The Journal of National Chiao Tung University
Vol. 2, December 1976
pp. 179-184

伯氏曲率張量為零的凱氏複流型

On a Kahlerian Manifold with Vanishing Bochner Curvature Tensor

余文能 Wen-Neng Yu*

Department of Applied Mathematics, N.C.T.U.

(Received September 17, 1976)

ABSTRACT — We shall consider a 2n real dimensional Kahlerian manifold M^{2n} $(n>2)$, with complex structure $(J_i{}^j)$, which admits a K torse-forming vector field (ξ^h), hence

$$\nabla_i \xi^h = a\delta_i{}^h + bJ_i{}^h + \alpha_i \xi^h + \beta_i \tilde{\xi}^h,$$

and its Christoffel symbol $\{{h \atop ji}\}$ takes the following form

$$\{{h \atop ji}\} = \rho_{(j}\delta_{i)}{}^h + \tilde{\rho}_{(j}J_{i)}{}^h + f_{ji}\xi^h - J_j{}^r f_{ir}\tilde{\xi}^h,$$

where a, b are functions, α_i, β_i, ρ_i are covariant vectors, $\delta_i{}^j$ is the Kronecker delta, f_{ji} is a symmetric tensor and $\tilde{\xi}^h = J_r{}^h \xi^r$, $\tilde{\rho}_k = -J_k{}^r \rho_r$. With

$$U_{kji} = \nabla_{[k} f_{j]i} + (f'_{[j} - \alpha_{[j})f_{k]i} - (f''_{[j} + \beta_{[j})f_{k]r}J_i{}^r,$$

$$f_j' = f_{jr}\xi^r, \quad f_j'' = f_{jr}\tilde{\xi}^r,$$

$$U_{jr}{}^r = \lambda|\xi|^2, \quad U_{ir\ell}\xi^j\xi^r\tilde{\xi}^\ell = -\sigma|\xi|^4,$$

and under the condition $2\lambda + (n+1)\sigma = 0$, M^{2n} has vanishing Bochner curvature tensor.

It has been noticed that the Bochner curvature tensor defined on Kahlerian manifold has similar properties corresponding to Weyl's conformal curvature tensor defined on a Riemannian manifold. While we know that Weyl's conformal curvature tensor is invariant under conformal transformation, we don't know what the new transformation, leaving the Bochner curvature tensor invariant, is. Recently S. Yamaguchi and T. Adati have introduced and studied the holomorphically subprojective manifold [2,3]. The purpose of this paper is to study a more general case and to obtain a condition for vanishing Bochner curvature tensor.

Let M^{2n} be a 2n real dimensional Kahlerian manifold with complex structure $(J_i{}^j)$. All indices will range from 1 to 2n. We use ∇_i to denote the covariant differentiation with respect to the i-th coordinate vector. The Einstein summation convention is assumed. Suppose that M^{2n} admits a K torse-forming vector field (ξ^h), [3], hence

---

* Partially suplorted by National Science Council

$$\nabla_i \xi^h = a\delta_i^{\ h} + bJ_i^{\ h} + \alpha_i \xi^h + \beta_i \tilde{\xi}^h,$$

and that the Christoffel symbol takes the form

$$\left\{ {h \atop ji} \right\} = \rho_j \delta_i^{\ h} + \rho_i \delta_j^{\ h} + \tilde{\rho}_j J_i^{\ h} + \tilde{\rho}_i J_j^{\ h} + f_{ji} \xi^h - J_j^{\ r} f_{ir} \tilde{\xi}^h,$$

where $\delta_i^{\ h}$ is the Kronecker delta, a, b are functions, $\alpha_i, \beta_j, \rho_i$ are covariant vectors, f_{ij} is a symmetric tensor and $\tilde{\xi}^h = J_r^{\ h} \xi^r$. $\tilde{\rho}_k = -J_k^{\ r} \rho_r$.

We shall use ( ) (resp. [ ]) for indices to mean the symmetric (resp. skew-symmetric) part of indices considered. First it follows from $\nabla_k J_j^{\ h} = 0$ that $\partial_k J_j^{\ h} = 0$, and from $\left\{ {h \atop j\,i} \right\} = \left\{ {h \atop ij} \right\}$ that

$$J_{[j}^{\ \ r} f_{i]r} = 0 \tag{1}$$

The curvature tensor $R_{kji}^{\ \ \ h}$ is obtained by a straightforward and rather complicated computation.

$$R_{kji}^{\ \ \ h} = u_{[kj]} \delta_i^{\ h} - u_{[k|r|} J_{j]}^{\ \ r} J_i^{\ h} + \delta_{[j}^{\ \ h} u_{k]i} - J_{[j}^{\ \ h} u_{k]r} J_i^{\ r}$$
$$+ \xi^h U_{kji} - \tilde{\xi}^h U_{kjr} J_i^r, \tag{2}$$

where

$$u_{ki} = \nabla_k \rho_i + \rho_k \rho_i - \tilde{\rho}_k \tilde{\rho}_i - (a - \rho') f_{ki} - (b + \rho'') f_{kr} J_i^{\ r},$$

$$\rho' = \rho_r \xi^r, \quad \rho'' = \rho_r \tilde{\xi}^r,$$

$$U_{kji} = \nabla_{[k} f_{j]c} + (f'_{[j} - \alpha_{[j}) f_{k]i} - (f''_{[j} + \beta_{[j}) f_{k]r} J_i^{\ r},$$

$$f'_j = f_{jr} \xi^r, \quad f''_j = f_{jr} \tilde{\xi}^r.$$

By quite the same method as in [2], we can prove that ρ_i is closed, so that $u_{[kj]} = 0$. It's obvious that

$$U_{(kj)i} = 0 \tag{3}$$

From (1) and (3) we have

$$U_{[kji]} = 0 \tag{4}$$

Since $g^{i\ell} R_{kji\ell} = 0$, we have

$$U_{kjr} \xi^r = 0 \tag{5}$$

Direct computation shows that

$$J_{[k}^{\ \ r} U_{ji]r} = 0 \tag{6}$$

holds.

Using (4) and (5) we can easily get

$$J^{rs}U_{\ell rs}\xi^{\ell}=0. \tag{7}$$

Now we note that the Ricci curvature tensor has three kinds of expression given by

$$R_{ji}=R_{kji}{}^{k}$$

$$=-2n\,u_{ji}-2J_j{}^rJ_i{}^tu_{rt}+U_{rji}\xi^r-J_i{}^rU_{\ell jr}\tilde{\xi}^{\ell}. \tag{8}$$

$$R_{ji}=g^{kh}R_{jkhi}$$

$$=-u_{ji}-u_r{}^rg_{ji}-J_j{}^rJ_i{}^{\ell}u_{r\ell}+U_{jr}{}^r\xi_i-U_{j\ell r}J^{\ell r}\tilde{\xi}_i. \tag{9}$$

$$R_{ji}=-\tfrac{1}{2}J_i{}^kR_{jkh\ell}J^{k\ell}$$

$$=-(n+1)u_{ji}-(n+1)J_j{}^rJ_i{}^{\ell}u_{r\ell}+J_i{}^rU_{jr\ell}\tilde{\xi}^{\ell}. \tag{10}$$

From (9), since $R_{[ji]}=0$, it follows that

$$(U_{jr}{}^r\xi_i-U_{ir}{}^r\xi_j)=(U_{j\ell r}J^{\ell r}\tilde{\xi}_i-U_{ir\ell}J^{\ell r}\tilde{\xi}_j). \tag{11}$$

With $U_{jr}{}^r\xi^j=\lambda|\xi|^2$, then transvecting (11) by ξ^j and using (7) we get

$$U_{ir}{}^r=\lambda\xi_i. \tag{12}$$

Substituting (12) into (11), we have

$$U_{j\ell r}J^{\ell r}\tilde{\xi}_i-U_{i\ell r}J^{\ell r}\tilde{\xi}_j=0. \tag{13}$$

With $U_{j\ell r}J^{\ell r}\tilde{\xi}^j=\mu|\xi|^2$, then transvecting (13) by $\tilde{\xi}^j$, we find

$$U_{i\ell r}J^{\ell r}=\mu\tilde{\xi}_i. \tag{14}$$

Using (4) we find $J^{kj}U_{kjr}\tilde{\xi}^r=-2\mu|\xi|^2$. However $J^{kj}U_{kjr}\tilde{\xi}^r=J^{kj}U_{kjr}J_t{}^r\xi^t$. Hence by (6) we get $J^{kj}U_{kjr}\tilde{\xi}^r=2\lambda|\xi|^2$. It follows that $\lambda+\mu=0$.

From (8)-(9) we find

$$(n-1)[u_{ji}-J_j{}^rJ_i{}^{\ell}u_{r\ell}]=U_{rji}\xi^r+J_i{}^rU_{r\ell j}\tilde{\xi}^{\ell}. \tag{15}$$

Contracting (15) by ξ^i and using (5) we get

$$u_{j\ell}\xi^{\ell}=J_j{}^ru_{r\ell}\tilde{\xi}^{\ell}. \tag{16}$$

Contracting (15) by $\tilde{\xi}^i$ and using (5) we get

$$u_{j\ell}\tilde{\xi}^{\ell}=-J_j{}^r u_{r\ell}\xi^{\ell}. \tag{17}$$

From (9)-(10), using (12), (14), we get

$$n(u_{ji}+J_j{}^r J_i{}^{\ell} u_{r\ell})-u_r{}^r g_{ji}+\lambda(\xi_j\xi_i+\tilde{\xi}_j\tilde{\xi}_i)-J_i{}^r \cup_{jr\ell}\tilde{\xi}^{\ell}=0. \tag{18}$$

Contracting (18) by ξ^i and $\tilde{\xi}^i$ resp., using (16), (17) we have

$$\cup_{jr\ell}\tilde{\xi}^r\tilde{\xi}^{\ell}=2n\ u_{j\ell}\xi^{\ell}+\varepsilon\xi_j, \tag{19}$$

$$\cup_{rj\ell}\xi^r\tilde{\xi}^{\ell}=2n\ u_{j\ell}\tilde{\xi}^{\ell}+\varepsilon\tilde{\xi}_j, \tag{20}$$

where $\varepsilon=\lambda|\xi|^2-u_r{}^r$.

Next, by considering $\delta_{[m}{}^{(\ell} R_{kj]}{}^{ih))}=0$, letting $\ell=m$, we have

$$\begin{aligned}&2n\ u_{[j|r|}J^{(i|r|}J_{k]}{}^{h)}+\delta_{[j}{}^{(h}u_{k]}{}^{i)}+u_{mr}\delta_{[k}{}^{(i}J^{h)r}J_{j]}{}^m\\ &+2n\cup_{kj}{}^{(i}\xi^{h)}+\delta_{[k}{}^{(i}\cup_{j]m}{}^{h}\xi^m+\delta_{[k}{}^{(h}\cup_{j]m}{}^{|m|}\xi^{i)}-2n\cup_{kjr}J^{(i|r|}\tilde{\xi}^{h)}\\ &-\delta_{[k}{}^{(i}\cup_{j]mr}J^{h)r}\tilde{\xi}^m-\delta_{[k}{}^{(h}\cup_{j]mr}\tilde{\xi}^{i)}J^{mr}=0\end{aligned} \tag{21}$$

Contracting (21) by ξ_h and bringing i down yields

$$\begin{aligned}&2n\ u_{[k|r|}\tilde{\xi}_{j]}J_i{}^r+2n\ u_{[j|r|}\tilde{\xi}^r J_{k]i}+\xi_{[j}u_{k]i}+g_{[j|i|}u_{k]r}\xi^r\\ &+J_{[j}{}^m g_{k]i}u_{mr}\tilde{\xi}^r+\xi_{[k}J_{j]}{}^m J_i{}^r u_{mr}+2n|\xi|^2\cup_{kji}+\xi_{[j}\cup_{|m|k]i}\xi^m\\ &+\xi_{[k}\cup_{j]m}{}^m\xi_i+|\xi|^2 g_{[k|i|}\cup_{j]m}{}^m-2n\tilde{\xi}_i\cup_{kjr}\tilde{\xi}^r-g_{[k|i|}\cup_{j]mr}\tilde{\xi}^m\tilde{\xi}^r\\ &-\xi_{[k}\cup_{j]mr}J_i{}^r\tilde{\xi}^m-\xi_{[k}\cup_{j]mr}J^{mr}\tilde{\xi}_i=0.\end{aligned} \tag{22}$$

Transvecting (22) by ξ^j yields

$$\begin{aligned}&2n\ u_{[i|\ell|}\tilde{\xi}^{\ell}\tilde{\xi}_{k]}+|\xi|^2(u_{ki}-J_k{}^m J_i{}^r u_{mr})+(2n-1)|\xi|^2\cup_{kri}\xi^r\\ &+(\lambda|\xi|^4-2n\ u_{r\ell}\xi^r\xi^{\ell}-\varepsilon|\xi|^2)g_{ki}-\lambda|\xi|^2\xi_k\xi_i+4n^2 u_{k\ell}\tilde{\xi}^{\ell}\tilde{\xi}_i\\ &+(2n\varepsilon+\mu|\xi|^2)\tilde{\xi}_i\tilde{\xi}_i+2n\ u_{k\ell}\xi^{\ell}\xi_i-2n\ u_{i\ell}\xi^{\ell}\xi_k+|\xi|^2 J_i{}^r\cup_{kmr}\tilde{\xi}^m=0.\end{aligned} \tag{23}$$

Transvecting (23) by $\tilde{\xi}^k$ yields

$$\begin{aligned}&(2n-1)|\xi|^2\cup_{kri}\tilde{\xi}^k\xi^r\\ &=-2n|\xi|^2 u_{i\ell}\tilde{\xi}^{\ell}+[4n(1-n)u_{r\ell}\xi^r\xi^{\ell}+(1-2n)\varepsilon|\xi|^2]\tilde{\xi}_i.\end{aligned} \tag{24}$$

From (24), using (4), (5), (20), we can get

$$u_{i\ell}{}^{\bullet}\tilde{\xi}^{\ell}=\tilde{\varepsilon}\tilde{\xi}_i, \tag{25}$$

where $u_{r\ell}\xi^r\xi^\ell=\tilde{\varepsilon}|\xi|^2$.

With $\sigma|\xi|^2=-(2n\tilde{\varepsilon}+\varepsilon)$ and using (25), it follows from (24) that

$$U_{\ell ri}\tilde{\xi}^{\ell}\xi^r=\sigma|\xi|^2\tilde{\xi}_i. \tag{26}$$

Now, by computing $\xi^j\xi^h R_{kj(ih)}=0$, we have

$$U_{kri}\xi^r=\sigma\tilde{\xi}_k\tilde{\xi}_i+\tilde{\varepsilon}g_{ki}-u_{ki}. \tag{27}$$

Contracting (22) by $\tilde{\xi}^j$ yields

$$-2n|\xi|^2U_{kri}\tilde{\xi}^r=2n|\xi|^2u_{kr}J_i{}^r+2n\tilde{\varepsilon}|\xi|^2J_{ki}$$
$$+(J^{mr}U_{m\ell r}\tilde{\xi}^\ell+2n\sigma|\xi|^2-\lambda|\xi|^2)\tilde{\xi}_i\xi_k. \tag{28}$$

Contracting (28) by J^{ki} yields

$$J^{mr}U_{m\ell r}\tilde{\xi}^\ell=\lambda|\xi|^2. \tag{29}$$

Substituting (29) back to (28) yields

$$U_{kri}\tilde{\xi}^r=\sigma\xi_k\tilde{\xi}_i+\tilde{\varepsilon}J_{ki}+u_{kr}J_i{}^r. \tag{30}$$

Using (27), (30), it follows from (15) that

$$u_{ji}=J_j{}^rJ_i{}^\ell u_{r\ell}. \tag{31}$$

By using (4), (30), (31), it follows from (18) that

$$2(n-1)u_{ji}=[2(n-1)\tilde{\varepsilon}+(\lambda+\sigma)|\xi|^2]g_{ji}-(\lambda+\sigma)(\xi_j\xi_i+\tilde{\xi}_j\tilde{\xi}_i). \tag{32}$$

From $R_{kj(ih)}\xi^h=0$, we finally have

$$2(n-1)|\xi|^2U_{kji}=-(\lambda+\sigma)|\xi|^2[g_{i[k}\xi_{j]}+J_{[j|i|}\tilde{\xi}_{k]}-2J_{kj}\tilde{\xi}_i]$$
$$+2[2\lambda+(n+1)\sigma]\tilde{\xi}_{[k}\xi_{j]}\tilde{\xi}_i. \tag{33}$$

Substituting (32), (33) into (2), we get

$$R_{kjih}=[\tilde{\varepsilon}+\frac{(\lambda+\sigma)}{2(n-1)}|\xi|^2][g_{i[k}g_{j]h}+J_{i[j}J_{k]h}+2J_{kj}J_{ih}]$$
$$+\frac{(\lambda+\sigma)}{2(n-1)}[2\tilde{\xi}_{[k}\xi_{j]}J_{ih}+\xi_i\xi_{[j}g_{k]h}+\tilde{\xi}_i\tilde{\xi}_{[j}g_{k]h}$$
$$+\xi_{[k}g_{j]i}\xi_h+\tilde{\xi}_{[j}J_{k]i}\xi_h+2J_{kj}\tilde{\xi}_i\xi_h$$

$$+\xi_{[k}J_{j]i}\tilde{\xi}_h+\tilde{\xi}_{[k}g_{j]i}\tilde{\xi}_h-2J_{kj}\xi_i\tilde{\xi}_h$$

$$+\tilde{\xi}_i\ \xi_{[j}J_{k]h}+\xi_i\tilde{\xi}_{[k}J_{j]h}]$$

$$+\frac{2\lambda+(n+1)\sigma}{(n-1)|\xi|^2}(\tilde{\xi}_{[k}\xi_{j]})\cdot(\tilde{\xi}_{[i}\xi_{h]}). \tag{34}$$

Hence the Ricci curvature tensor R_{ji} is given by

$$R_{ji}=-[2(n+1)\tilde{\varepsilon}-\frac{n(\lambda+\sigma)}{(n-1)}|\xi|^2]g_{j}$$

$$+[\frac{(n+2)(\lambda+\sigma)}{(n-1)}-\frac{2\lambda+(n+1)\sigma}{(n-1)|\xi|^2}|\xi|^2](\xi_j\xi_i+\tilde{\xi}_j\tilde{\xi}_i). \tag{35}$$

The Bochner curvature tensor K_{kjih}, [1], is given by

$$K_{kjih}=R_{kjih}+\frac{1}{(n+4)}[L_{ki}g_{jh}-L_{ji}g_{kh}+g_{ki}L_{jh}-g_{ji}L_{kh}$$

$$+M_{ki}J_{jh}-M_{ji}J_{kh}+J_{ki}M_{jh}-J_{ji}M_{kh}+2M_{kj}J_{ih}+2J_{kj}M_{ih}], \tag{36}$$

where $L_{ji}=R_{ji}-\frac{R}{2(n+2)}g_{ji}$, $M_{ji}=J_j{}^rL_{ri}$.
Assuming $2\lambda+(n+1)\sigma=0$ and substituting (34), (35) into (36), then after long but strainghtforward calculation we can easily get $K_{kjih}=0$. Thus we have the following:

MAIN THEOREM: Let M^{2n} $(n>2)$ be a 2n real dimensional Kahlerian manifold such that M^{2n} admits a K torse-forming vector field (ξ^h) [3] and the Christoffel symbol takes the following form

$$\left\{{h \atop ji}\right\}=\rho_{(j}\delta_{i)}{}^h+\tilde{\rho}_{(j}J_{i)}{}^h+f_{ji}\xi^h-J_j{}^rf_{ir}\tilde{\xi}^h$$

Then, if $2\lambda+(n+1)\sigma=0$, M^{2n} has vanishing Bochner curvature tensor.

COROLLARY: Let M^{2n} be as in Main Theorem, then if $\lambda=\sigma=0$, M^{2n} is a space of constant holomorphically sectional curvature.

Acknowledgment

The auther wishes to express his sincere thanks to Professor S. Yamaguchi who gave suggestions and encouragement.

References

(1) S. Tachibana, "On the Bochner curvature tensor," Nat. Sci. Rep. Ochanomizu Univ., 18, 15-19, (1967).

(2) S. Yamaguchi and T. Adati, "On Holomorphically subprojective Kahlerian manifolds 1, 11, 111, to appear.

(3) S. Yamaguchi, "On Kahlerian torse-forming vector fields", to appear.