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常曲率空間的示性

A Characterization of Spaces of Constant Curvature

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Abstract — We shall obtain a characterization of space of constant curvature and space of constant holomorphic curvature in terms of conformal Killing tensors of degree 2 and F-conformal Killing tensors of degree 2, respectively.

1. Let M^n be an n-dimensional Riemannian manifold with metric tensor g_{ij}. Indices i, j, k,..... run over the range 1, 2, ..., n. Let ∇_i denote the operator of covariant differentiation with respect to the Riemannian connection. The Riemannian curvature tensor, the Ricci tensor and the scalar curvature are denoted by $R_{kji}{}^h$, $R_{ji} = R_{\ell ji}{}^\ell$ and $R = g^{ij} R_{ij}$ respectively. $\delta_j{}^i$ is the Kronecker delta.

M^n is called a space of constant curvature if its Riemannian curvature tensor is given by

$$R_{kji}{}^h = \frac{R}{n(n-1)}(\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki}).$$

A vector field v^i is called a conformal Killing vector field if there exists a function ρ such that

$$\nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}. \tag{1}$$

When ρ vanishes identically v^i is called a Killing vector field. For a conformal Killing vector field v^i, we have

$$\nabla_k \nabla_j v_i = -R_{\ell kji} v^\ell - g_{kj}\rho_i + g_{ki}\rho_j + g_{ji}\rho_k. \tag{2}$$

A skew symmetric tensor field u_{ji} is called a conformal Killing tensor of degree 2, [3], if there exists a vector field p^i such that

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2 p_i g_{kj} - p_k g_{ji} - p_j g_{ki}. \tag{3}$$

Such a vector field p^i is called an associated vector field of u_{ji}, and is given by

$$(n-1) p_i = \nabla^\ell u_{\ell i}. \tag{4}$$

S. Tachibana studied such a tensor and got the following

Theorem 1 [3]: In a Riemannian manifold M^n of constant curvature, the covariant derivative $\nabla_j v_i$ of any Killing vector v_i is a conformal Killing tensor.

C. H. Chen dealt with the converse of Theorem 1, i.e. he got the following

Theorem 2 [2]: In a Riemannian manifold $M^n(n > 2)$, if (i) the Lie algrbra of all local Killing vectors v^i is transitive, i.e. for any point p and any direction V^i at p, there exists a local Killing vector v^i satisfying $v^i(p) = V^i$, and (ii) the covariant derivative $\nabla_j v_i$ of any Killing vector v_i is a conformal Killing tensor, then M^n is a space of constant curvature.

In this section, we shall generalize Theorem 2 as follows:

Theorem 1: Let M^n $(n > 2)$ be a Riemannian manifold of dimension n satisfying

i) the Lie algebra of all local conformal Killing vectors is transitive,

ii) for any conformal Killing vector v^i, set $u_{ji} = \nabla_j v_i - \rho g_{ji}$, then u_{ji} is a conformal Killing tensor of degree 2, then M^n is a space of constant curvature.

Proof: Let v^i be a conformal Killing vector and assume that $u_{ji} = \nabla_j v_i - \rho g_{ji}$ is conformal Killing. Taking account of (2), we have from (4)

$$p_i = -\frac{1}{(n-1)} R_{\ell i} v^{\ell} - \rho_i \ , \tag{5}$$

Substituting $u_{ji} = \nabla_j v_i - \rho g_{ji}$ into (3) and making use of (2) and (5) in what follows, we have $(R_{\ell kji} + R_{\ell jki}) v^{\ell} = \frac{1}{(n-1)} (2R_{\ell i} g_{kj} - R_{\ell k} g_{ji} - R_{\ell j} g_{ki}) v^{\ell}$. Since above equation is valid for any v^{ℓ}, we have by assumption i)

$$(R_{\ell kji} + R_{\ell jki}) = \frac{1}{(n-1)} (2R_{\ell i} g_{kj} - R_{\ell k} g_{ji} - R_{\ell j} g_{ki}). \tag{6}$$

Transvecting (6) with $g^{\ell i}$ we have

$$R_{kj} = \frac{R}{n} g_{kj} \ , \tag{7}$$

Substituting (7) into (6), we find

$$R_{\ell kji} + R_{\ell jki} = \frac{R}{n(n-1)} (2g_{\ell i} g_{kj} - g_{\ell k} g_{ji} - g_{\ell j} g_{ki}) \tag{8}$$

and taking account of first Bianchi identity, we finally got

$$R_{\ell jki} = \frac{R}{n(n-1)} (g_{\ell i} g_{jk} - g_{\ell k} g_{ji}) .$$

Thus Theorem 1 is proved.

2. In this Section, we shall study the analogous fact in a Kählerian manifold. A Kählerian manifold M^n is an even dimensional Riemannian manifold with a mixed tensor $F_j{}^i$ satisfying following conditions:

$$F_k{}^r F_r{}^j = -\delta_k{}^j \ , \quad F_k{}^{\ell} F_j{}^r g_{\ell r} = g_{kj} \ ,$$

$$\nabla_k F_j{}^i = 0 \qquad , \quad F_{kj} = F_k{}^r g_{rj} = -F_{jk}.$$

It is well-known that the following relations hold

$$R_{kjir} F_h{}^r = R_{kjhr} F_i{}^r \ , \quad R_{kj} = R_{\ell r} F_k{}^{\ell} F_j{}^r \ , \tag{9}$$

$$R_k{}^r F_r{}^j = -\tfrac{1}{2} R_{\ell r k}{}^j F^{\ell r} \ .$$

If we define tensor S_{kj} by

$$S_{kj} = \tfrac{1}{2} R_{kj\ell r} F^{\ell r}, \tag{10}$$

then $R_{kr} F_j{}^r = S_{kj}$, $S_{kr}F_j{}^r = - R_{kj}$, $R = - S_{\ell r} F^{\ell r}$. (11)

M^n is called a space of constant holomorphic sectional curvature if its Riemannian curvature tensor satisfies

$$R_{kji}{}^h = \frac{R}{n(n+2)} (\delta_k{}^h g_{ji} - g_{ki} \delta_j{}^h + F_k{}^h F_{ji} - F_{ki}F_j{}^h - 2F_{kj}F_i{}^h). \tag{12}$$

For a skew symmetric tensor field w_{ji} in M^n, if there exist two vector fields p^i and q^i such that

$$\nabla_k w_{ji} + \nabla_j w_{ki} = 2p_i g_{kj} - p_k g_{ji} - p_j g_{ki} + 3(q_k F_{ji} + q_j F_{ki}), \tag{13}$$

then w_{ji} is called an F-conformal Killing tensor and p^i and q^i are the associated vectors of w_{ji}, [1], [4].

Corresponding to Theorem 1 and Theorem 2, the following two theorems are known.

Theorem 3, [1]: In a space of constant holomorphic sectional curvature the covariant derivative $\nabla_j v_i$ of any Killing vector v_i is an F-conformal Killing tensor.

Theorem 4. [2]: In a Kählerian manifold $M^n (n > 2)$, if (i) the Lie algebra L of all local Killing vectors v^j is transitive and (ii) the covariant derivative $\nabla_j v_i$ of any Killing vector v_i is an F-conformal Killing tensor, then M^n is a space of constant holomorphic sectional curvature.

In the following we shall generallize Theorem 4, i.e. we have

Theorem 2: Let $M^n (n>2)$ be a Kählerian manifold satisfying

i) the Lie algebra of all local conformal Killing vectors is transitive,

ii) for any conformal Killing vector v^i, set $w_{ji} = \nabla_j v_i - \rho g_{ji}$,

then w_{ji} is an F-conformal Killing tensor,

then M^n is a space of constant holomorphic sectional curvature.

Proof: Let v^i and w_{ji} be the ones stated in (ii).

Taking $w_{ji} = \nabla_j v_i - \rho g_{ji}$ in (13) and making use of (2), then (13) becomes

$$- (R_{rkji} + R_{rjki}) v^r - 2g_{kj} \rho_i + g_{ki} \rho_j + g_{ji} \rho_k \tag{14}$$

$$= 2 p_i g_{kj} - p_k g_{ji} - p_j g_{ki} + 3(q_k F_{ji} + q_j F_{ki}).$$

Transvecting (14) with g^{kj} we have

$$- R_{ri} v^r - (n-1) \rho_i = (n-1) p_i + 3 q^r F_{ri}. \tag{15}$$

Transvecting (14) with F^{ji} yields

$$- S_{rk} v^r - \rho^r F_{kr} = p^r F_{kr} + (n+1) q_k.$$

Transvecting above equation with $F_i{}^k$, we get

$$R_{ri}\, v^r + \rho_i = - p_i - (n+1)\, q^r F_{ri} . \tag{16}$$

(15) + (16) impiles

$$p_i = q^r F_{ri} - \rho_i . \tag{17}$$

Making use of (17), equation (15) implies

$$p_i = - \frac{1}{n+2} R_{ri}\, v^r - \rho_i , \tag{18}$$

and equation (16) implies

$$q_i = - \frac{1}{n+2} S_{ri}\, v^r . \tag{19}$$

Substituting (18), (19) back to (14) and using assumption (i) we obtain

$$R_{rkji} + R_{rjki} = \frac{1}{(n+2)} [2R_{ri}g_{kj} - R_{rk}g_{ji} - R_{rj}g_{ki} + 3(S_{rk}\, F_{ji} + S_{rj}\, F_{ki})] . \tag{20}$$

Transvecting (20) with g^{ri} and taking account of (11) we find

$$R_{kj} = \frac{R}{n} g_{kj} .$$

Substituting above equation into (20) we find

$$R_{rkji} + R_{rjki} = \frac{R}{(n+2)} [2g_{ri}g_{kj} - g_{rk}g_{ji} - g_{rj}g_{ki} + 3\,(F_{rk}F_{ji} + F_{rj}F_{ki})] , \tag{21}$$

from which we finally get

$$R_{rjki} = \frac{R}{n(n+2)} (g_{ri}g_{jk} - g_{rk}g_{ji} + F_{ri}F_{jk} - F_{rk}F_{ji} - 2F_{rj}F_{ki}) .$$

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