The Journal of National Chiao Tung University
Vol. 3, September 1977
pp. 67-76

相關不良品檢驗對生產過程管制的影響

The Effect of Correlated Defective Inspection Scheme for Process Control

黃俊雄 **Jun S. Huang**

Department of Applied Mathematics, N. C. T. U.

楊照崑 **Mark C. K. Yang**

Department of Statistics, University of Florida

(Received July 15, 1977)

Abstract –– In a continuous production process, samples of the products are taken and inspected at regular time intervals. A statistic of each sample is plotted on a chart to control the production process. In many cases, the inspection of one attribute X of the product is too costly (e.g. destructive inspection), and one would try to take a less costly but less reliable inspection of another attribute Y of the product. This paper tries to answer the question that under what conditions this new attribute Y should be inspected.

A simple Markov model for the production process is presented and the cost factors including cost of a defective, cost of inspection and cost of taking corrective action are considered. An approximate relationship between two different inspection costs corresponding to X and Y is found when both inspections are under optimal inspection schemes.

Key Words: Markov process   Attribute Inspection   Optimal Stopping Time

I. Introduction

In a continuous production process, samples of products are taken and inspected at regular time intervals, and a statistic of each sample is plotted on a chart which porvides the information to control the production process. In general, one may face a problem that the inspection of a produced item may be too costly and sometimes destructive. In such a situation, one would like to try to find another inspection method which is less costly although may be less reliable to control the production process. This paper deals with the comparison of two inspection methods inspecting two different attributes of the product when their inspection costs and reliabilities are different.

There are many publications (e. g. Baker [1], Bather [2], and Duncan [3], [4]) discussing the economic design of control charts for the measurable quality characteristics. The proposed models vary from one another. For the non-measurable quality characteristic, Lave [6] has proposed a simple Markov model and by way of linear programming, he finds the optimal decision rule for the Sequence plans. In his plan the inspector inspects a definite number of consecutive produced items at the beginning of every regular time interval. The resultant output pattern consists of batches of inspected items alternated with batches of non-inspected items. In his other paper [7], he uses computer simulation to discuss some different sampling plans. Here, his model is simplified and developed in a different way. An analytical approach to find the relationship between two correlated inspection costs is proposed. Thus some special conditions are imposed. First, a Sequence(1) plan and a single-sample decision rule are applied to maintain the production process in control. Thus we sample and inspect the last item produced when the process produces m items (m: positive integer), and we take corrective action if the inspected item is defective. Secondly, the fraction of the defectives produced is assumed to be very small, if the production process is in the desired quality level; and is assumed to be large enough if the production process is in the undesired quality level. Thirdly, the optimal scheme is to minimize the expected total cost per unit time in the long run, or, the ratio of the expected total cost divided by the expected stopping time.

The Bayesian decision approach for process control has been studied by Girshick and Rubin [5]. They assume the process has two quality levels and four states. Then they derive an integral equation which is too complicated to be solved analytically. Here, we do not try to find a complicated Bayes' rule, but propose a simple rule which is practically reasonable. This will enable us to find some approximate solutions and simplify the evaluation of the correlated inspection scheme.

Our main object is to compare the two correlated inspection schemes in process control. The correlated inspection has been studied by Owen, Mcintire and Seymour [8] in a problem to increase acceptable products. The problem can be formulated as follows : If the product has performance variable Y, which is normally distributed with known mean and known standard deviation. A one-sided lower specification limit, L, is given on Y, i.e., all products with Y values below L are not acceptable. Suppose the proportion of acceptable product in the population is now γ, and it is necessary to raise the proportion of the acceptable product to δ, $\delta > \gamma$. It is not possible to measure Y directly, because the inspection of Y is destructive. Suppose there is a second variable X_1 which is correlated with Y and X_1 can be measured easily. As a result, we can use measurements on X_1 to screen out a sufficiently large number of products which have a high probability of having values of Y below the specification limits, L, so as to raise the proportion of acceptable products in the long run from γ to δ. They publish some tables corresponding to different values of parameters in this problem. A similar situation about the correlated measurements has been studied by Tenenbein [9], [10], for estimation of binomial data. However, in process control, the use of the correlated variable (or, screen variable) makes the problem far more complicated. Yet, we try to find a way to compare two different inspection costs corresponding to two correlated quality variables of the product in maintaining the process in control.

II. The Average Cost for one Inspection Scheme

Suppose that a non-aging production process has two quality levels, based on the attribute X of its products, and when the production process is running, there are certain probabilities of the process shifting from one quality level to the other. By non-aging we mean that the Markov probability transition matrix P of the process shift from time t to time t+1 is independent of t, t = 0, 1, 2, ... Let P be given by

$$\begin{array}{c c} & \text{quality level 1} \quad \text{quality level 2} \\ \begin{array}{c}\text{quality level 1}\\ \text{quality level 2}\end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{bmatrix} \end{array}$$

where $0 < \alpha < 1$.

The two quality levels 1, 2 are generally referred to as the desired and undesired levels respectively. It is clear from the transition matrix P that once the process is in level 2, it remains there unless some corrective action is taken. The probability of the process shifting from level 1 to level 2 in a unit time is α. Now consider a Sequence (1) plan where the inspector inspects the last item produced when the process produces m items, m being a positive integer. The inspector will take corrective action if the inspected item is defective, based on the attribute X. Let λ_1 and λ_2 be the fractions of defectives produced corresponding to quality levels 1 and 2 respectively. It is assumed that λ_1 is very small and $\lambda_2 >> \lambda_1$. The assumption is reasonable since in many cases, an efficient production process will keep λ_1 to a very small value unless some part of the machine breaks down, and when some part of the machine breaks down, it will produce high percentage of defectives.

Let $\beta = 1 - (1-\alpha)^m$, i.e. β is the probability of the process shifting from level 1 to level 2 within time interval $0 < t \leq m$. Let N be the stopping random variable corresponding to the time of taking a corrective action. Then the probability of $N = n \cdot m$ for any positive integer n is given by:

$$P(N=nm) = \frac{\lambda_1}{1-\lambda_1} r_1^n + \frac{\lambda_2\beta}{r_2 - r_1}(r_2^n - r_1^n),$$

where $r_1 = (1-\lambda_1)(1-\beta)$ and $r_2 = 1 - \lambda_2$. The expected value E(N) of N, or the expected stopping time is given by:

$$E(N) = \frac{m(\lambda_2 + \beta - \lambda_2\beta)}{\lambda_2(\lambda_1 + \beta - \lambda_1\beta)} \tag{1}$$

We see that if $\alpha \to 1$ then $\beta \to 1$ or $E(N) \to \frac{m}{\lambda_2}$; if $\alpha \to 0$ then $E(N) \to \frac{m}{\lambda_1}$; if α increases then E(N) decreases; and if α decreases then E(N) increases.

Now, we define the cost model of the process. Let C_b be the cost of producing a defective item, C_I be the cost of inspecting an item based on the attribute X, and C_s be the cost of corrective action, or the stopping cost. Then the expected total cost until the time action is taken, $E(C_N)$, is given by:

$$E(C_N) = \lambda_2 C_b E(N) + C_b + C_s + \frac{C_I}{m} E(N) - \frac{\lambda_2 C_b}{m} E(N) + \frac{(\beta - \alpha)(\lambda_1 - \lambda_2) C_b}{\alpha(1 - r_1)} \tag{2}$$

Thus the ratio of $E(C_N)$ divided by E(N), the expected cost per unit time in running the production process is given by:

$$R_1 = \frac{E(C_N)}{E(N)} = \lambda_2 C_b + \frac{C_I}{m} + \frac{C_s}{E(N)} + \frac{\lambda_2 D_b \beta(\alpha - 1)}{\alpha m(\lambda_2 + \beta - \lambda_2\beta)} \tag{3}$$

where $D_b = (\lambda_2 - \lambda_1)C_b$.

We are interested in finding the optimal sampling time interval m_0 so that R_1 is minimized. Differentiating R_1 with respect to m and set the result to zero, i.e. $\frac{dR_1}{dm} = 0$ we have the optimal equation given by:

$$a_1 x^2 + a_2 x + a_4 - a_3 x \log x = 0 \tag{4}$$

where $x = (1-\alpha)^m$, $0 \leq x \leq 1$ and

$$a_1 = \alpha C_I (1-\lambda_2)^2 + \alpha C_s \lambda_2 (1-\lambda_1)(1-\lambda_2) + \lambda_2(1-\lambda_2)(\alpha - 1) D_b,$$

$$a_2 = -2\alpha C_I (1-\lambda_2) - \alpha C_s \lambda_2 (2 - \lambda_1 - \lambda_2) - \lambda_2(2-\lambda_2)(\alpha-1) D_b,$$

$$a_3 = -\alpha C_s \lambda_2(\lambda_2 - \lambda_1) - \lambda_2^2(\alpha - 1) D_b,$$

$$a_4 = \alpha C_I + \alpha\lambda_2 C_s + \lambda_2(\alpha - 1) D_b.$$

Let m_0 be the solution of (4) and $\delta = (1-\alpha)^{m_0}$. It can be seen from (4) that $\delta \to 1$ as $\alpha \to 0$, or $m_0\alpha \to 0$ as $\alpha \to 0$.

Rewriting (4) in terms of β, we have:

$$\alpha C_I[\lambda_1 + (1-\lambda_2)\beta]^2 + \alpha\lambda_2 C_s[(1-\lambda_1)(1-\lambda_2)\beta^2 - m\alpha^*(1-\beta)(\lambda_2 - \lambda_1) +$$

$$\beta(\lambda_1+\lambda_2-2\lambda_1\lambda_2)+\lambda_1\lambda_2]+\lambda_2(\alpha-1)D_b\,[(1-\lambda_2)\beta^2+\lambda_2\beta-\lambda_2(1-\beta)\,m\alpha^*]=0$$

where $\alpha^* = -\log(1-\alpha) = \alpha + \frac{\alpha^2}{2} + \cdots \geq 0$. Since $m\alpha^* = \beta + \frac{\beta^2}{2} + O(\beta^3)$, let $\alpha \to 0$ in the above equation. We have:

$$\lim_{\alpha\to 0} m_0\sqrt{\alpha} = \sqrt{\frac{\lambda_2(C_I+\lambda_1 C_s)}{\frac{(1-\lambda_2)D_b}{2}}}$$

Rewriting (4) in terms of $\frac{dR_1}{dm}$, we get:

$$\alpha m^2(\lambda_2+\beta-\lambda_2\beta)^2\frac{dR_1}{dm} = ((1-\lambda_2)x-1)(a_4-a_5x)+a_3\,x\log x \tag{5}$$

where $a_5 = a_4 - \alpha\lambda_2(C_I+\lambda_1 C_s)$. There are following possibilities:

Case 1 $a_5 \geq 0$ which implies $a_4 > a_5 \geq 0$.

(A) If $a_3 < 0$ then $a_3\,x\log x \leq a_3(x-1) \leq ((1-\lambda_2)x-1)(a_5x-a_4)$. Right hand side of (5) becomes less than or equal to zero; hence $m_0 = \infty$ is the solution of (4).

(B) If $a_3 \geq 0$, then since $0 \leq x \leq 1$, the right hand side of (5) is less than or equal to 0, or $m_0 = \infty$.

Case 2 $a_5 < 0$ which implies $a_3 > 0$, $-a_5x \geq 0$ and $a_3\,x\log x \leq 0$

(A) If $a_4 \geq 0$, then $((1-\lambda_2)x-1)(a_4-a_5x) < 0$, or $\frac{dR_1}{dm} < 0$ i.e. $m_0 = \infty$.

(B) If $a_4 < 0$ then $|a_4| \leq |a_5|$, which implies $\frac{dR_1}{dm} = 0$ has exactly one solution $m_0 < \infty$ except when $\alpha = 0$. For $\alpha = 0$, we see that $m_0 = \infty$ from (3). Hence the necessary and sufficient condition for (4) having a finite solution m_0 is

$$0 < \alpha < \frac{\lambda_2 D_b}{\lambda_2 D_b + C_I + \lambda_2 C_s}\,.$$

This is clear since if C_I is much larger than C_b and α is not small, then no inspection is needed.

To prove m_0 minimizes R_1, note that when m changes from 0 to ∞, $x = (1-\alpha)^m$ changes from 1 to 0 and the right hand side of (5) changes from negative to positive, i.e. R_1 decreases then increases. This shows that m_0 minimizes R_1.

In summary we see that for the production model and Sequency (1) plan described as before, the expected stopping time (or the expected running time until the time action is taken) is given by (1) and the expected cost per unit time, R_1, is given by (3). The optimal sampling time interval m_0 satisfying (4) has following properties:

1) $\lim_{\alpha\to 0} m_0\sqrt{\alpha} = \sqrt{\frac{\lambda_2(C_I+\lambda_1 C_s)}{\frac{(1-\lambda_2)D_b}{2}}}$

2) m_0 is finite if and only if $0 < \alpha < \frac{\lambda_2 D_b}{\lambda_2 D_b + C_I + \lambda_2 C_s}$.

Since the optimal equation (4) is non-linear, we want to get some approximate solution to it. Let $x\log x = \frac{x^2}{2} - ½ + O(x-1)^3$. Then (4) becomes : $a_1x^2 + a_2x + a_4 - a_3\left[\frac{x^2}{2} - ½\right] = 0$. Let the solution of this equation be denoted as x_1 and $\beta_1 = 1 - x_1$. We have:

$$\beta_1 = 1 - x_1 = \frac{b + \sqrt{b^2 - ac}}{|a|}$$

where $a = \alpha C_I(1-\lambda_2)^2 + \alpha C_s\left(1 - \frac{3}{2}\lambda_1 - \frac{\lambda_2}{2} + \lambda_1\lambda_2\right)\lambda_2 + \lambda_2\left[1 - \frac{\lambda_2}{2}\right](\alpha-1)D_b$,

$$b = \alpha\lambda_2 (1 - \lambda_2)(C_I + \lambda_1 C_s) ,$$

$$c = \alpha\lambda_2^2 (C_I + \lambda_1 C_s) ,$$

and $b^2 - ac = (C_I + \lambda_1 C_s)[\lambda_2 D_b(1 - \alpha) - \alpha C_s(\lambda_2 - \lambda_1)]\lambda_2^2(1 - \frac{\lambda_2}{2})\alpha$.

The approximate solution m_1 to m_0 is given by

$m_1 = \frac{\log x_1}{\log(1 - \alpha)} = \frac{\log(1 - \frac{b + \sqrt{b^2 - ac}}{a})}{\log(1 - \alpha)}$, where α is small enough to make $\beta_1 < 1$. The approximation error is $m_1 - m_0 = O(\sqrt{\alpha})$.

Substituting m_1 back into (3) and dropping αC_I and small terms, we get an approximation R_a to the optimal cost per unit time R_0, i.e.

$$R_a = \lambda_2 C_b + D_b \left[\frac{\sqrt{\alpha b_1}\,\lambda_2 D_b + 2\sqrt{e(1 - \frac{\lambda_2}{2})}\,[D_b(3\alpha - 1) + 3\alpha C_s]}{\sqrt{\alpha b_1}\,(4 - 3\lambda_2)D_b + 2\sqrt{e(1 - \frac{\lambda_2}{2})}\,[D_b(1 - 2\alpha) - 2\alpha C_s \frac{(\lambda_2 - \lambda_1)}{\lambda_2}]} \right] \tag{6}$$

where $b_1 = C_I + \lambda_1 C_s$ and $e = \lambda_2 D_b (1 - \alpha) - \alpha C_s(\lambda_2 - \lambda_1)$. The approximation error: $R_a - R_0 = O(\alpha)$.

From the above equation, we see that in case α is small enough, C_I increasing implies R_a increasing; C_b increasing implies R_a increasing, and C_s increasing implies R_a increasing.

III. The Second Inspection Scheme Based on Another Attribute Y

Suppose there is a second quality attribute Y which is correlated to the attribute X discussed in the previous section, and we assume that X can determine the items being good or bad without error. Let the relationship between X and Y be:

$$P(Y = \text{good} \mid X = \text{good}) = p_1$$
$$P(Y = \text{bad} \mid X = \text{bad}) = p_2 \tag{7}$$

where p_1 and p_2 are constants, independent of time and the process levels. The word 'bad' means defective. Let us adopt the same procedure discussed in Section 2 except that we inspect the attribute Y instead of X on each sampled item. Define λ_1^* as the probability of an inspected item being bad according to Y when the process is in level 1. Then $\lambda_1^* = \lambda_1 p_2 + (1 - \lambda_1)(1 - p_1)$. Similarly, we define $\lambda_2^* = P(Y = \text{bad} \mid \text{process is in level 2}) = \lambda_2 p_2 + (1 - \lambda_2)(1 - p_1)$. If we denote the stopping random variable (the time until action is taken) by N^* in this second inspection scheme, then just like in Section 2, we have the expected stopping time

$$E(N^*) = \frac{m(\lambda_2^* + \beta - \lambda_2^* \beta)}{\lambda_2^*(\lambda_1^* + \beta - \lambda_1^* \beta)} . \tag{8}$$

Similarly, define C_I^* to be the new inspection cost, then the expected cost $E(C_N^*)$ of the production process until the time a corrective action is taken, is given by

$$E(C_N^*) = C_s + \frac{C_I^* E(N^*)}{m} + \lambda_2 C_b E(N^*) + \frac{(\lambda_2 - \lambda_1)\beta(\alpha - 1)C_b}{\alpha(\lambda_1^* + \beta - \lambda_1^* \beta)} . \tag{9}$$

Thus, the expected cost per unit time R_1^* is given by:

$$R_1^* = \lambda_2 C_b + \frac{C_I^*}{m} + \frac{C_s}{E(N^*)} + \frac{\lambda_2^* \beta (\alpha - 1) D_b}{\alpha m(\lambda_2^* + \beta - \lambda_2^* \beta)} \tag{10}$$

NOTE: if we put *'s on λ_1, λ_2, C_I, N except $\lambda_2 C_b$ in (3) we get (10) exactly.

Next, we let $\frac{dR_1^*}{dm} = 0$ in order to find the optimal solution m_0^* of the sampling time interval. The optimal equation is given by:

$$a_1^* x^2 + a_2^* x + a_4^* - a_3^* x \log x = 0 \tag{11}$$

where $x = (1 - \alpha)^m$, $a_1^* = \alpha C_I^*(1 - \lambda_2^*)^2 + \alpha C_s \lambda_2^*(1 - \lambda_1^*)(1 - \lambda_2^*) + \lambda_2^*(1 - \lambda_2^*)(\alpha - 1) D_b$,

$$a_2^* = -2\alpha C_I^*(1 - \lambda_2^*) - \alpha C_s \lambda_2^*(2 - \lambda_1^* - \lambda_2^*) - \lambda_2^*(2 - \lambda_2^*)(\alpha - 1) D_b,$$

$$a_3^* = -\alpha C_s \lambda_2^*(\lambda_2^* - \lambda_1^*) - \lambda_2^{*2}(\alpha - 1) D_b,$$

$$a_4^* = \alpha C_I^* + \alpha \lambda_2^* C_s + \lambda_2^*(\alpha - 1) D_b.$$

This equation is similar to (4). Following the same arguments as before, we have the following summary.

For the production model and the new Sequence (1) plan (i.e. inspecting attribute Y) described as before, the expected stopping time is given by (8) and the expected cost per unit time R_1^* is given by (10). The optimal sampling time interval m_0^* satisfying (11) has the following properties:

1) $\lim_{\alpha \to 0} m_0^* \sqrt{\alpha} = \sqrt{\dfrac{\lambda_2^*(C_I^* + \lambda_1^* C_s)}{(1 - \frac{\lambda_2^*}{2}) D_b}}$

2) m_0^* is finite if and only if $0 < \alpha < \dfrac{\lambda_2^* D_b}{\lambda_2^* D_b + C_I^* + \lambda_2^* C_s}$

Using the same procedure in the previous section, we get an approximate solution m_1^* to m_0^*,

$$m_1^* = \log\left(1 - \frac{b^* + \sqrt{d^*}}{|a^*|}\right) / \log(1 - \alpha),$$

where $a^* = \alpha C_I^*(1 - \lambda_2^*)^2 + \alpha C_s \lambda_2^*(1 - \frac{3}{2}\lambda_1^* - \frac{\lambda_2^*}{2} + \lambda_1^* \lambda_2^*) + \lambda_2^*(1 - \frac{\lambda_2^*}{2})(\alpha - 1) D_b$,

$$b^* = \alpha \lambda_2^*(1 - \lambda_2^*)(C_I^* + \lambda_1^* C_s),$$

$$d^* = (C_I^* + \lambda_1^* C_s)(\lambda_2^* D_b (1 - \alpha) - \alpha(\lambda_2^* - \lambda_1^*) C_s) \lambda_2^{*2}(1 - \frac{\lambda_2^*}{2}) \alpha .$$

The approximation error $m_1^* - m_0^* = O(\sqrt{\alpha})$.

Substituting m_1^* back into R_1^* and dropping αC_I^*, $\alpha \lambda_1^* C_s$, α^2, α^3 terms, we have the approximate optimal cost per unit time R_a^* to the optimal cost per unit time R_0^*:

$$R_a^* = \lambda_2 C_b + D_b \left(\frac{\sqrt{\alpha b_1^*}\, \lambda_2^* D_b + 2\sqrt{e^*(1 - \frac{\lambda_2^*}{2})}\, \{D_b (3\alpha - 1) + 3\alpha C_s\}}{\sqrt{\alpha b_1^*}\,(4 - 3\lambda_2^*) D_b + 2\sqrt{e^*(1 - \frac{\lambda_2^*}{2})}\, \{D_b(1 - 2\alpha) - 2\alpha C_s \frac{\lambda_2^* - \lambda_1^*}{\lambda_2^*}\}} \right). \tag{12}$$

where $b_1^* = C_I^* + \lambda_1^* C_s$ and $e^* = \lambda_2^* D_b(1-\alpha) - \alpha C_s(\lambda_2^* - \lambda_1^*)$. The approximation error is $R_a^* - R_0^* = O(\alpha)$. Thus, if $p_1 = 1$, then $\lambda_1^* = p_2\lambda_1$, $\lambda_2^* = p_2\lambda_2$, $b_1^* = C_I^* + p_2\lambda_1 C_s$, $e^* = p_2 e$ and

$$R_a^* = \lambda_2 C_b + D_b\left\{\frac{\sqrt{\alpha b_1^*}\, p_2\lambda_2 D_b + 2\sqrt{p_2(1-\frac{p_2\lambda_2}{2})}\, e\{D_b(3\alpha-1)+3\alpha C_s\}}{\sqrt{\alpha b_1^*}(4-3p_2\lambda_2)D_b + 2\sqrt{p_2(1-\frac{p_2\lambda_2}{2})}\, e\{D_b(1-2\alpha)-2\alpha C_s\frac{\lambda_2-\lambda_1}{\lambda_2}\}}\right\}$$

This simplifies R_a^* to some degree.

IV. The Relationship Between Two Inspection Costs

We will derive a formula to compare the two inspection schemes. Letting ξ be the exact solution of C_I^* in the equation $R_0 - R_0^* = 0$, and ξ_a be the solution of C_I^* in $R_a = R_a^*$ when C_I is given. We have, from (12) and (6),

$$\xi = e^*(1-\frac{\lambda_2^*}{2})H^2 - \lambda_1^* C_s + O(\alpha),$$

where

$$H = \frac{\sqrt{C_I+\lambda_1 C_s}\,\{\alpha C_s(3-\frac{\lambda_2}{4}) - D_b(1-(3-\frac{\lambda_2}{4})\alpha)\} + \sqrt{\alpha e(1-\frac{\lambda_2}{2})[(1-p_1)(\lambda_2-\lambda_1)C_s/(\lambda_2\lambda_2^*)]}}{\sqrt{\alpha(C_I+\lambda_1 C_s)}\,[\frac{\lambda_2-\lambda_2^*}{2}]D_b + [\alpha C_s(3-\frac{\lambda_2^*}{4}) - D_b(1-(3-\frac{\lambda_2^*}{4})\alpha)]\sqrt{e(1-\frac{\lambda_2}{2})}}$$

In particular, if $p_1=1$, then

$$H = \frac{[\alpha C_s(3-\frac{\lambda_2}{4}) - D_b(1-(3-\frac{\lambda_2}{4})\alpha)]\sqrt{(C_I+\lambda_1 C_s)/\lambda_2 D_b}}{[\alpha C_s(3-\frac{\lambda_2 p_2}{4}) - D_b(1-(3-\frac{\lambda_2 p_2}{4})\alpha)]\sqrt{1-\frac{\lambda_2}{2}} + \sqrt{\alpha(C_I+\lambda_1 C_s)\lambda_2 D_b}\cdot\frac{1-p_2}{2}}$$

or, $\xi = -\lambda_1 p_2 C_s + p_2[1-\frac{p_2\lambda_2}{2}]\,[\frac{C_I+\lambda_1 C_s}{1-\lambda_2/2}]\,[\frac{1}{1+k}]^2 + O(\alpha)$

$$\text{where } K = [\frac{1-p_2}{2}]\left\{\frac{\frac{\alpha\lambda_2}{2}(C_s+D_b) + \sqrt{\alpha(C_I+\lambda_1 C_s)\lambda_2 D_b/(1-\frac{\lambda_2}{2})}}{\alpha C_s(3-\frac{\lambda_2}{4}) - D_b[1-(3-\frac{\lambda_2}{4})\alpha]}\right\}.$$

If we drop all α terms, and keep all $\alpha^{1/2}$ term, the approximate equation becomes

$$\xi_a = p_2\left[\frac{(1-\frac{p_2\lambda_2}{2})(C_I+\lambda_1 C_s)}{1-\frac{\lambda_2}{2} - (1-p_2)\sqrt{\alpha(C_I+\lambda_1 C_s)\lambda_2(1-\frac{\lambda_2}{2})/D_b}} - \lambda_1 C_s\right] \quad (13)$$

From this equation, we see that suppose α is small, ξ_a is approximately equal to ξ and if $p_2 \simeq 1$ then $\xi_a \simeq p_2 C_I$; ξ_a increases as p_2 decreases; ξ_a increases as λ_2 increases, and if $p_2 = 1$ then $\xi_a = C_I$. An important application of (13) is that if the inspection cost of the second inspection scheme is less than ξ_a computed from (13), then we should use the second inspection scheme rather than the first because the second inspection scheme has less minimum cost per unit time.

V. An Example

Let us consider a fast production process. Since the unit time for producing an item is small, we may assume that α, the probability of the process shifting from level 1 to level 2 in a unit time is very small. Let $\lambda_1 = .001$, $\lambda_2 = .1$, $C_b = 1$, $C_s = 100$ and $C_I = 1.5$ which is greater than C_b, i.e. we can assume it is destructive inspection. Table 1 compares the different approximate solutions to the exact optimal solutions at different α values.

Table 1: Percentage error rates
between R_0 and R_a for different values of α

α	10^{-4}	5×10^{-5}	10^{-5}	5×10^{-6}	10^{-6}
m_0	158	208	431	601	1321
m_1	158.1	207.9	431.5	601.4	1321.3
E(N)	10821	20057	84525	154404	556458
R_0	.03266	.02210	.00956	.00689	.00354
R_a	.03522	.02231	.00935	.00676	.00350
% Error Rate	7.80	.93	2.20	2.96	.98

Table 2 gives a comparison between $C_I = 1.5$ and various ξ and p_2 when $\alpha = 10^{-5}$ and 10^{-6}. Here ξ_a is the approximation to ξ given by Eq. (13).

Table 1 tells us the percentage error rates between R_0 and R_a are pretty small except when $\alpha = 10^{-4}$. Thus the approximation R_a is satisfactory.

Table 2 tells us the percentage error rates between ξ_a and ξ are pretty small, and the error rate increases when p_2 decreases. Thus (13) is a good approximation to ξ when α is small.

Table 2: Percentage error rates
between the exact ξ and the approximated $\xi = \xi_a$ given by (13)

α	p_2	.95	.9	.85	.8	.75	.7
10^{-5}	$E(N^*)$	84505	84463	84418	84375	84332	84291
	m_0^*	409	386	364	342	319	297
	ξ	1.4021	1.3425	1.2644	1.1867	1.1094	1.0326
	ξ_a	1.4293	1.3582	1.2866	1.2145	1.1420	1.0690
% Error Rate		.58	1.17	1.76	2.35	2.94	3.53

α	p_2	.95	.9	.85	.8	.75	.7
10^{-6}	$E(N^*)$	576158	575389	574580	574011	573230	572574
	m_0^*	1253	1184	1114	1047	978	910
	ξ	1.4246	1.3444	1.2643	1.1878	1.1094	1.0325
	ξ_a	1.4291	1.3577	1.2860	1.2138	1.1412	1.0681
% Error Rate		.32	.99	1.7	2.2	2.8	3.4

Table 2 can be used to choose a better scheme. Suppose $p_2 = .9$, $\alpha = 10^{-5}$, and the second inspection cost C_I^* is less than 1.3425, then it is obvious from Table 2 that we should use the second inspection scheme rather than the first one.

VI. Comment

Although samples of size 1 and a single-sample decision rule are studied here, the formulas and equations derived in this paper are valuable. The generalization to samples of size k, $k \geq 1$, has been studied and will be published later.

References

1 Baker, K. R. "Two process models in the economic design of an $\bar{X}$ chart". AIIE Transactions, 3, 257-263, (1971).

2 Bather, J. A. "Control charts and the minimization of costs". J. Roy. Statist. Soc., Series B, 25, 49-80, (1963).

3 Duncan, A. J. "The economic design of $\bar{X}$-charts used to maintain current control of a process". J. Amer. Statist. Assoc., 51, 228-242, (1956).

4 Duncan, A. J. "The economic design of $\bar{X}$-charts when there is a multiplicity of assignable causes". J. Amer. Statist. Assoc., 66, 107-121, (1971).

5 Girshick, M. A., and Rubin, Herman. "A Bayes approach to a quality control model". Annals of Mathematical Statistics, 23, 114-25, (1952).

6 Lave, R. E. "A Markov decision process for economic quality control". IEEE Transactions on Systems Science and Cybernetics, SSC-2, 45-54, (1966).

7 Lave, R. E. "A markov model for quality control plan selection". AIIE Transactions, 1, 139-145, (1969).

8 Owen, D. B., Mcintire, D., and Seymour, E. "Tables using one or two screening variables to increase acceptable product under one-sided specifications". J. of Quality Technology, 7, 127-138, (1975).

9 Tenenbein, A. "A double sampling scheme for estimating from misclassified binomial data". J. Amer. Statist. Assoc., 65, 1350-1361, (1970).

10 Tenebein, A. "A double sampling scheme for estimating from binomial data with misclassifications: sample size determination". Biometrics, 27, 935-944, (1971).