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哥氏空間上運算子之弱緊緻性

Weak Compactness of Operators on Grothendieck Spaces

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ABSTRACT — Weak compactness of operators on Grothendieck spaces is investigated. If Y is a Banach space satisfying one of the following properties: (a) the unit ball of Y* is weak* sequentially compact; (b) Y=C(K) with K a countably compact scattered space; (c) Y is an $\mathcal{L}^1$-space; then every bounded linear operator from a Grothendieck space into Y is weakly compact.

I. Notation and Introduction

For a Banach space X, denote by X* its conjugate space and B_X its unit ball. A subspace of X shall refer to a closed infinite-dimensional linear submanifold.

S denotes a compact Hausdorff space unless otherwise specified. S is said to be Stonian (resp. σ-Stonian) if every open set (resp. open F_σ-set) has open closure. S is an F-space if disjoint open F_σ subsets of S have disjoint closures. If K is a completely regular space, C(K) denotes the space of all bounded (real or complex valued) continuous functions on K endowed with the supremum norm. K is scattered whenever it contains no non-empty perfect subset.

A Banach space X is called a Grothendieck space if every weak* convergent sequence in X* is weakly convergent. The first examples of non-reflexive Grothendieck spaces were exhibited in [5, p.168], where it was proved that if S is stonian, then C(S) is a Grothendieck space. Later these results were improved by Andô [2], Semadeni [11] for σ-Stonian spaces and Seever [12] for F-spaces. It seems to be difficult to establish an internal characterization of Grothendieck spaces; even for Grothendieck space of C(S) type, a topological characterzation of S is still left open. However, a simple necessary and sufficient condition was given in terms of weak compactness of operators [5]. In the sequel, we shall investigate weak compactness of operators on Grothendieck spaces. If a Banach space Y satisfies one of the following properties: (a) B_{Y*} is weak* sequentially compact; (b) Y=C(K) with K a countably compact scattered space; (c) Y is an $\mathcal{L}^1$-space in the sense of [7]; then every bounded linear operator from a Grothendieck space into Y is weakly compact.

II. Weak Compactness of Operators on Grothendieck Spaces

Let X, Y, be Banach spaces. We shall denote by B(X,Y) the Banach space of bounded linear operators from X to Y, and W(X,Y) the subspace of weakly compactly operators.

THEOREM 1. Let X be a Banach space; then the following statements are equivalent:

(i) X is a Grothendieck space;

(ii) B(X,Y)=W(X,Y) whenever Y is a Banach space such that the unit ball of Y* is weak* sequentially compact;

(iii) $B(X,c_o)=W(X,c_o)$.

Proof:

(i) ⟹ (ii): Let X be a Grothendieck space and Y be a Banach space such that the unit ball B_{Y*} of Y* is weak* sequentially compact. For each TεB(X,Y), consider its adjoint operator T*εB(Y*,X*). Since B_{Y*} is weak* sequentially compact and T* is weak* continuous, $T^*(B_{Y*})$ is also weak* sequentially compact. But X is a Grothendieck space; $T^*(B_{Y*})$ is then weakly sequentially compact; hence it is conditionally weakly compact by Eberlein's Theorem. Therefore TεW(X,Y).

(ii) ⟹ (iii) is trivial.

(iii) ⟹ (i): Suppose (x_n^*) is a sequence in X* which converges to zero in the weak* topology. Define TεB(X,c_o) by $T(x)=(x_n^*(x))\varepsilon c_o$ for each xεX. Let $P_n\varepsilon B_{c_o^*}$ be the canonical projection of c_o defined by $P_n^o(\xi)=\xi_n$ for $\xi\varepsilon c_o$, then $T^*(P_n)=x^*{}_n$. Thus (x_n^*) is in the set $T^*(B_{c_o^*})$; but T is weakly compact, hence so is T*, and $T^*(B_{c_o^*})$ is weakly sequentially compact by Eberlein's Theorem. Since (x_n^*) was weak* convergent, it is weakly convergent.

Remark. The proof of (i) ⟹ (ii) is a modification of Grothendieck's proof in [5] for separable Banach spaces Y. (iii) ⟹ (i) is due to Grothendieck; as we shall see, it also follows immediately from the representation theorem quoted in the proof of Theorem 5.

COROLLARY 2. Every Banach space that is a continuous linear image of a Grothendieck space is a Grothendieck space. In particular, every complemented subspace and every quotient space of a Grothendieck space is a Grothendieck space.

Proof: Follows from the equivalence of (i) and (iii) in Theorem 1.

A Banach space Y is said to be weakly compactly generated (briefly, WCG) if it contains a weakly compact set whose linear span is dense in Y. Obviously, separable and reflexive spaces are WCG.

PROPOSITION 3. If Y is a WCG Banach space, then the unit ball of Y* in the weak* topology is homeomorphic to a weakly compact set of some Banach space [1].

COROLLARY 4. Let X be a Grothendieck space, and Y be a WCG space. Then $B(X,Y)=W(X,Y)$.

Proof: By Proposition 3, the unit ball of Y* in the weak* topology is homeomorphic to a weakly compact set of some Banach space, hence weakly sequentially compact by Eberlein's Theorem. The result then follows from Theorem 1.

In the next theorem and its corollaries, we assume that K is a completely regular Hausdorff space.

THEOREM 5. Let X be a Grothendieck space, and let K be sequentially compact. Then $B(X,C(K))=W(X,C(K))$.

Proof: By the representation theorem in [13], a bounded linear operator $T: X \to C(K)$ is weakly compact if and only if the function $\tau: K \to X^*$ defined by $\tau(k)(x)=Tx(k)$ (τ is then continuous as a map into X* with the weak* topology) maps K onto a conditionally compact subset in the weak topology of X*. Now suppose $T \in B(X,C(K))$ is given and τ is defined as above; then $\tau(K)$ is weak* sequentially compact, for K is assumed to be sequentially compact. Hence, by Eberlein's Theorem and the assumption that X is a Grothendieck space, we conclude that $\tau(K)$ is weakly conditionally compact. It then follows from the representation theorem that T is weakly compact.

The following proposition was proved by Baker [3], while our proof here is considerably simpler.

PROPOSITION 6. Let K be countably compact and scattered; then K is sequentially compact.

Proof: Suppose $(k_n) \subset K$ has no convergent subsequence; since K is countably compact, the set of cluster points of (k_n) is nonempty and perfect, which contradicts the assumption that K is scattered. Therefore, K must be sequentially compact.

COROLLARY 7. Let X be a Grothendieck space, and let K be countably compact and scattered; then $B(X,C(K))=W(X,C(K))$.

COROLLARY 8. If K is sequentially compact (which is in particular the case when S is countably compact and scattered) then C(K) contains no subspace isomorphic to ℓ^∞.

Proof: Otherwise there would exist an operator from ℓ^∞ into C(K) that is not weakly compact, which contradicts Theorem 5.

LEMMA 9. Let S be a compact Hausdorff space and Y a weakly complete Banach space; then $B(C(S),Y)=W(C(S),Y)$ [4, p.494].

THEOREM 10. Every bounded linear operator from a Grothendieck space into an $\mathcal{L}^1$-space is weakly compact.

Proof: Suppose X is a Grothendieck space and Y is an $\mathcal{L}^1$-space. Let $T\varepsilon B(X, Y)$ be given; consider its adjoint $T^*\varepsilon B(Y^*,X^*)$. Since weak* and weak sequential convergences are equivalent in X*, it is easily verified that X* is weakly complete. Furthermore, Y*, being the dual space of an $\mathcal{L}^1$-space, is an injective space [7 , p. 308] hence it can be embedded as a complemented subspace in a suitable C(S) space.

Now T* can be extended to a linear operator on C(S), which must be weakly compact by Lemma 9. It follows that T is weakly compact.

Remark 1. If S is a compact Hausdorff space, then C(S) is WCG if and only if S is homeomorphic to a weakly compact set in some Banach space [1]. For the space $L^1(\mu)$ to be WCG, it is necessary an sufficient that μ be σ-finite [8, p.240]. Hence Theorems 5 and 9 are not consequences of Corollary 4. In particular, let λ be an uncountable ordinal and τ_λ be the scattered space $\{\alpha:\alpha\leq\lambda\}$ in the order topology. Then since the topological closure and sequential closure are not equivalent in τ_λ, $C(\tau_\lambda)$ cannot be WCG.

Remark 2. By Corollary 4, a non-reflexive WCG space can never be a Grothendieck space, nor can it contain a subspace that is a non-reflexive Grothendieck space.

The following theorem is listed for the sake of completeness. We refer to [6, p. 108] for its terminologies and proof.

THEOREM 11. If X is a Grothendieck space and Y is isomorphic to a conjugate Banach space with RNP, then $B(X,Y)=W(X,Y)$.

It is desirable to characterize the Banach space Y such that every bounded linear operator from a Grothendieck space to Y is weakly compact. We remark here that most of the known Grothendieck spaces are continuous linear images of C(S) with S an F-space; for such spaces X, the above problem has been completely settled by the following result of Pełczyński and Rosenthal [9, p. 32] [10], which asserts that if Y contains no subspace isomorphic to ℓ^∞ then $B(X, Y)=W(X, Y)$. This arises a question that whether Banach spaces contain no subspace isomorphic to ℓ^∞ will achieve the same purpose for the weak compactness of operators on general Grothendieck spaces.

THEOREM 12. (a) Let S be a compact Hausdorff space, Y a Banach space and $T: C(S) \to Y$ a bounded linear operator which is not weakly compact. Then there exists a subspace Z_o of C(S), isometric to c_o, such that $T|_{Z_o}$ is an isomorphism. (b) If in addition, S is a σ-Stonian space, then there exists a subspace Z of C(S) isometric to ℓ^∞ such that $T|_Z$ is an isomorphism. (c) IF S is a compact F-space, and X is a non-reflexive continuous linear image of C(S), then ℓ^∞ is a continuous linear image of X. (d) Assuming the Continuum Hypothesis, the ássertion (b) holds when S is merely a compact F-space.

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