The Journal of National Chiao Tung University
Vol. 2, December 1976
pp.151-156

e^{-z} 之帕德近似函數之討論

On the Padé Approximants to e^{-z}

倪維城 Wei-Chen Ni

Department of Applied Mathematics, N. C. T. U.

(Received October 11, 1976)

ABSTRACT — In this paper, we establish a lower bound as well as an upper bound for the error of the $(n-\mu,n)$-th Padé approximant to e^{-z}. Moreover, we find a condition under which $\{R_{n-\mu,n}(x)\}_{n=1}^{\infty}$ can not converge geometrically to e^{-z} on $[0, +\infty)$.

I. Introduction

It has been proved in [2, Prop. 2.4] that there exist positive constants A_1 and A_2 such that for all $n>1$, $\frac{A_1}{n}$ and $\frac{A_2 \ln n}{n}$ serve as lower and upper bounds respectively for the error term $\eta_{n-1,n}$ of the $(n-1,n)$-th Padé approximant to e^{-x}; i.e.,

$$\frac{A_1}{n} \le \eta_{n-1,n} \le \frac{A_2 \ln n}{n} \text{ for all } n>1.$$

In this paper we shall improve and generalize the above result by showing that for ν fixed and $0 \le \nu \le n$, there exist positive constants A and B such that

$$\frac{A}{n^{\nu}} \le \eta_{n-\nu,n} \le \frac{B}{n^{\nu}} .$$

Saff and Varga have also established in [2] a sufficient condition for the geometric convergence of a sequence of Padé approximants to e^{-x} on $[0,+\infty)$ in the uniform norm. We, however, shall give a sufficient condition under which the Padé approximants $\{R_{n-\mu,n}\}_{n=1}^{\infty}$ cannot converge geometrically in the uniform norm to e^{-x}.

II. Results and Discussion

Let π_m denote the set of all complex polynomials in the variable z with degree at most m, and let $\pi_{\nu,n}$ denote the set of all complex rational functions

$R_{\nu,n}(z)$ of the form $R_{\nu,n}(z)=\frac{Q_{\nu,n}(z)}{P_{\nu,n}(z)}$, where $Q_{\nu,n}(z)\varepsilon\pi_\nu$, $P_{\nu,n}\varepsilon\pi_n$ and $P_{\nu,n}(0)=1$. For any function $f(z)=\sum_{k=0}^{\infty}a_k z^k$ analytic in a neighborhood of z=0, and for any nonnegative integers ν and n, the (ν,n)-th Padé approximant to f(z) is defined as the element $R_{\nu,n}\varepsilon\,\pi_{\nu,n}$ for which

$$f(z)-R_{\nu,n}(z)=\mathcal{O}(|z|^m) \tag{1}$$

as $|z|\to 0$ is valid for the largest integer m. In[1], it is known that for f(z) $=e^{-z}$, its (ν,n)-th Padé approximant $R_{\nu,n}(z)=\frac{Q_{\nu,n}(z)}{P_{\nu,n}(z)}$ is given explicitly by

$$Q_{\nu,n}(z)\equiv\sum_{k=0}^{\nu}\frac{(\nu+n-k)!\,\nu!\,(-z)^k}{(\nu+n)!\,k!\,(\nu-k)!} \tag{2}$$

and

$$P_{\nu,n}(z)\equiv\sum_{k=0}^{n}\frac{(\nu+n-k)!\,n!\,z^k}{(\nu+n)!\,k!\,(n-k)!} \tag{3}$$

We shall denote $\sup_{0<x<+\infty}|R_{n-\mu(n),n}(x)-e^{-x}|$ by $\eta_{n-\mu(n),n}$. Our object is to find a lower bound for $\bar{\eta}_{n-\mu(n),n}$ with the condition $\lim_{n\to\infty}\frac{(\mu(n))^2}{n}=0$ imposed on μ(n).

We first state a few lemmas.

LEMMA 1. For $n\geq m$, $\prod_{j=0}^{m-1}(1-\frac{j^2}{n^2})=\frac{1}{n^{2m}}\cdot\frac{(n+m-1)!\,n}{(n-m)!}$.

Proof:

$$\prod_{j=0}^{m-1}(1-\frac{j^2}{n^2})=\frac{1}{n^{2m}}\prod_{j=0}^{m-1}(n^2-j^2)$$

$$=\frac{1}{n^{2m}}\prod_{j=0}^{m-1}(n-j)(n+j)$$

$$=\frac{1}{n^{2m}}(n-m+1)\ldots(n-1).n.n.(n+1)\ldots(n+m-1)$$

$$=\frac{1}{n^{2m}}\frac{(n+m-1)!\,n}{(n-m)!}.$$

LEMMA 2. For $n\geq m+1$, $\prod_{j=1}^{m}(1-\frac{j^2}{n^2})=\frac{(n+m)!}{n^{2m}.n.(n-1-m)!}$.

Proof: A similar argument as in the proof of Lemma 1 yields the desired result.

LEMMA 3. If m is a fixed non-negative integer, and μ(n) is finite, then

$$\lim_{n\to\infty}\frac{(n-m)!\,(n+m)}{(n-\mu(n)-m)!\,n^{\mu(n)+1}}=1.$$

Proof: This follows directly from the fact that $\lim_{n\to\infty}\frac{n-j}{n}=1$ for any fixed j and $\mu(n)$ finite.

LEMMA 4. If m is a fixed positive integer, $\lim_{n\to\infty}\frac{\mu(n)^2}{n}=0$, and $\lim_{n\to\infty}\mu(n)=\infty$, then $\lim_{n\to\infty}\frac{(n-m)!(n+m)}{(n-\mu(n)-m)!n^{\mu(n)+1}}=1$.

Proof: It is not difficult to derive the following inequality:

$$\frac{n+m}{n}\geq\frac{(n-m)!(n+m)}{(n-\mu(n)-m)!n^{\mu(n)+1}}\geq\left(1-\frac{\frac{(\mu(n)+m)\mu(n)}{n}}{\mu(n)}\right)^{\mu(n)}$$

Now since m is a fixed positive integer and $\lim_{n\to\infty}\frac{\mu(n)^2}{n}=0$, by basic properties of logarithm, we have that

$$\lim_{n\to\infty}\left(1-\frac{\frac{(\mu(n)+m)\cdot\mu(n)}{n}}{\mu(n)}\right)^{\mu(n)}=1.$$

Together with $\lim_{n\to\infty}\frac{n+m}{n}=1$, we have $\lim_{n\to\infty}\frac{(n-m)!(n+m)}{(n-\mu(n)-m)!n^{\mu(n)+1}}=1$ as desired.

It is straight forward to establish the following:

LEMMA 5. If $\lim_{n\to\infty}\frac{\mu(n)^2}{n}=0$, then given any $\varepsilon>0$, there exists an integer N such that for all $n>N$,

$$\left|\sum_{m=0}^{n}\frac{(n-\mu(n)+m)!n^{\mu(n)-1}}{(n+m-1)!m!}\prod_{j=0}^{m-1}\left(1-\frac{j^2}{n^2}\right)-\sum_{m=0}^{\infty}\frac{1}{m!}\right|<\varepsilon.$$

Similarly, we have

LEMMA 6. If $\lim_{n\to\infty}\frac{\mu(n)}{n}=0$, then given $\varepsilon>0$, there exists an N such that for all $n>N$,

$$\left|\sum_{m=0}^{n-\mu(n)}(-1)^m\frac{(n-m)!(n+m)}{(n-\mu(n)-m)!n^{\mu(n)+1}m!}\prod_{j=0}^{m-1}\left(1-\frac{j^2}{n^2}\right)-e^{-1}\right|<\varepsilon.$$

From (3),

$$P_{n-\mu(n),n}(x)=\sum_{m=0}^{n}\frac{(2n-\mu(n)-m)!n!}{(2n-\mu(n))!m!(n-m)!}x^m.$$

$$=\frac{n!}{(2n-\mu(n))!}\sum_{m=0}^{n}\frac{(2n-\mu(n)-m)!}{m!(n-m)!}x^m.$$

Letting $k=n-m$, the above equation becomes

$$P_{n-\mu(n),n}(x)=\frac{n!\,x^n}{(2n-\mu(n))!}\sum_{m=0}^{n}\frac{(n-\mu(n)+m)!}{(n-m)!m!x^m}. \qquad (4)$$

Similarly, we have from (2) that

$$Q_{n-\mu(n),n}(x) = \sum_{k=0}^{n-\mu(n)} (-1)^k \frac{(2n-\mu(n)-k)!(n-\mu(n))!}{(2n-\mu(n))!k!(n-\mu(n)-k)!} x^k$$

$$= \frac{(n-\mu(n))!}{(2n-\mu(n))!} x^{n-\mu(n)} \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n+m)!}{(n-\mu(n)-m)!m!x^m} \quad (5)$$

where $m=n-k-\mu(n)$. Thus if we devide (5) by (4), we obtain

$$\frac{Q_{n-\mu(n),n}(x)}{P_{n-\mu(n),n}(x)} = \frac{(n-\mu(n))! \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n+m)!}{(n-\mu(n)-m)!m!x^m}}{n!\, x^{\mu(n)} \sum_{m=0}^{n} \frac{(n-\mu(n)+m)!}{(n-m)!m!x^m}}.$$

Now, replacing x by n^2,

$$\frac{Q_{n-\mu(n),n}(n^2)}{P_{n-\mu(n),n}(n^2)} = \frac{(n-\mu(n))! \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n+m)!}{(n-\mu(n)-m)!m!n^{2m}}}{n!\, n^{2\mu(n)} \sum_{m=0}^{n} \frac{(n-\mu(n)+m)!}{(n-m)!m!n^{2m}}}$$

$$= \frac{(n-\mu(n))! \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n-m)!(n+m)\cdot n\cdot(n+m-1)!}{(n-\mu(n)-m)!m!n\cdot(n-m)!n^{2m}}}{n!n^{2\mu(n)} \sum_{m=0}^{n} \frac{(n-\mu(n)+m)!n\cdot(n+m-1)!}{(n+m-1)!m!n\cdot(n-m)!n^{2m}}} \quad (6)$$

Lemmas 1 and 2 permits us to write (6) as

$$\frac{Q_{n-\mu(n),n}(n^2)}{P_{n-\mu(n),n}(n^2)} = \frac{(n-\mu(n))! \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n-m)!(n+m)}{(n-\mu(n)-m)!\cdot n\cdot m!} \prod_{j=0}^{m-1} (1-\frac{j^2}{n^2})}{n!n^{2\mu(n)} \sum_{m=0}^{n} \frac{(n-\mu(n)+m)!}{(n+m-1)!m!n} \sum_{j=0}^{m-1} (1-\frac{j^2}{n^2})}$$

$$= \frac{(n-\mu(n))! \sum_{m=0}^{n-\mu(n)} (-1)^m \frac{(n-m)!(n+m)}{(n-\mu(n)-m)!n^{\mu(n)+1}} \prod_{j=0}^{n-1} (1-\frac{j^2}{n^2})}{n! \sum_{m=0}^{n} \frac{(n-\mu(n)+m)!n^{\mu(n)-1}}{(n+m-1)!m!} \prod_{j=0}^{m-1} (1-\frac{j^2}{n^2})}.$$

By Lemmas 5 and 6, given $\varepsilon>0$, there exists an N such that for all $n>N$,

$$\frac{Q_{n-\mu(n),n}(n)^2}{P_{n-\mu(n),n}(n^2)} \geq \frac{(n-\mu(n))! \left(\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!}-\varepsilon\right)}{n! \left(\sum_{m=0}^{\infty} \frac{1}{m!}+\varepsilon\right)}.$$

Hence given an arbitrary ε with $\frac{1}{2e} \geq \varepsilon > 0$,

$$\left\| \frac{Q_{n-\mu(n),n}(x)}{P_{n-\mu(n),n}(x)} \right\|_{L_\infty[0,+\infty)} > \frac{(n-\mu(n))!}{4n!e^2}$$

as $n > N$, providing $\lim_{n\to\infty} \frac{\mu(n)^2}{n} = 0$. We have proved the following Theorem.

THEOREM 7. For any $\mu(n)$ with $0 \leq \mu(n) \leq n$ and $\lim_{n\to\infty} \frac{\mu(n)^2}{n} = 0$, $\eta_{n-\mu(n),n} \geq \frac{(n-\mu(n))!}{4n!e^2}$ for all a sufficiently large.

COROLLARY 8. If μ is a fixed non-negative integer, then

$$\eta_{n-\mu,n} \geq \frac{A}{n^\mu}$$

where A is a positive constant.

Proof: This is an immediate consequence of Thm. 7.

Combining Corollary 8 and [3; Thm. 1], we have established the following result.

THEOREM 9. For any μ fixed with $0 \leq \mu \leq n$, there exist positive constants A and B such that

$$\frac{A}{n^\mu} \leq \eta_{n-\mu,n} \leq \frac{B}{n^\mu}.$$

THEOREM 10. If $0 \leq \mu(n) \leq n$ and $\lim_{n\to\infty} \frac{\mu(n)^2}{n} = 0$, then the sequence of Padé approximants, $\{R_{n-\mu(n),n}(x)\}_{n=1}^{\infty}$, cannot converge geometrically to e^{-x} on $[0, +\infty)$; i.e.,

$$\overline{\lim_{n\to\infty}} (\eta_{n-\mu(n),n})^{\frac{1}{n}} = 1.$$

Proof: From [3, Thm. 1], we know that $\eta_{n-\mu(n),n} \leq 1$. Hence it is enough to show that

$$\overline{\lim_{n\to\infty}} (\eta_{n-\mu(n),n})^{\frac{1}{n}} \geq 1.$$

By Theorem 7, it suffices to show that

$$\lim_{n\to\infty} \left(\frac{(n-\mu(n))!}{n!\, e^3}\right)^{\frac{1}{n}} \geq 1.$$

Application of stirling's formula directly gives the desired result.

References

1. O. Perron, "Die Lehre von den Kettenbrücken II", Teubuer, Leipzig; 1929.
2. E. B. Saff and R. S. Varga, "Convergence of Padé Approximants to e^{-z} on Unbounded Sets"

J. Approximation Theory, 13,470-488, (1975).

3. E. B. Saff, R. S. Varga and W. C. Ni, "Geometric convergence of Rational Approximations to e^{-z} in Infinite Sectors" Numer. Mathematik, 26, 211-225 (1976).