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弱收縮變換之雙不變子空間
Bi-invariant Subspaces of Weak Constrictions

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Abstract — For a bounded linear operator T acting on a complex, separable Hilbert space, let Lat T, Lat" T and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. In this paper we characterize the elements of Lat" T, in terms of the characteristic function of T, when T is a completely non-unitary weak contraction with finite defect indices. We show that if the defect indices of T are $n < \infty$ and Θ_T denotes the characteristic function of T, then a subspace in Lat T belongs to Lat" T if and only if the intermediate space of its corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$ is of dimension n. As corollaries, necessary and sufficient conditions that two of these lattices of subspaces be equal to each other are obtained. In particular, if T_1, T_2 are completely non-unitary C_{11} contractions with finite defect indices which are quasi-similar to each other, then Lat" T_1 is isomorphic to Lat" T_2. Whether this is true for weak contractions is still unknown.

For a bounded linear operator T acting on a complex, separable Hilbert space H, let Alg T, $\{T\}''$ and $\{T\}'$ denote the weakly closed algebra generated by T and I, the double commutant and the commutant of T, respectively. A subspace K of H is bi-invariant (resp. hyperinvariant) for T if K is invariant for every operator in $\{T\}''$ (resp. $\{T\}'$). Let Lat T, Lat''T and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. The following trivial relations hold: Alg $T \subseteq \{T\}'' \subseteq \{T\}'$ and Lat $T \supseteq$ Lat'' $T \supseteq$ Hyperlat T. In this note we announce some results concerning these algebras and lattices when T is a completely non-unitary (c.n.u.) weak contraction with finite defect indices. In particular, we obtain specific descriptions of the elements of Lat''T, which are analogous to the results we obtained earlier in [2], [3] and [4]. The results will be developed first for C_{11} contractions and then for weak contractions.

For an arbitrary contraction T, let Θ_T denote its characteristic function. If T is c.n.u. and has defect indices $d_T = d_{T*} \equiv n < \infty$, then T may be considered as defined on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w\colon w \in H_n^2\}$ by $T(f \oplus g) = P(e^{it}f \oplus e^{it}g)$, where $\Delta = (I - \Theta_T^* \Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. For the details the readers are referred to [1].

We start with the following

Lemma 1. Let T be a c.n.u. C_{11} contraction with defect indices $n < \infty$ and let U be the operator of multiplication by e^{it} on $\overline{\Delta L_n^2}$. If $S = P\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ is an operator in $\{T\}''$, then C is in $\{U\}''$.

Lemma 2. For j=1,2, let T_j be a c.n.u. C_{11} contraction with finite defect indices and let $\Delta_j = (I - \Theta_{T_j}^* \Theta_{T_j})^{1/2}$. Then the following are equivalent to each other:

(1) T_1 is quasi-similar to T_2;

(2) rank $\Delta_1(t)$ = rank $\Delta_2(t)$ a.e.

Using Lemma 2 we may prove

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Lemma 3. Let T be as in Lemma 1. If $K_1, K_2 \in \mathrm{Lat}''\, T$, $K_1 \subseteq K_2$ and $T|K_1$ is quasi-similar to $T|K_2$, then $K_1=K_2$.

Lemma 4. Let T be as in Lemma 1 and let V be the operator of multiplication by e^{it} on $\overline{\Delta_* L_n^2}$, where $\Delta_* = (I - \Theta_T \Theta_T^*)^{1/2}$. Let $X: H \to \overline{\Delta_* L_n^2}$ be defined by $X(f \oplus g) = -\Delta_* f + \Theta_T g$. Then X is a quasi-affinity intertwining T and V.

The next theorem gives our main result for C_{11} contractions.

Theorem 5. Let T, V and X be as in Lemma 4. Let $K \subseteq H$ be an invariant subspace for T with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$. Then the following statements are equivalent:

(1) K is bi-invariant for T;

(2) $T \mid K$ is of class C_{11};

(3) the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension n;

(4) $K = X^{-1}(L)$ for some bi-invariant subspace $L \subseteq \overline{\Delta_* L_n^2}$ for V.

The main body of the proof is to show (4) $\Rightarrow$ (1) by using Lemma 1 and (2) $\Rightarrow$ (4) by Lemma 3.

Corollary 6. Let T, V and X be as in Lemma 4. Then $\mathrm{Lat}''T \cong \mathrm{Lat}''\, V$. Moreover the isomorphisms may be implemented by the mappings $K \to \overline{XK}$ and $L \to X^{-1}(L)$, where $K \in \mathrm{Lat}''\, T$ and $L \in \mathrm{Lat}''\, V$. In this case the C_{11} contraction $T \mid K$ is quasi-similar to the unitary operator $V \mid \overline{XK}$.

Corollary 7. Let T_1, T_2 be c.n.u. C_{11} contractions with finite defect indices. If T_1 is quasi-similar to T_2, then $\mathrm{Lat}''T_1 \cong \mathrm{Lat}''T_2$.

Generalizing theorem 5 to weak contractions we have

Theorem 8. Let T be a c.n.u. weak contraction with defect indices $n < \infty$ on H, and let H_0, H_1 be invariant subspaces for T such that $T_0 = T \mid H_0$ and $T_1 = T \mid H_1$ are the C_0 and C_{11} parts of T, respectively. Let $K \subseteq H$ be an invariant subspace for T with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$. Then the following statements are equivalent:

(1) K is bi-invariant for T;

(2) $T \mid K$ is a weak contraction;

(3) the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension n;

(4) $K = K_0 \vee K_1$, where $K_0 \subseteq H_0$, $K_1 \subseteq H_1$ are bi-invariant subspaces for T_0, T_1, respectively.

Corollary 9. Let T, T_0 and T_1 be as in Theorem 8. Then the following lattices are isomorphic: $\mathrm{Lat}''T$, $\mathrm{Lat}''T_0 \oplus \mathrm{Lat}''T_1$ and $\mathrm{Lat}''(T_0 \oplus T_1)$.

We should point out that whether two quasi-similar c.n.u. weak contractions with finite defect indices have isomorphic bi-invariant subspace lattices is still unknown. The difficulty lies in that we don't know whether this holds for $C_0(N)$ contractions.

From Theorem 8 we can derive necessary and sufficient conditions that two of Lat T, Lat'' T and Hyperlat T be equal to each other. One sample is the following

Theorem 10. Let T be as in Theorem 8. Then the following statements are equivalent:

(1) $\mathrm{Alg}\, T = \{T\}''$;

(2) $\mathrm{Lat}\, T = \mathrm{Lat}''T$;

(3) for any $K \in \mathrm{Lat}\, T$, $T \mid K$ is a weak contraction;

(4) the intermediate space of any reqular factorization of Θ_T is of dimension n;

(5) $\Theta_T(t)$ is isometric for t in a set of positive Lebesgue measure.

The proofs of these results will appear elsewhere.

References

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