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不用「線」之單磁極理論
Monopole Theory without String

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ABSTRACT — We propose a Schrödinger equation for the non-relativistical charge-monopole system without using the string and demonstrate that it is a correct equation for this system.

I. Introduction

It is well known that to quantize a classical system, i.e., to obtain a quantum mechanical system from a classical system, usually takes the following steps:

Classical equations of motion $\xrightarrow{\text{Step I}}$ Lagrangian of the classical system $\xrightarrow{\text{Step II}}$ Hamiltonian of the classical system $\xrightarrow{\text{Step III}}$ Quantum mechanical system (Schrödinger equation).

Many physically interesting quantum mechanical systems can be obtained from the corresponding classical system by using the above standard method. In the first step, there are three possible classes of systems as follows

Class 1: systems with no Lagrangian whose Euler-Lagrange equation is the classical equation of motion.

Class 2: systems with no singularity-free Lagrangian whose Euler-Lagrange equation is the classical equation of motion. But, if we divide the configuration space into many overlapped regions, then for each region, we can find a singularity-free Lagrangian whose Euler-Lagrange equation is the classical equation of motion.

Class 3: systems with Lagrangian whose Euler-Lagrange equation is the classical equation of motion.

In this paper, we are interested in considering the quantization of the classical systems which belong to class (2). It is a very interesting problem at least theoretically and it is not treated completely enough in the

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past. The classical electric charge-magnetic monopole system is a good example [1].

We give the proposition to quantize this kind of classical system in section II, that is, we propose a Schrodinger equation with boundary condition for this kind of classical system. In section III, we use the results in section II to treat the monopole-charge case in detail and in section IV, we give some remarks.

II. Quantization

We consider a classical system whose equation of motion can not be obtained from a singularity-free Lagrangian. But, if we divide the configuration space into, say, two overlapped regions R_1 and R_2 (we can easily extend the results to the case of more than two overlapped regions), then we can find singularity-free Lagrangian L_1 in region R_1 and L_2 in region R_2, such that the equation of motion can be obtained from these Lagrangians. In the region $R_1 \cap R_2$, since the equation of motion obtained from Lagrangian L_1 and L_2 are the same, Lagrangian L_1 can differ from Lagrangian L_2 only up to a total time derivative of some function $f(q,t)^2$, that is,

$$L_1 = L_2 + \frac{df(q,t)}{dt} \quad \text{in region } R_1 \cap R_2 \tag{1}$$

Now if we follow the usual procedure to construct the Hamiltonian, we get Hamiltonian H_1 in region R_1 and Hamiltonian H_2 in region R_2. In the region $R_1 \cap R_2$, we have

$$H_2(p,q,t) = H_1(p - \frac{\partial f}{\partial q}, q, t) - \frac{\partial f}{\partial t} \tag{2}$$

If we follow the usual procedure, we would get the following Schrödinger Equation for the wave function $\Psi(q,t)$

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = H_1\Psi(q,t) \quad \text{in region } R_1 \tag{3}$$

$$i\hbar\frac{\partial\Psi(q,t)}{\partial t} = H_2\Psi(q,t) \quad \text{in region } R_2 \tag{4}$$

It is inconsistent in the overlapped region $R_1 \cap R_2$ due to the fact of Equation (2), that is, in this region, if we use H_1 as the Hamiltonian, we get Equation (3), and if we use H_2 as the Hamiltonian, we get Equation (4), or, using Equation (2), we can convert it into

$$i\hbar\frac{\partial}{\partial t}(e^{i\frac{-f(q,t)}{\hbar}}\psi(q,t)) = H_1(e^{-\frac{if(q,t)}{\hbar}}\Psi(q,t)) \tag{5}$$

It is interesting to note that the wave function in Equation (5) differs from the wave function in Equation (2) only up to a phase factor, or a gauge transformation of the second kind. Due to this fact, it is very nature to propose the following Schrödinger equation for this classical system. First we choose a surface σ in the overlapped region $R_1 \cap R_2$. The surface σ divides the configuration space into region R_1' and R_2', where $R_1' \subset R_1$ and $R_2' \subset R_2$. Now the wave function $\Psi(q,t)$ in region R_1', called $\Psi_1(q,t)$, satisfies the following equation

$$i\hbar \frac{\partial \Psi_1}{\partial t} = H_1 \Psi_1(q,t) \qquad q \in R_1' \tag{6}$$

and the wave function in region R_2', called $\Psi_2(q,t)$, satisfies the following equation

$$i\hbar \frac{\partial \Psi_2}{\partial t} = H_2 \Psi_2(q,t) \qquad q \in R_2' \tag{7}$$

On the surface σ, we assign the following boundary conditions

$$\Psi_1(q,t)\Big|_{q\in\sigma} = e^{\frac{if(q,t)}{\hbar}} \psi_2(q,t)\Big|_{q\in\sigma} \tag{8}$$

$$\frac{\partial}{\partial n}(\psi_1(q,t))\Big|_{q\in\sigma} = \frac{\partial}{\partial n}\left(e^{\frac{if(q,t)}{\hbar}} \psi_2(q,t)\right)\Big|_{q\in\sigma} \tag{9}$$

These boundary conditions are not the usual "wave function continuity conditions", that is, we "twist" the wave function $\Psi_2(q,t)$ and then connect it to the wave function $\Psi_1(q,t)$.

From the proposed Schrödinger equation for the wave function, i.e., Eqs. (6), (7), (8) and (9), we know that the wave function $\Psi(q,t)$ surely depends on the choice of surface σ. But it must be noted that the operators which represent physical observables also depend on the choice of surface σ [1,2]. For example, the operator representing the velocity $\dot{q}$ will be different in region R_1' and R_2' even in the overlapped region $R_1 \cap R_2$. The over all effect of the choice of surface σ is that the expectation value of any observable does not depend on the choice of surface σ [2].

We conclude that the proposed Schrödinger equations are well defined and physically sensible.

III. The Monopole-Charge Case

We consider a particle of mass m with electric charge e moving in a magnetic field which is generated by a magnetic monopole of magnetic charge g resting on the coordinate origin [3,4]. The classical equations of motion are

$$m\ddot{\vec{r}} = \frac{eg}{c}\,\dot{\vec{r}} \times \frac{\vec{r}}{r^3} \tag{10}$$

It is easy to see that the Euler-Lagrange equation of the following Lagrangian L

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - \frac{eg}{c}\,\dot{\vec{r}} \cdot \vec{A} \tag{11}$$

are

$$m\ddot{\vec{r}} = \frac{eg}{c}\,\dot{\vec{r}} \times (\vec{\nabla}\times\vec{A}) \tag{12}$$

Thus if we can find a vector potential $\vec{A}$ such that

$$\vec{\nabla}\times\vec{A} = \frac{\vec{r}}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right) \tag{13}$$

then we can use this vector potential $\vec{A}$ to construct the Lagrangian in Equation (11), and in turn we can obtain Equation (10). But it is well known that we can not have a singularity-free vector potential $\vec{A}$ which satisfies Equation (13) over the space with the origin removed. Thus it is impossible to construct a singularity-free Lagrangian of the form as in Equation (11). As point out by Wu and Yang [5], we can divide the space into two overlapped region R_1 and R_2. In each region we can obtain a singularity-free vector potential $\vec{A}$ which satisfies Equation (13). In fact, we choose

$$R_1: 0<r,\ 0\le\theta<\frac{\pi}{2}+\delta,\ 0\le\phi<2\pi,\ A_{1r}=A_{1\theta}=0,\ A_{1\phi}=\frac{1-\cos\theta}{r\sin\theta}$$

$$R_2: 0<r,\ \frac{\pi}{2}-\delta<\theta\le\pi,\ 0\le\phi<2\pi,\ A_{2r}=A_{2\theta}=0,\ A_{2\phi}=\frac{-1-\cos\theta}{r\sin\theta}$$

$$0<\delta<\frac{\pi}{2} \tag{14}$$

Now we can write down the Lagrangian.

$$L_1 = \frac{m}{2}[\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2] - \frac{eg}{c}\, r\sin\theta\dot{\phi}\left(\frac{1-\cos\theta}{r\sin\theta}\right) \tag{15}$$

$$L_2 = \frac{m}{2}[\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2] - \frac{eg}{c}\, r\sin\theta\dot{\phi}\left(\frac{-1-\cos\theta}{r\sin\theta}\right) \tag{16}$$

It is easy to see that

$$L_1 = L_2 - \frac{2eg}{c}\, r\sin\theta\dot{\phi}\,\frac{1}{r\sin\theta} = L_2 - \frac{2eg}{c}\frac{d}{dt}\phi \tag{17}$$

The Hamiltonians are

$$H_1 = \frac{1}{2m}\left[P_r^2 + \frac{P_\theta^2}{r^2} + \frac{1}{r^2\sin^2\theta}\left(P_\phi + \frac{eg}{c}(1-\cos\theta)\right)^2\right] \tag{18}$$

$$H_2 = \frac{1}{2m}\left[P_r^2 + \frac{P_\theta^2}{r^2} + \frac{1}{r^2\sin^2\theta}\left(P_\phi + \frac{eg}{c}(-1-\cos\theta)\right)^2\right] \tag{19}$$

Now we choose the surface σ as $\theta=\frac{\pi}{2}$ and obtain the proposed Schrödinger equation from section II.

$$\text{Region } R_1': 0<r,\quad 0\le\theta<\frac{\pi}{2},\quad 0\le\phi<2\pi$$

$$i\hbar\frac{\partial\Psi_1}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi_1 + \frac{\frac{eg}{c}(1-\cos\theta)}{mr^2\sin^2\theta}\,\frac{\hbar}{i}\frac{\partial\Psi_1}{\partial\phi} + \frac{\left(\frac{eg}{c}(1-\cos\theta)\right)^2}{2mr^2\sin^2\theta}\Psi_1 \tag{20}$$

Region R_2': $0<r$, $\frac{\pi}{2}<\theta\leq\pi$, $0\leq\phi<2\pi$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi_2+\frac{\frac{eg}{c}(-1-\cos\theta)}{mr^2\sin^2\theta}\frac{\hbar}{i}\frac{\partial\Psi_2}{\partial\phi}+\frac{(\frac{eg}{c}(-1-\cos\theta))^2}{2mr^2\sin^2\theta}\Psi_2 \quad (21)$$

The boundary conditions at the surface $\theta=\frac{\pi}{2}$ are

$$\Psi_1(r,\theta,\phi,t)\Big|_{\theta=\frac{\pi}{2}}=e^{-i\frac{2eg}{\hbar c}\phi}\Psi_2(r,\theta,\phi,t)\Big|_{\theta=\frac{\pi}{2}} \quad (22)$$

$$\frac{\partial}{\partial\theta}\psi_1(r,\theta,\phi,t)\Big|_{\theta=\frac{\pi}{2}}=\frac{\partial}{\partial\theta}[e^{-i\frac{2eg}{\hbar c}\phi}\Psi_2(r,\theta,\phi,t)]\Big|_{\theta=\frac{\pi}{2}} \quad (23)$$

If we assume $\frac{eg}{\hbar c}=\mu$, then from the above equations, we can see that (single valued wave function)

$$2\mu=n,\quad n \text{ is an integer} \quad (24)$$

which is Dirac's charge quantization condition.

Now let us try to solve Equation (20) and (21) subjecting to the boundary conditions, Equations (22) and (23). We set

$$\Psi_1(r,\theta,\phi,t)=\exp(-\frac{iE_1t}{\hbar})f_1(r)g_1(\theta)h_1(\phi) \quad (25)$$

$$\Psi_2(r,\theta,\phi,t)=\exp(-\frac{iE_2t}{\hbar})f_2(r)g_2(\theta)h_2(\phi) \quad (26)$$

where

$$h_1(\phi)=e^{im_1\phi},\ h_2(\phi)=e^{im_2\phi},\ m_1 \text{ and } m_2 \text{ are integers} \quad (27)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dg_1(\theta)}{d\theta})-\frac{1}{\sin^2\theta}[(m_1+\mu(1-\cos\theta)]^2g_1(\theta)+\lambda_1g_1(\theta)=0 \quad (28)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dg_2(\theta)}{d\theta})-\frac{1}{\sin^2\theta}[m_2+\mu(-1-\cos\theta)]^2g_2(\theta)+\lambda_2g_2(\theta)=0 \quad (29)$$

and

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{df_1(r)}{dr})-\frac{\lambda_1}{r^2}f_1(r)+\frac{2mE_1}{\hbar}f_1(r)=0 \quad (30)$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{df_2(r)}{dr})-\frac{\lambda_2}{r^2}f_2(r)+\frac{2mE_2}{\hbar^2}f_2(r)=0 \quad (31)$$

We can write down solutions for Equations (28) and (29). (without lose of generality, we will assume $\mu>0$.) We also set $z=\frac{1-\cos\theta}{2}$ and $z'=\frac{1+\cos\theta}{2}$.

Case 1: $0\leq m_1$

$$g_1(\theta)=Z^{\frac{m_1}{2}}(Z-1)^{\frac{m_1}{2}+\mu}\ {}_2F_1[(\mu+m_1)+\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+4\lambda_1+4\mu^2},$$

$$(\mu+m_1)+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2}\ ;\ m_1+1;\ Z]$$

Case 2: $-2\mu\le m_1<0$

$$g_1(\theta)=Z^{-\frac{m_1}{2}}(Z-1)^{\frac{m_1}{2}+\mu}\ {}_2F_1[\mu+\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2}\ ,$$

$$\mu+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2}\ ;\ -m_1+1;\ Z]$$

Case 3: $m_1<-2\mu$

$$g_1(\theta)=Z^{-\frac{m_1}{2}}(Z-1)^{-(\frac{m_1}{2}+\mu)}\ {}_2F_1[-(\mu+m_1)+\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2}\ ,$$

$$-(\mu+m_1)+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2}\ ;\ -m_1+1;\ Z]$$

and for $g_2(\theta)$

Case 1: $2\mu\le m_2$

$$g_2(\theta)=(Z')^{\frac{m_2}{2}}(Z'-1)^{\frac{m_2}{2}-\mu}\ {}_2F_1[(m_2-\mu)+\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}\ ,$$

$$(m_2-\mu)+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}\ ;\ m_2+1;\ Z']$$

Case 2: $0\le m_2<2\mu$

$$g_2(\theta)=(Z')^{\frac{m_2}{2}}(Z'-1)^{\mu-\frac{m_2}{2}}\ {}_2F_1[\mu+\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}\ ,$$

$$\mu+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}\ ;\ m_2+1;\ Z']$$

Case 3: $m_2<0$

$$g_2(\theta)=(Z')^{-\frac{m_2}{2}}(Z'-1)^{\mu-\frac{m_2}{2}}\ {}_2F_1[(\mu-m_2)+\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}$$

$$(\mu-m_2)+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_2+4\mu^2}\ ;\ 1-m_2;\ Z']$$

Thus the general solution for the wave functions Ψ_1 and Ψ_2 are

$$\Psi_1(r,\theta,\phi,t)=\sum_{E_1,\lambda_1,m_1} a_{E_1\lambda_1m_1}\, e^{\frac{iE_1t}{\hbar}}\, f_1(E_1,\lambda_1;\, r)\, g_1(\lambda_1,m_1;\theta)\, h_1(m_1;\phi) \tag{32}$$

$$\Psi_2(r,\theta,\phi,t)=\sum_{E_2,\lambda_2,m_2} a_{E_2\lambda_2m_2}\, e^{-\frac{iE_2t}{\hbar}}\, f_2(E_2,\lambda_2;\, r)$$

$$g_2(\lambda_2, m_2; \theta) h_2(m_2; \phi) \tag{33}$$

Substituting Equation (32) and (33) into the boundary conditions Equations (22) and (23), we get

$$m_1 = m_2 - 2\mu = m, \quad E_1 = E_2 = E, \quad \lambda_1 = \lambda_2 = \lambda \tag{34}$$

and the following important relations

Case 1: $m_1 = m_2 - 2\mu = m \geq 0$

$$(\mu+m) + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2} = -n_1, \quad n_1 = 0,1,2,\ldots$$

or $$\lambda = (\mu+m+n_1)(\mu+m+n_1+1) - \mu^2 \tag{35}$$

Case 2: $0 > m = m_1 = m_2 - 2\mu \geq -2\mu$

$$(\mu+\frac{1}{2}) - \frac{1}{2}\sqrt{1+4\lambda_1+4\mu^2} = -n_2, \quad n_2 = 0,1,2,\ldots$$

or $$\lambda = (\mu+n_2)(\mu+n_2+1) - \mu^2 \tag{36}$$

Case 3: $m_1 = m_2 = 2\mu = m < -2\mu$

$$-(\mu+m) + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda+4\mu^2} = -n_3, \quad n_3 = 0,1,2,\ldots$$

or $$\lambda = (n_3-\mu-m)(n_3-\mu-m+1) - \mu^2 \tag{37}$$

To see how we obtain condition (35), we rewrite the solution for case (1) as follows

$$m_1 = m_2 - 2\mu = m \geq 0$$

$$g_1 = Z^{\frac{m}{2}} (Z-1)^{\frac{m}{2}+\mu} {}_2F_1 \left[(\mu+m) + \frac{1}{2} + \frac{1}{2}\sqrt{1+4\lambda+4\mu^2}, \right.$$
$$\left. (\mu+m) + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda+4\mu^2}; \ m+1; \ Z \right]$$

$$g_2 = (Z')^{\frac{m}{2}+\mu} (Z'-1)^{\frac{m}{2}} {}_2F_1 \left[(\mu+m) + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda+4\mu^2}, \right.$$
$$\left. (\mu+m) + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda+4\mu^2}; \ m+2\mu+1; \ Z' \right]$$

As a function of z, g_1 and g_2 satisfy the same second order linear differential equation, in which, g_1 is analytic at z=0, but g_2 is analytic at z=1 and from the boundary condition g_1 and g_2 take the same value and same derivative at the point z=1/2. Thus, the only possibility is that g_1 and g_2 as a function of z are polynorminals. So we obtain the condition (35). The same argument can be applied for conditions (36) and (37).

We see from Equations (35), (36) and (37) that λ can takes only the following values

$$\lambda = (\ell+\mu)(\ell+\mu+1)-\mu^2$$

where ℓ is positive integer or zero. It is easy to see that for a given ℓ, we can have $-\ell-2\mu \leq m \leq \ell$ corresponding to the same eigen value ℓ. These solutions were discussed in reference 3 and 4, but using the singular potential $\vec{A}_1$.

IV. Concluding Remarks

1. We have been successfully proposed a Schrödinger equation for a classical system, which possesses no singularity-free Lagrangian over the whole configuration space, but if we divide the configuration space into overlapped regions, then for each region, the classical system possesses singularity-free Lagrangian.

2. We have been demonstrated that the proposed Schrödinger equation for the charge-monopole case gives Dirac's charge quantization conditions $2\mu=n$ and leads to quantization of the angular-momentum-like quantity $\lambda=(\ell+\mu)(\ell+\mu+1)-\mu^2$.

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