for rising or lowing the indices a, b, c, etc. So from (26) and (28) we know that the gauge fields $A^{ab}{}_{\mu}$ associated with Lorentz group are the gravitational fields and the geometrical meaning of $A^{ab}{}_{\mu}$ can be quite easily seen from formula (26).

The advantage that we describe gravitational field by the Lorentz group vector bundle lies on the fact that we can formulate the interaction between spinor field and gravitational field. The generators of spinor representation of $E_{(ab)}$ are

$$T_{(ab)} = \frac{1}{4}[\gamma_a, \gamma_b] \; , \qquad (29)$$

where γ_a's are the Dirac matrices. So the covariant derivative of spinor field is [5]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \frac{1}{8} A^{ab}{}_\mu [\gamma_a, \gamma_b]\psi \quad , \tag{30}$$

and the Dirac equation on the Riemanian space is

$$i\gamma^a \iota_a{}^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{i}{8}\gamma^a \iota_a{}^\mu A^{cd}{}_\mu [\gamma_c, \gamma_d]\psi - m\psi = 0 \tag{31}$$

The proof of covariance of equation (31) is given in appendix.

Appendix

PROOF OF COVARIANCE OF DIRAC EQUATION (31)

Let $\{\iota'_a\}$ be another moving Lorentz frame on M which related to the old one by

$$\iota'_a = \iota_b L^b{}_a \quad , \tag{B.1}$$

where $L=(L^b{}_a)$ is a Lorentz matrix function on M, that is, they satisfy

$$\eta_{ab}\, L^a{}_c\, L^b{}_d = \eta_{cd} \quad . \tag{B.2}$$

Let ψ' be the new spinor field which describe to ι' the same physical state as ψ to ι. ψ and ψ' is related by

$$\psi = S(L)\psi \quad , \tag{B.3}$$

where S(L) is the spinor representation of L and has the properties

$$L^a{}_b \gamma^b = S^{-1}(L)\gamma^a S(L) ,$$

$$(L^{-1})^b{}_a\, \gamma_b = S^{-1}(L)\gamma_a\, S(L) \tag{B.4}$$

For our purpose it is more convenient to introduce a set of new Christoffel symbols $\Gamma^c_{a\,b}$ by

$$\nabla_{\iota_a} \iota_b = \Gamma^c_{ab}\, \iota_c \quad . \tag{B.5}$$

It is easy to find that

$$\Gamma^c_{ab} = \eta_{bd}\, \iota_a{}^\mu\, A^{dc}{}_\mu \quad . \tag{B.6}$$

The transformation property between $\Gamma^c_{a\,b}$ and $\Gamma'^c_{a\,b}$, where $\Gamma'^c_{a\,b}$ is defined by

$$\nabla_{\iota'_a}\iota'_b = \Gamma'^{c}{}_{ab}\,\iota'_c \,, \tag{B.7}$$

is

$$\Gamma'^{c}{}_{ab} = L^e{}_a (L^{-1})^c{}_d\, \iota_e{}^\mu \frac{\partial L^d{}_b}{\partial x^\mu} + L^e{}_a L^f{}_b (L^{-1})^c{}_d \Gamma^d{}_{ef} . \tag{B.8}$$

The Dirac equations in these two moving frames ι and ι' are

$$i\gamma^a \iota_a{}^\mu \frac{\partial\psi}{\partial x^\mu} - \frac{i}{8}\gamma^a\eta^{bd}\,\Gamma^c{}_{ab}[\gamma_d,\gamma_c]\psi - m\psi = 0 \,. \tag{B.9}$$

$$i\gamma^a \iota'_a{}^\mu \frac{\partial\psi'}{\partial x^\mu} - \frac{i}{8}\gamma^a\eta^{bd}\,\Gamma'^c{}_{ab}[\gamma_d,\gamma_c]\psi' - m\psi' = 0 \,, \tag{B.10}$$

, respectively. By reexpressing the Dirac equation (B.10) of ι' in terms of ψ with the aid of (B.1), (B.3) and (B.8), we find

$$\begin{aligned} & i\gamma^a L^b{}_a \iota_b{}^\mu \frac{\partial}{\partial x^\mu}(S^{-1}(L)\psi) \\ & - \frac{i}{8}\gamma^a\eta^{bd}\Big(L^e{}_a (L^{-1})^c{}_h \iota_e{}^\mu \frac{\partial L^h{}_b}{\partial x^\mu} + L^e{}_a L^f{}_b (L^{-1})^c{}_h \Gamma^h{}_{ef}\Big) \\ & [\gamma_d,\gamma_c]\, S(L^{-1})\psi - mS(L^{-1})\psi = 0 \,. \end{aligned} \tag{B.11}$$

This is form invariance, that is, identical with (B.9), provided that we can prove

$$\begin{aligned} & i\gamma^a L^b{}_a S^{-1}(L)\, \iota_b{}^\mu \frac{\partial\psi}{\partial x^\mu} \\ & - \frac{i}{8}\gamma^a\eta^{bd} L^e{}_a L^f{}_b (L^{-1})^c{}_h \Gamma^h{}_{ef}[\gamma_d,\gamma_c] S^{-1}(L)\psi - mS^{-1}(L)\psi \\ & = S^{-1}(L)\Big(i\gamma^b \iota_b{}^\mu \frac{\partial\psi}{\partial x^\mu} - \frac{i}{8}\gamma^e\eta^{fd}\Gamma^h{}_{ef}[\gamma_d,\gamma_h]\psi - m\psi\Big) \\ & = 0 \,, \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} & i\gamma^a L^b{}_a \iota_b{}^\mu \frac{S^{-1}(L)}{\partial x^\mu}\,\psi \\ & - \frac{i}{8}\gamma^a \eta^{bd} L^e{}_a (L^{-1})^c{}_h \iota_e{}^\mu \frac{\partial L^h{}_b}{\partial x^\mu}[\gamma_d,\gamma_c] S^{-1}(L)\psi \end{aligned}$$

$$= 0 \tag{B.13}$$

The last equality of (B.12) is true if we can prove (B.13) and that is Dirac equation in ι moving frames.

To prove the first equality of (B.12), we have to show they are equal term by term. It is trivial to see that this is true for the first and the third term. For the second term:

$$\frac{1}{8}\gamma^a\eta^{bd}L^e{}_a L^f{}_b (L^{-1})^c{}_h \Gamma^h{}_{ef}[\gamma_d,\gamma_c]\, S^{-1}(L)$$

$$=\frac{1}{8}S^{-1}(L)\gamma^e S(L)\, L^f{}_b (L^{-1})^c{}_h \Gamma^h{}_{ef}[\gamma^b,\gamma_c]\, S^{-1}(L)$$

$$=\frac{1}{8}S^{-1}(L)\gamma^e S(L)\Gamma^h{}_{ef}\,[L^f{}_b\gamma^b,(L^{-1})^c{}_h\gamma_c]\, S^{-1}(L)$$

$$=\frac{1}{8}S^{-1}(L)\gamma^e S(L)\Gamma^h{}_{ef}[S^{-1}(L)\gamma^f S(L),\, S^{-1}(L)\gamma_h S(L)]S^{-1}(L)$$

$$=\frac{1}{8}S^{-1}(L)\gamma^e\eta^{fd}\Gamma^h{}_{ef}\,[\gamma_d,\gamma_h]$$

Hence (B.12).

Finally we prove (B.13). It is equivalent to prove that

$$\frac{1}{8}[\iota_e{}^\mu \frac{\partial L^h{}_b}{\partial x^\mu}\gamma^b,(L^{-1})^c{}_h\gamma_c]\, S^{-1}(L)=\iota_e{}^\mu\frac{\partial S^{-1}(L)}{\partial x^\mu}\,. \tag{B.14}$$

By using (B.4), we have

LHS of (B.14)

$$=\frac{1}{8}[\iota_e{}^\mu\frac{\partial}{\partial x^\mu}(S^{-1}(L)\gamma^h S(L)),S^{-1}(L)\gamma_h\, S(L)]\, S^{-1}(L)$$

$$=\frac{1}{8}\,(\iota_e{}^\mu\frac{\partial}{\partial x^\mu}(S^{-1}(L)\gamma^h S(L))S^{-1}(L)\gamma_h$$

$$-S^{-1}(L)\gamma_h S(L)\iota_e{}^\mu\frac{\partial}{\partial x^\mu}(S^{-1}(L)\gamma^h S(L))S^{-1}(L))$$

$$=\frac{1}{8}(4\iota_e{}^\mu\frac{\partial S^{-1}(L)}{\partial x^\mu}+S^{-1}(L)\gamma^h\iota_e{}^\mu\frac{\partial S(L)}{\partial x^\mu}\, S^{-1}(L)\gamma_h$$

$$-S^{-1}(L)\gamma_h\, S(L)\iota_e{}^\mu\frac{\partial S^{-1}(L)}{\partial x^\mu}\gamma^h-4S^{-1}(L)\iota_e{}^\mu\frac{\partial S(L)}{\partial x^\mu}S^{-1}(L))$$

$$= \iota_e{}^{\mu} \frac{\partial S^{-1}(L)}{\partial x^{\mu}} - \frac{1}{4} S^{-1}(L) \gamma^h \, S(L) \iota_e{}^{\mu} \frac{\partial S^{-1}(L)}{\partial x^{\mu}} \gamma_h$$

$$= \iota_e{}^{\mu} \frac{S^{-1}(L)}{\partial x^{\mu}} \, ,$$

where the last equality comes from the fact that

$$S(L) \iota_e{}^{\mu} \frac{\partial S^{-1}(L)}{\partial x^{\mu}} \sim \Sigma \; f^{\mu\nu} \sigma_{\mu\nu} \, ,$$

and the identity

$$\gamma^h \gamma_{\mu} \gamma_{\nu} \gamma_h = 0 \, , \quad \text{if } \mu \neq \nu \, .$$

Hence we prove (B.13)

References

1. C. N. Yang and R. L. Mills, Phys. Rev. 96, p. 191 (1954).
2. Ryoyn Utiyama, Phys. Rev. 101, p. 1597 (1956).
3. T. W. B. Kibble, J. Math. Phys. 2, p. 212 (1961).
4. S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, Vol. I, John Wiley & Sons (1963).
5. P. A. M. Dirac, "Interacting Gravitational and Spinor Fields", in "Recent Developments in General Relativity", p. 191, Pergamon Press, Oxfor, 1962.