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銳截止單調響應曲線之低通網路設計
On Designing Sharp-cutoff Low-pass Filter with Monotonic Response

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Abstract –– Effects of introducing multiple pairs of coincident $j\omega$–axis zeros in the class L filter functions are investigated. It is shown that for the same order n, modified filters with finite zeros provide sharper cutoff rate than the conventional all pole filters. Expressions for cutoff slope and minimum attenuation in the stopband are derived in terms of the order n, the number of zeros m and their locations. A design example is given to illustrate the procedures for finding out the proper transfer functions.

I. Introduction

Considerable attention has been given in the recent past to the design of sharp-cutoff low-pass filters, obtained by introducing finite zeros in the well-known Butterworth and Chebyshev filters. Budak and Aronhime [1] considered the case of a maximally flat design with one pair of $j\omega$-axis zeros, while Dutta Roy [2] generalized this to multiple (m) pairsof $j\omega$-axis zeros. Agarwal and Sedra [3] extended the technique to Chebyshev functions.

For the same order n, the same number of pairs of zeros m and the same locations of zeros, the cutoff slape and stopband attenuation characteristics of the modified (finite zeros) Chebyshev filter are much better than those of the modified (finite zeros) Butterworth filter. As is known, the Butterworth type filters provide maximally flat magnitude in the passband, while the Chebyshev type filters provide a much sharper cutoff slope at the expense of ripples in the passband. However, in many applications, i.e. when the transient response is also considered, a high ripple in the passband is undesirable. One then uses the class L filter [4] as a compromise between the Butterworth type and Chebyshev type filters. The class L filter has the property that its magnitude characteristic decreases monotonically with ω in the passband with the greatest possible cutoff rate. In this paper, we study the effects of introducing multiple (m) pairs of $j\omega$-axis zeros in the transfer function of the conventional class L filter.

II. Filter Transfer Function

The magnitude squared function for the normalized class L filter is given as [4]

$$|M(j\omega)|^2 = \frac{1}{1+L_n(\omega^2)} \tag{1}$$

where $L_n(\omega^2)$ is a monotonically increasing polynomial, and

$$\left.\frac{d\,L_n(\omega^2)}{d\omega}\right|_{\omega=1} = L_n'(1) = \text{maximum}$$

$$L_n(1)=1,\ L_n(0)=0.$$

The polynomial $L_n(\omega^2)$ for n=2 to 10 are given in Table I [4] for reference.

Table I

The Function $L_n(\omega^2)$ at its first derivative at $\omega = 1$.

n	$L_n(\omega^2)$	$\frac{dL_n(1)}{d\omega}$
2	ω^4	4
3	$3\omega^6-3\omega^4+\omega^2$	8
4	$6\omega^8-8\omega^6+3\omega^4$	12
5	$20\omega^{10}-40\omega^8+28\omega^6-8\omega^4+\omega^2$	18
6	$50\omega^{12}-120\omega^{10}+105\omega^8-40\omega^6+6\omega^4$	24
7	$175\omega^{14}-525\omega^{12}+615\omega^{10}-355\omega^8+105\omega^6-15\omega^4+\omega^2$	32
8	$490\omega^{16}-1680\omega^{14}+2310\omega^{12}-1624\omega^{10}+615\omega^8-120\omega^6+10\omega^4$	40
9	$1764\omega^{18}-7056\omega^{16}+1170\omega^{14}-10416\omega^{12}+5376\omega^{10}-1624\omega^8+276\omega^6-24\omega^4+\omega^2$	50
10	$5292\omega^{20}-23520\omega^{18}+44100\omega^{16}-45360\omega^{14}+27860\omega^{12}-10416\omega^{10}+2310\omega^8-280\omega^6+15\omega^4$	60

To introduce m pairs of identical zeros at $\pm j\omega_o$ to the class L filter, the magnitude squared function is modified as

$$|T(j\omega)|^2 = \frac{(\omega^2-\omega_o^2)^{2m}}{(\omega^2-\omega_o^2)^{2m}+\lambda L_n(\omega^2)} \tag{2}$$

where $\omega_o > 1$ and $n \geqslant 2m+1$.

If the cutoff frequency is also 1 rad/sec., i.e. $|T(j\omega)|^2_{\omega=1} = ½$, then $\lambda = (\omega_o^2-1)^{2m}$. Equation (2) can then be rewritten as

$$|T(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega_o^2-1}{\omega^2-\omega_o^2}\right)^{2m} L_n(\omega^2)} \tag{3}$$

For $\omega < 1$, it is clear that

$$1+\left(\frac{\omega_o^2-1}{\omega^2-\omega_o^2}\right)^{2m} L_n(\omega^2) < 1+L_n(\omega^2)$$

and hence, $|T(j\omega)| > |M(j\omega)|$. Thus, the error in the passband of the modified class L filter is always less than that of the conventional class L filter.

For the design of filters, we need to find out the network transfer function T(s). By the usual procedure of analytic continuation, the transfer function T(s) will be of the form [5]

$$T(s) = \frac{(s^2+\omega_o^2)^m}{(\omega_o^2-1)^m D(s)} \tag{4}$$

where D(s) is the product of the left-half s-plane factors of the polynomial

$$D(s)D(-s)=L_n(-s^2)+(\frac{s^2+\omega_o^2}{\omega_o^2-1})^{2m} \tag{5}$$

Unfortunately, it has not been possible to derive analytical solution for the roots of (5). The roots of (5) can only be obtained by numerical techniques.

III. Cutoff Slope of the Magnitude Function

Let $g(\omega)=(\omega^2-\omega_o^2)^{2m}$, then (2) becomes

$$|T(j\omega)|^2 = \frac{g(\omega)}{g(\omega)+\lambda L_n(\omega^2)} \tag{6}$$

Differentiating (6) with respect to ω gives

$$|T(j\omega)|' \triangleq \frac{d|T(j\omega)|}{d\omega} = \frac{1}{2|T(j\omega)|} \frac{\lambda[g'(\omega)L_n(\omega^2)-g(\omega)L_n(\omega^2)]^2}{[g(\omega)+\lambda L_n(\omega^2)]^2} \tag{7}$$

At the cutoff frequency, $\omega=1$, $|T(j1)| = \frac{1}{\sqrt{2}}$ and $g'(1)=-4m(\omega_o^2-1)^{2m-1}$. Thus, we obtain the cutoff slope

$$S_{nm} = -[\frac{L_n'(1)}{4\sqrt{2}} + \frac{m}{\sqrt{2}(\omega_o^2-1)}] \tag{8}$$

Equation (8) clearly shows that the slope increases as the number of zeros is increased and also when ω_o is brought closer to unity.

Usually, it is convenient to express the cutoff slope in the unit dB/octave. In general, the slope in dB/octave is given by

$$\begin{matrix}\text{cutoff slope}\\ \text{(dB/octave)}\end{matrix} = -\frac{3}{10} \frac{d[20\log|T(j\omega)|]}{d\log\omega}\Big|_{\omega=1} \tag{9}$$

Combining (9) and (8), we obtain

$$\begin{aligned}\begin{matrix}\text{cutoff slope}\\ \text{(dB/octave)}\end{matrix} &= -6\sqrt{2}\,\frac{d|T(j\omega)|}{d\omega}\Big|_{\omega=1} \\ &= -6[\frac{L_n'(1)}{4} + \frac{m}{\omega_o^2-1}]\end{aligned} \tag{10}$$

For the convenience of comparison, the magnitude squared functions and cutoff slopes of the modified (finite zeros) Butterworth [2], Chebyshev [3] and class L filters are all given in Table II. It is easily seen that the cutoff slope of the modified class L filter is between those of the other two types of filter.

IV. Stopband Attenuation Characteristics

Let us compare the stopband attenuations of the modified class L filter with that of the conventional class L filter. Comparing the functions in (1) and (3) for $1<\omega<\omega_o$, it is seen that

$$1+(\frac{\omega_o^2-1}{\omega^2-\omega_o^2})^{2m} L_n(\omega^2) > 1+L_n(\omega^2)$$

and hence, $|T(j\omega)|$ is always less than $|M(j\omega)|$.

For $\omega >> \omega_o$, $1+(\frac{\omega_o^2-1}{\omega^2-\omega_o^2})^{2m} L_n(\omega^2) < 1+L_n(\omega^2)$ and thus, $|T(j\omega)|$ is larger than $|M(j\omega)|$.

Table II

Magnitude square function and cutoff slope of the modified filter with finite zeros

	normalized magnitude square function	Cutoff slope
modified Butterworth filter [2]	$\frac{1}{1+(\frac{\omega_o^2-1}{\omega^2-\omega_o^2})^{2m}\omega^{2m}}$	$-[\frac{n}{2\sqrt{2}}+\frac{m}{\sqrt{2}(\omega_o^2-1)}]$
modified Chebyshev filter [3]	$\frac{1}{1+(\frac{\omega_o^2-1}{\omega^2-\omega_o^2})^{2m} C_n(\omega^2)}$	$-[\frac{n^2}{2\sqrt{2}}+\frac{m}{\sqrt{2}(\omega_o^2-1)}]$
modified class L filter	$\frac{1}{1+(\frac{\omega_o^2-1}{\omega^2-\omega_o^2})^{2m} L_n(\omega^2)}$	$-[\frac{L_n'(1)}{4\sqrt{2}}+\frac{m}{\sqrt{2}(\omega_o^2-1)}]$

The slope of the magnitude function, given in (7), can be rewritten as

$$|T(j\omega)|' = \frac{\lambda[g'(\omega)L_n(\omega^2)-g(\omega)L_n'(\omega^2)]}{2[g(\omega)]^{1/2}[g(\omega)+\lambda L_n(\omega^2)]^{3/2}}$$

For $\omega < \omega_o$, it is clear that

$$g(\omega) > 0,\ g'(\omega) < 0,\ L_n'(\omega^2) > 0$$

and hence $|T'(j\omega)|' < 0$. For $\omega >> \omega_o$, $g'(\omega) > 0$ and thus, there may exist some point in this frequency range such that $|T(j\omega)|' > 0$. In general, the transmission characteristics of the filter under consideration will be of the form shown in Fig. 1. In the stopband, there will be a peak at some frequency $\omega_p > \omega_o$. If $T_p \triangleq |T(j\omega_p)|$, the

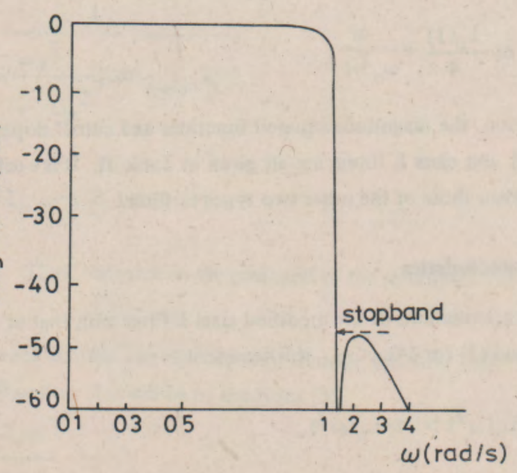

Fig. 1 Transmission characteristic of a modified class L filter with n=7, m=2 and ω_o=1.41

minimum attenuation in the stopband is $-20 \log T_p$ dB. The frequency ω_p is obtained from the condition $|T(j\omega)|'_{\omega=\omega_p}=0$, i.e.

$$g'(\omega_p)\, L_n(\omega_p^2) = g(\omega_p)\, L'_n(\omega_p^2) \tag{11}$$

Therefore, for a given n and m, the minimum attenuation in the stopband is

$$-20 \log T_p = 10 \log \left[1+\left(\frac{\omega_o^2-1}{\omega^2-\omega_o^2}\right)^{2m} L_n(\omega_p^2)\right] \tag{12}$$

V. An Example of Design

Suppose it is required to design a sharp cutoff low-pass filter with a cutoff slope higher than 60 dB/octave. The minimum order of the required conventional class L filter is 8. With the modified class L filter, several solutions are possible by solving the inequality $\frac{L'_n}{4}+\frac{m}{\omega_o^2-1} > 10$. Some of these results are given in Table III. The values of

Table III

Some possible filters for obtaining a cutoff slope higher than 60 dB/octave

n	m	required ω_o	minimum attenuation in the stopband (dB)
4	1	1.06	Very small
5	1	1.16	8.1
	2	1.11	6.1
6	1	1.11	20.6
	2	1.22	28.6
7	1	1.22	35.0
	2	1.41	48.4
	3	1.58	48.5

n and m to be selected depend on the minimum attenuation required in the stoppand. For a given set of (n, m, ω_o), the frequency ω_p can be obtained by solving (11). The standard subroutine program called POLRT is employed to find the roots of a polynomial. If the minimum attenuation of 40 dB is required, then n=7, m=2 and ω_o=1.41 are a good choice. The final step is to find out the transfer function T(s) from (4) and (5). The roots of (5) are also obtained by the subroutine POLRT. From our computation, $D_7(s)=(s+0.6520)(s^2+1.0283s+0.5665)(s^2+0.1958s+2.0368)(s^2+0.5422s+1.7831)$. Therefore, the required transfer function is

$$T(s)=\frac{(s^2+1.98)^2}{(0.98)^2\, D_7(s)}$$

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