The Journal of National Chiao Tung University
Vol. 4, April 1978
pp. 33-38

局部凸空間上的歌西問題
Cauchy Problem in Locally Convex Spaces

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(Received September 1, 1977)

Abstract —— Let S be a locally convex space. We use the product integration to show that Cauchy initial value problem: $u'(x) = Au(x)$, $u(o) = p$, has a unique solution, where A is a function from S into itself and u is a continuously differentiable function from $[o, \infty)$ into S. The conditions required on A are that A is dissipative, and there exists an open subset C of S such that A is continuous on C and the range of $(I- \varepsilon A)$ contains C as ε is sufficiently small.

I. Introduction

Let S be a Hausdorff locally convex topological vector space, and let A be a mapping from S into S. We shall consider the solution of the Cauchy initial value problem:

$$u'(t) = Au(t), \quad u(o) = p, \tag{1}$$

where u is a continuously differentiable function from $[o, \infty)$ into S and p is a point in S.

In recent years there appeared many works on nonlinear evolution equations in Hilbert or Banach spaces. Y. Kōmura [1] M. Crandall and A. Pazy [3] have considered (1), in which S is a Hilbert space. In [2], T. Kato has shown that (1) has a unique solution, if A is maximal dissipative and S* is uniformly convex. By definition A is maximal dissipative if it is dissipative and the range of $I- \lambda A$ is the whole of S for each (or equivalently, for some) $\lambda > 0$. G. F. Webb [4] has shown that (1) has a solution if the following three conditions hold: (i) S is a Banach space, (ii) A is dissipative, continuous on a domain which contains some open subset C of S, and (iii) the range of $(I- \varepsilon A)$ contains C, where $0 < \varepsilon < \alpha$, for some $\alpha > 0$.

For other investigations of nonlinear evolution equations we refer to F. Browder [5], J. Neuberger [6] and M. I. Hazan [7].

Through this paper we let S be a sequantially complete Hausdorff locally convex topological vector space with the property that there exists an open bounded neighborbood of 0. Let X be a family of convex symmetric bounded neighborboods of 0 which forms a basis for the system of 0. Let I be the identity mapping, $(I - \varepsilon A)^{-1}$ the inverse map of $(I- \varepsilon A)$, and let D(A) be the domain of A, R(A) the range of A.

Let $\{L_i \mid i \in N\}$ be a sequence of mappings from S into S. We define $\pi_{i=m}^{n} L_i = I$, if $m > n$; and $\pi_{i=m}^{n} L_i = L_n \pi_{i=m}^{n-1} L_i$ if $m \leq n$.

The aim of this paper is to obtain a unique solution to (1) in S by means of product integration which is defined as follows. Suppose that $p \in S$, $x > o$ and q is a point in S and for any given $W \in X$, there exists a chain $\{s_i\}_{i=0}^{m}$ from 0 to x, such that if $\{t_i\}_{i=0}^{n}$ is a refinement of $\{s_i\}_{i=0}^{m}$, then

$$\{q - \pi_{i=1}^{n} [I - (t_i - t_{i-1}) A]^{-1} p\} \in W. \tag{2}$$

We call q the product integration of A with respect to p from o to x, and denoted it by $\pi_o^x (I\text{-}dIA)^{-1} p$.

II. Theorem

Suppose that A is a mapping from S into S, satisfying the following conditions: (I) A is dissipative on D(A), that is, if w, v $\in$ D(A) and k > o then for each W $\in$ X, w-v $\notin$ W implied that (I - kA) w - (I - kA) v $\notin$ W. (II) There exists an open subset C of S such that C $\subseteq$ D(A) and there is a positive number α such that C $\subseteq$ R (I - ε A) for all $0 < \varepsilon < \alpha$. (III) A is continuous on C, that is, for any W $\in$ X, there exists a V $\in$ X such that if w, v $\in$ C and w-v $\in$ V then Aw - Av $\in$ W.

In this paper we always let A satisfy (I), (II) and (III). From this assumption, we can easily see that (I - εA) is one to one on D(A) for $0 < \varepsilon < \alpha$. We denote $(I - \varepsilon A)^{-1}$ by L(ε) on R (I - εA). From (I), L(ε) is X - nonexpansive, namely, L(ε)w - L(ε) v $\in$W whenever w - v $\in$ W.

Since C is an open set, p $\in$ C and S contains a bounded set U $\in$ X, there exists a positive number $\gamma_p = \text{Sup}\{\eta > 0 \mid p + \eta U \subset C\}$, dependent on p.

Theorem: Let A satisfy conditions (I), (II) and (III). If p $\in$ C and $\gamma_p = \text{Sup}\{\eta > 0 \mid p + \eta U \subset C\}$, then there is a unique continuously differentiable function u_p from $[o, \gamma_p)$ into S such that $u_p(o) = p$ and if $o < x < \gamma_p$, $U'_p(x) = Au_p(x)$.

We shall prove our theorem by a series of lemmas under the hypotheses of the theorem.

Lemma 1: If p $\in$ C, Ap $\in$ U, then AL(y)p $\in$ U for $o < y < \alpha$.

Proof: By definition we have (I - yA) L(y)p = p and hence

$$yAL(y)p = p - L(y)p. \tag{3}$$

By the assumption Ap $\in$ U, we have p - (p - yAp) $\in$ yU. Now from the X-nonexpansivity of L(y), we obtain that

$$L(y)p - p = L(y)p - L(y)\,[I - yA]\,p \in yU. \tag{4}$$

(3) and (4) imply that AL(y)p $\in$ U.

Lemma 2: If p $\in$ C and $o < y < x < \alpha$, Ap $\in$ U, then L(x)p - L(y)p $\in$ (x - y)U.

Proof: Let q = L(y)p, then p = (I - yA)q. Since p - (x -y)Aq = (I -yA)q - (x - y)Aq = [(I - yA) - (x - y)A] q = (I - xA)q, therefore L(y)p = L(x) [p - (x - y) Aq]. By lemma 1, p - [p - (x - y) Aq] = (x - y) Aq $\in$ (x - y) U. Hence L(x)p - L(y)p $\in$ (x - y)U.

Lemma 3: Let p $\in$ C, Ap $\in$ U, $o < x < \gamma_p$ and let $\{s_i\}_{i=0}^{m}$ be a chain from o to x such that $\text{Max}\,\{(s_i - s_{i-1})\}_{i=1}^{m} < \alpha$. If j is an integer in [1, m], then

$$\pi_{i=1}^{j-1} L(s_i - s_{i-1})\, p \in C \tag{5}$$

$$\pi_{i=1}^{j} L(s_i - s_{i-1})\, p - p \in s_j U \tag{6}$$

$$A\pi_{i=1}^{j} L(s_i - s_{i-1})\, p \in U. \tag{7}$$

Proof: The proof is by induction. For j = 1, it follows from the definition of L(x) and lemma 2 that $\pi_{i=1}^{1-1} L(s_i - s_{i-1})p = p \in C$, $\pi_{i=1}^{1} L(s_i - s_{i-1})p - p = L(s_1 - s_0)p - p \in s_1 U$. From lemma 1, we have $A\pi_{i=1}^{1} L(s_i - s_{i-1})p = AL(s_1 - s_0)p \in U$. Suppose that for some j in [1, m-1], we have

$$\pi_{j=1}^{j-1} L(s_i - s_{i-1})\, p \in C \tag{8}$$

$$\pi_{i=1}^{j} L(s_i - s_{i-1})\, p - p \in s_j U \tag{9}$$

$$A\, \pi_{i=1}^{j} L(s_i - s_{i-1})\, p \in U. \tag{10}$$

Then by (6) $\pi_{i=1}^{j} L(s_i - s_{i-1})p = \pi_{i=1}^{j} L(s_i - s_{i-1})p - p + p \in s_j U + p \subset p + \gamma_p U \subset C$.

By (8), (10) and lemma 2, we have $\pi_{i=1}^{j+1} L(s_i - s_{i-1})\, p - p = \Sigma_{k=1}^{j+1} [\, \pi_{i=1}^{k} L(s_i - s_{i-1})p - \pi_{i=1}^{k-1} L(s_i - s_{i-1})p] \in \Sigma_{k=1}^{j+1} (s_k - s_{k-1})U = s_{j+1} U$. From lemma 1 and the facts that $\pi_{i=1}^{j} L(s_i - s_{i-1})p \in C$ and $A\, \pi_{i=1}^{j} L(s_i - s_{i-1})p \in U$, we have

$$A\, \pi_{i=1}^{j+1} L(s_i - s_{i-1})\, p = AL(s_{j+1} - s_j)\, \pi_{i=1}^{j} L(s_i - s_{i-1})\, p \in U.$$

Lemma 4: Let $p \in C$, $0 < x < \gamma_p$ and let $\{t_i\}_{i=0}^{n}$ be a chain from 0 to x. If j is an integer in [1, n], then

$$\pi_{i=j}^{n} L(t_i - t_{i-1})p - p = \Sigma_{i=j}^{n} (t_i - t_{i-1})\, A\, \pi_{k=j}^{i} L(t_k - t_{k-1})\, p.$$

Proof:

$$\pi_{i=j}^{n} L(t_i - t_{i-1})\, p - p$$

$$= \Sigma_{i=j}^{n} [\, \pi_{k=j}^{i} L(t_k - t_{k-1})p - \Sigma_{k=j}^{i-1} L(t_k - t_{k-1})\, p]$$

$$= \Sigma_{i=j}^{n} (t_i - t_{i-1})\, A\, \pi_{k=j}^{i} L(t_k - t_{k-1})\, p.$$

Given a $W \in X$, let $p \in C$, $Ap \in U$, and m a nonnegative integer. The number sequence $\{s_i\}_{i=0}^{m}$ is said to have property P_w if the following conditions hold: (i) $s_o = 0$, $s_m < \gamma_p$, (ii) $\{s_i\}_{i=0}^{m}$ is increasing and (iii) if h is an integer in [0, m-1], $s_h < x < s_{h+1}$, $\{t_i\}_{i=0}^{n}$ is a chain from s_h to x, and j is an integer in [0, n], then

$$A\, \pi_{i=1}^{j} L(t_i - t_{i-1})\, \pi_{i=1}^{h} L(s_i - s_{i-1})\, p - A\, \pi_{i=1}^{n} L(t_i - t_{i-1})\, \pi_{i=1}^{h} L(s_i - s_{i-1})\, p \in \overline{W}. \tag{11}$$

Lemma 5: Given a $W \in X$, let $p \in C$, $Ap \in U$ and $\{s_i\}_{i=0}^{m}$ have property P_w. Then there is a number s_{m+1} such that $s_m < s_{m+1} < \gamma_p$ and $\{s_i\}_{i=0}^{m+1}$ has property P_w.

Proof: Let $q = \pi_{i=1}^{m} L(s_i - s_{i-1})\, p$. By lemma 3, we have $q \in C$ and $Aq \in U$. Since A is continuous at q, there exists a $V \in X$ such that for all $v \in C$ and $v\text{-}q \in V$, $Av - Aq \in \frac{1}{2}W$. We choose y such that $(y - s_m)\, U \subset V$ and $s_m < y < \gamma_p$. Let $\{t_i\}_{i=0}^{n}$ be a chain from s_m to y. By lemma 3, we have $\pi_{i=1}^{j} L(t_i - t_{i-1})q - q \in (t_j - s_m)\, U \subset V$, for all integer j in [0, n]. Therefore, $A\pi_{i=1}^{j} L(t_i - t_{i-1})q - Aq \in \frac{1}{2}W$, which implies that $A\, \pi_{i=1}^{j} L(t_i - t_{i-1})q - A\, \pi_{i=1}^{n} L(t_i - t_{i-1})q = A\, \pi_{i=1}^{j} L(t_i - t_{i-1})q - Aq + Aq - A\, \pi_{i=1}^{n} L(t_i - t_{i-1})q \in \frac{1}{2}W + \frac{1}{2}W = W$. This completes the proof.

Lemma 6: Given a $W \in X$, let $p \in C$, $Ap \in U$ and let $\{s_i\}_{i=0}^{m}$ have property P_w, B be a subset of (s_m, γ_p), y = Sup B. Suppose that for any b in B, $\{s_i\}_{i=0}^{m} \cup \{b\}$ has propoerty P_w. Then $\{s_i\}_{i=0}^{m} \cup \{y\}$ has property P_w.

Proof: Let $q = \pi_{i=1}^{m} L(s_i - s_{i-1})p$, and $\{t_i\}_{i=0}^{n}$ be a chain from s_m to y. Given any $\varepsilon > 0$, there is a $V \in X$ such that if $v \in C$ and $v - \pi_{i=1}^{n} L(t_i - t_{i-1})\, q \in V$, then $Av - A\, \pi_{i=1}^{n} L(t_i - t_{i-1})q \in \varepsilon W$. There is also a positive number b in

B such that $t_{n-1} < b < t_n = y$, and $(t_n - b)\,U \subset V$. By lemmas 2 and 3,

$$L(b - t_{n-1})\,\pi_{i=1}^{n-1} L(t_i - t_{i-1})\,q - \pi_{i=1}^{n} L(t_i - t_{i-1})\,q \subset (t_n - b)\,U \subset V.$$

Then if j is an integer in $[0, n-1]$, $A\,\pi_{i=1}^{j} L(t_i - t_{i-1})q - A\,\pi_{i=1}^{n} L(t_i - t_{i-1})q = A\,\pi_{i=1}^{j} L(t_i - t_{i-1})q - AL(b - t_{n-1})\,\pi_{i=1}^{n-1} L(t_i - t_{i-1})q + AL(b - t_{n-1})\,\pi_{i=1}^{n-1} L(t_i - t_{i-1})q - A\,\pi_{i=1}^{n} L(t_i - t_{i-1})q \in \overline{W} + \epsilon W$. If j=n, it is clear that $A\pi_{i=1}^{j} L(t_i - t_{i-1})q - A\pi_{i=1}^{n} L(t_i - t_{i-1})q = 0 \in W$. Since ϵ is arbitrary, therefore, if j is an integer in $[0,n]$, $A\pi_{i=1}^{j} L(t_i - t_{i-1})q - A\pi_{i=1}^{n} L(t_i - t_{i-1})q \in \overline{W}$.

Lemma 7: Given a $W \in X$, let $p \in C$, $Ap \in U$ and $\{s_i\}_{i=0}^{\infty}$ be an infinite increasing number sequence such that $\lim_{i\to\infty} s_i < \gamma_p$. If $\{s_i\}_{i=0}^{n}$ has property P_w for all n, then there is a positive integer m and a sequence $\{r_i\}_{i=0}^{m}$ such that $r_i = s_i$ for all integers $i \in [0, m]$, $r_{m+1} = \lim_{i\to\infty} s_i$ and $\{r_i\}_{i=0}^{m+1}$ has property P_w.

Proof: Let $q_o = p$ and if n is positive integer, let $q_n = L(s_n - s_{n-1})q_{n-1}$. Then $q_n - q_{n-1} = L(s_n - s_{n-1})\,q_{n-1} - q_{n-1} \in (s_n - s_{n-1})\,U$. Since $\{s_i\}_{i=0}^{\infty}$ is a convergent sequence, $\{q_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Let $s = \lim_{i\to\infty} s_i < \gamma_p$, $q = \lim_{i\to\infty} q_i$. By lemma 3, $q_n - p \in s_n U$ and hence $q - p \in sU$. There is a $V \in X$ such that if $v \in C$ and $v - q \in V$ then $Av - Aq \in \frac{1}{4}W$. Let m be a positive integer such that $q - q_m \in V$ and $(s - s_m)U \subset V$. Let $0 < x < s - s_m$, and let $\{t_i\}_{i=0}^{n}$ be a chain from 0 to x and let j be an integer in $[0, n]$. By lemma 3, $\pi_{i=1}^{j} L(t_i - t_{i-1})q_m - q_m \in t_j U \subset V$ and so $A\,\pi_{i=1}^{j} L(t_i\ t_{i-1})\,q_m - Aq_m \in \frac{1}{4}W$. Hence for any integer $j \in [0, n]$, we have that

$$A\,\pi_{i=1}^{j} L(t_i - t_{i-1})q_m - A\,\pi_{i=1}^{n} L(t_i - t_{i-1})q_m \tag{12}$$

$$= A\,\pi_{i=1}^{j} L(t_i - t_{i-1})q_m - Aq + Aq - A\,\pi_{i=1}^{n} L(t_i - t_{i-1})q_m$$

$$\in \tfrac{1}{2}W + \tfrac{1}{2}W = W, \text{ since } q - q_m \in V \text{ implies } Aq - Aq_m \in \tfrac{1}{4}W.$$

Lemma 8: Given a $W \in X$, let $p \in C$, $Ap \in U$ and $0 < x < \gamma_p$. There is a chain $\{s_i\}_{i=0}^{m}$ from 0 to x with property P_w.

Proof: By lemma 5 there is an infinite increasing number sequency $\{s_i\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} s_i < \gamma_p$ and $\{s_i\}_{i=0}^{n}$ has property P_w for all n. Let M denote the set of all such sequences. If $s = \{s_i\}_{i=0}^{\infty}$ is in M, let Z(s) denote the limit of s. For $s, t \in M$, we define $s \leqslant t$ if $s = t$ or if n is the greatest nonnegative integer such that for all $i \in [0, n]$, $s_i = t_i$, then $Z(s) \leqslant t_{n+1}$. Therefore " $\leqslant$ " is a partial ordering on M.

Assume that there exists no element s of M such that $Z(s) > x$. Let L be a linearly ordered subset of M and let y be the smallest positive number such that if s is in L, $Z(s) \leqslant y$. Let $\{s_i(0)\}_{i=0}^{\infty}$, $\{s_i(1)\}_{i=0}^{\infty}$, ... be increasing sequences of points in L such that Z(s(0)), Z(s(1)), ... converges to y. For each nonnegative integer i, we define $y_i = \text{Sup}_k\, s_i(k)$. Then $y_i \leqslant y_{i+1}$ and $\lim_{i\to\infty} y_i = y$.

Case 1: Suppose that there is a positive integer n such that $y_n = y$. Then there is a least positive integer n such that $y_n = y$ and there must exist an integer $\bar{k}$ such that $s_i(\bar{k}) = s_i(j)$ for each integer $i \in [0, n-1]$ and $j > \bar{k}$. In this case $s_n(\bar{k})$, $s_n(\bar{k}+1)$, ... converges to y and by lemma 6 $\{s_i\}_{i=0}^{n}$, whenever $s_i = s_i(\bar{k})$ for $i \in [0, n-1]$ and $s_n = y$, has property P_w. Furthermore, since $y < \gamma_p$, we have, by lemma 5, that $\{s_i\}_{i=0}^{n}$ may be extended to $\{s_i\}_{i=0}^{\infty}$ of M, and hence $\{s_i\}_{i=0}^{\infty}$ is an upper bound of L.

Case 2: If there is no positive integer n such that $y_n = y$, then $y_n < y$ for all n, $\{y_n\}_{n=0}^{\infty}$ is in M, and $\{y_n\}_{n=0}^{\infty} \geqslant s(k)$ for every k. Thus $\{y_n\}_{n=0}^{\infty}$ is an upper bound of L.

Hence if L is a linearly ordered subset of M, then L is bounded by a member of M. By Zorn's lemma there exists a $\ell \in M$ such that ℓ is maximal. But then we have a contradiction, since $Z(\ell) < x < \gamma_p$ and by lemma 7, there exists a $t \in M$ such that $\ell < t$. Hence there exists a $s \in M$ such that $Z(s) > x$. In the proof of lemma 5 we can restrict all such $s_{m+1} < x$, hence $Z(s) < x$, and the lemma is proved.

Lemma 9: Given $W \in X$, let $p \in C$, $Ap \in U$ and $0 < x < \gamma_p$. There is a chain $\{s_i\}_{i=0}^m$ from 0 to x such that if $\{t_i\}_{i=0}^n$ is a refinement of $\{s_i\}_{i=0}^m$, then $\pi_{i=1}^n L(t_i - t_{i-1})p - \pi_{i=1}^m L(s_i - s_{i-1})p \in W$.

Proof: Let $\{s_i\}_{i=0}^m$ be a chain from 0 to x such that $\{s_i\}_{i=0}^m$ has property $P_{w/x}$. Let $\{t_i\}_{i=0}^n$ be a refinement of $\{s_i\}_{i=0}^m$. Then there is an increasing sequence $\{f_i\}$ from $\{0, 1, 2, \ldots m\}$ into $\{0, 1, 2, \ldots n\}$ such that $f_o = 0$, $f_m = n$ and if i is an integer in $[0, m]$, $s_i = t_{f_i}$. If i is an integer in $[1, m]$, let $K_i = \pi_{j=f_{i-1}+1}^{f_i} L(t_j - t_{j-1})$, $J_i = \pi_{j=1}^{i} L(s_j - s_{j-1})$ and $K_{m+1} = I$. Then

$$\pi_{i=1}^n L(t_i - t_{i-1})p - \pi_{i=1}^m L(s_i - s_{i-1})p = \pi_{i=1}^m K_i p - J_m p \tag{13}$$

$$= \Sigma_{i=1}^m [\pi_{j=i}^m K_j J_{i-1} p - \pi_{j=i+1}^m K_j J_j p].$$

Now we have

$$[I - (s_i - s_{i-1}) A]\, K_i J_{i-1} p - [I - (s_i - s_{i-1}) A]\, J_i p \tag{14}$$

$$= K_i J_{i-1} p - (s_i - s_{i-1}) A K_i J_{i-1} p - J_{i-1} p$$

$$= K_i J_{i-1} p - J_{i-1} p - (s_i - s_{i-1}) A K_i J_{i-1} p$$

$$= [\pi_{j=f_{i-1}+1}^{f_i} L(t_j - t_{j-1})\, J_{i-1} p - J_{i-1} p] - (s_i - s_{i-1}) A K_i J_{i-1} p$$

$$= [\Sigma_{j=f_{i-1}+1}^{f_i} L(t_j - t_{j-1})\, A\, \pi_{k=f_{i-1}+1}^{j} L(t_k - t_{k-1})\, J_{i-1} p] - (s_i - s_{i-1}) A K_i J_{i-1} p$$

$$= \Sigma_{j=f_{i-1}+1}^{f_i} (t_j - t_{j-1})\, [A\, \pi_{k=f_{i-1}+1}^{j} L(t_k - t_{k-1})\, J_{i-1} p - A K_i J_{i-1} p].$$

Since there is a chain $\{s_i\}_{i=0}^m$ from 0 to x such that $\{s_i\}_{i=0}^m$ has property $P_{w/x}$, i.e, $A\pi_{k=f_{i-1}+1}^{f_i} L(t_k - t_{k-1}) J_{i-1} p - AK_i J_{i-1} p \in W/x$, from (14), we have the $[I - (s_i - s_{i-1}) A]\, K_i J_{i-1} p - [I - (s_i - s_{i-1}) A]\, J_i p \in ((s_i - s_{i-1})/x)W$.

Now since A is dissipative, $K_i J_{i-1} p - J_i p \in ((s_i - s_{i-1})/x)W$. It follows from the X-nonexpansivity of $L(\varepsilon)$, that $\pi_{j=i}^m K_j J_{i-1} p - \pi_{j=i+1}^m K_j J_j p \in ((s_i - s_{i-1})/x)W$. By (13), we get $\pi_{i=1}^n L(t_i - t_{i-1})p - \pi_{i=1}^m (s_i - s_{i-1})p \in \Sigma_{i=1}^m ((s_i - s_{i-1})/x)W = W$.

III. Proof of the Theorem

Let $p \in C$. It is clear that if x=o, then $\pi_o^x (I - dIA)^{-1} p = p$, and if $o < x < \gamma_p$, then $\pi_o^x (I - dIA)^{-1} p$ exists by lemma 9. If $o < x < \gamma_p$, we define $u_p(x) = \pi_o^x (I - dIA)^{-1} p$. By lemma 3, we see that u_p is continuous on $[o, \gamma_p)$. For $o < x < \gamma_p$, let $o < y < \gamma_p - x$. We see that $u_p(x+y) = u_{u_p(y)}(x)$. What is left is to show $u'_p(x) = Au_p(x)$. Given a $W \in X$, let $o < x < \gamma_p$ and $Ap \in U$. By lemma 3, there is a positive number $z < \gamma_p - x$ such that if $o < y < z$ and $\{s_i\}_{i=o}^m$ is a chain from o to y, then $A\, \pi_{i=1}^m L(s_i - s_{i-1}) u_p(x) - Au_p(x) \in$ ½W. Let $\{t_i\}_{i=0}^n$ be a chain from o to y such that $\pi_{i=1}^n L(t_i - t_{i-1}) u_p(x) - u_p(x+y) \in$ (½) yW. Then

$$1/y\, [u_p(x+y) - u_p(x)] - Au_p(x)$$

$$= 1/y\, [u_p(x+y) - \pi_{i=1}^n L(t_i - t_{i-1}) u_p(x)] + 1/y [\pi_{i=1}^n L(t_i - t_{i-1}) u_p(x) - u_p(x) - yAu_p(x)]$$

$$= 1/y[u_p(x+y) - \pi_{i=1}^{n} L(t_i - t_{i-1})u_p(x)] + 1/y[\Sigma_{i=1}^{n}(t_i - t_{i-1})A\,\pi_{j=1}^{i}(t_j - t_{j-1})u_p(x) - yAu_p(x)]$$

$$= 1/y[u_p(x+y) - \pi_{i=1}^{n} L(t_i - t_{i-1})u_p(x)] + 1/y\ \Sigma_{i=1}^{n}(t_i - t_{i-1})[A\,\pi_{j=1}^{i}(t_j - t_{j-1})u_p(x) - Au_p(x)]$$

$$\epsilon\ 1/y(½)yW + 1/y\,\Sigma_{i=1}^{n}(t_i - t_{i-1})\cdot ½\,W = W,$$

and so $u'_p(x) = Au_p(x)$ on $[0, \gamma_p)$. Since A and u_p are continuous, $u_p(x)$ has a continuous right derivative on $[0, \gamma_p)$ and hence $u_p(x)$ is a continuously differentiable function on $[0, \gamma_p)$. So the existence theorem is proved. In particular let $s_i - s_{i-1} = x/n$ for a positive integer n, we have the exponential formula $u_p(x) = \lim_{n\to+\infty} [I - (x/n)A]^{-n}p$.

Now we show the uniqueness. Let u_1, u_2 satisfy $u'(t) = Au(t)$ and let $f(t) = u_1(t) - u_2(t)$. We want to prove that if $f(t) \epsilon W$ for a $W \epsilon X$, then $f(s) \epsilon \overline{W}$ for all $s > t$. Let $x_o = \inf\{x \mid t < x, f(x) \notin \overline{W}\}$. By the continuity of f, we have $f(x_o) \epsilon \overline{W}$ and $f(x_o) \notin W$, hence $f(x_o) - kf'(x_o) \notin W$ for all $k > o$. We define $hf'(x_o) = \lim_{t\to o} [f(x_o+th) - f(x_o)]/t$ for all $h \epsilon R$. Thus as t is sufficiently small, $f(x_o) - [f(x_o + tk) - f(x_o)]/t \notin W$ implies $f(x_o + tk) - f(x_o) \notin tf(x_o) + |t| W$. Since $f(x_o + tk) \notin \overline{W}$, we can assume that $f'(x_o)$ is in the hyperplane H which has the property that for all vector $v \epsilon f(x_o) + H$, v is not in $\overline{W}$ unless $v = f(x_o)$. Since $\lim_{t\to o} [f(x_o - th) - f(x_o)]/t = -hf'(x_o) \epsilon H$, for $h > o$, we have $f(x_o - t_1 h) - f(x_o) \epsilon H$ for small $t_1 > o$, that is, $f(x_o - t_1 h) \epsilon f(x_o) + H$. Since $f(x_o - t_1 h) \epsilon \overline{W}$, this implies $f'(x_o) = 0$. Thus $\lim_{t\to o} [f(x_o+tk) - f(x_o)]/t = kf'(x_o) = 0$, for all $k > o$, and for any $V \epsilon X$, there exists $d > o$ such that if $o < t < d$, we have $f(x_o + tk) - f(x_o) \epsilon tV$ for all $k > o$. Now we know that there is a $x > x_o$ such that $f(x) \notin \overline{W}$, hence we have a $V(x) \epsilon X$ such that $f(x) - f(x_o) \notin V(x)$. Let $k = (1/t)(x - x_o)$. This implies $f(x_o + tk) - f(x_o) = f(x) - f(x_o) \notin V(x)$, thus $V(x) \subset tV$ for all $o < t < d$, which is impossible. Therefore $x_o = +\infty$. This completes the uniqueness of Cauchy initial value problem (1).

Corollary: Let A be a mapping from the Hausdorff sequentially complete locally convex topological vector space S into S such that the followings are true: (1) A is dissipative on S, (2) $R(I - \varepsilon A) = S$ for each $\varepsilon > o$, (3) A is continuous on S. If $p \epsilon S$ then there is a unique continuously differentiable function u_p from $[o, \infty)$ into S such that $u_p(o) = p$, $u'_p(x) = Au_p(x)$ for $x > o$, and $u_p(x) = \pi_o^x(I - dIA)^{-1}p$.

Proof: The proof follows immediately from the theroem if one observes that $\alpha = +\infty$ and $\gamma_p = +\infty$.

By virtue of the corollary one may define for each $x \geq o$, the transformation T(x) from S into S as follows: $T(x)p = u_p(x)$ for each $p \epsilon S$. T is then a continuous semi-group of nonlinear X - nonexpansive transformations on S, that is, (i) $T(x+y)p = T(x)T(y)p$ for $x, y > o$, $p \epsilon S$, (ii) $T(o)p = p$, (iii) $T(x)p$ is continuous on $[o, \infty)$ for any fixed p.

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