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A Note on A Family of Hyperbolic Secant Distributions

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Abstract — We show that a class of hyperbolic secant distributions is not a member of exponential family. A related family of distributions is proved to be complete.

I. Introduction

By a hyperbolic secant distribution we mean a density function of the form

$$f_\theta(x)=\frac{\theta}{\pi\cosh x\theta}, \quad -\infty<x<\infty \quad , \tag{1}$$

where θ is a scale parameter such that $0<\theta<\infty$.

Champernowne [1] had used (1) to calculate the income power (logarithm of the income). The general properties of the above distributions have been reviewed by Lai [2].

The primary aim of this note is to show that although the family of hyperbolic secant distributions is a member of Meixner classes (Meixner [3] & [4]) it does not have an exponential representation. We then show that the family of distributions of $|X_\theta|$ is complete (X_θ denotes the random variable that corresponds to (1)).

II. Exponential Representation

Theorem 1: The family $\{f_\theta(x), -\infty<x<\infty, 0<\theta<\infty\}$ defined in (1) does not belong to the exponential family of distributions.

Proof: Suppose on the contrary that $\{f_\theta(x), 0<\theta<\infty\}$ belongs to the exponential family, we then could write $f_\theta(x)$ in the form

$$f_\theta(x)=\frac{\theta}{\pi}\operatorname{sech}x\theta=c(\theta)\exp\{-Q(x)R(\theta)\}\ h(x), \tag{2}$$

where Q(x), h(x) are functions of x only and $c(\theta)$, $R(\theta)$ are functions of θ only. Now

$$\frac{\pi}{\theta}f_\theta(x)=\operatorname{sech}x\theta=\frac{\pi}{\theta}c(\theta)\exp\{-Q(x)R(\theta)\}\ h(x).$$

Since $\operatorname{sech}x\theta$ is symmetric in x and θ, it is easy to see that we must have $\frac{\pi}{\theta}c(\theta)=h(\theta)$, $Q(x)=R(x)$ and hence

$$\operatorname{sech}x\theta=h(x)h(\theta)\exp\{-Q(x)Q(\theta)\}. \tag{3}$$

Put $\theta = 0$ in (3) we obtain

$$h(x) = \exp\{Q(x)Q(0)\}/h(0)$$

from which we deduce at once that

$$\operatorname{sech} x\theta = \exp\{Q(x)Q(0) + Q(\theta)Q(0) - Q(x)Q(\theta)\}/h(0)^2. \tag{4}$$

By putting $\theta = 1$ in (4) we get

$$\operatorname{sech} x = \lambda \exp \beta\, Q(x) \tag{5}$$

where $\lambda = \exp\{Q(1)Q(0)\}/h(0)^2 > 0$ and $\beta = Q(0) - Q(1)$.

It is clear that λ and β are constants which are independent of both of x and θ. It follows directly from (5) that

$$Q(x) = -\frac{1}{\beta}\log \lambda \cosh x, \qquad Q'(x) = -\frac{1}{\beta}\tanh x. \tag{6}$$

By differentiating both sides of (4) with respect to x we get

$$-\theta \operatorname{sech} x\theta \tanh x\theta = \operatorname{sech} x\theta \;\{Q'(x)[Q(0) - Q(\theta)]\}. \tag{7}$$

By letting $x \to 0$ in (7) we obtain

$$\beta = [Q(0) - Q(\theta)]/\theta^2 \tag{8}$$

Since $Q'(x) = -\frac{1}{\beta}\tanh x$, Q(x) can not be a polynomial of x of degree 2 and hence β cannot be a constant independent of θ. This contradicts our earlier derivation that β is a constant. Hence $\{f_\theta(x): 0 < \theta < \infty\}$ cannot be an one-parameter exponential family of distributions.

We note that since the range of the above distribution does not depend on θ and $\{f_\theta(x): 0 < \theta < \infty\}$ does not belong to the exponential family of distributions, we conclude from the result of Pitman [5] that there exists no sufficient statistic.

III. Completeness

The second aspect of this note is to discuss the problem of completeness of the family $\{g_\theta(y), 0 \leq y < 0, 0 < \theta < \infty\}$, the density functions of $|X_\theta|$ where X_θ, $0 < \theta < \infty$, denote the random variables that correspond to $f_\theta(x)$.

It is well known that a family $F = \{g_\theta(y): \theta \in \Omega\}$ of probability distributions is complete if $E_\theta[\varphi(Y)] = 0$ (for all $g_\theta \in F$) implies $\varphi(y) = 0$ a.e. (see for example, Silvey [6, p29-30]). In our case $g_\theta(y)$ is given by

$$g_\theta(y) = \frac{2\theta}{\pi}\operatorname{sech}\theta y, \quad 0 \leq y < \infty, 0 < \theta < \infty \tag{9}$$

We mention in advance that the discussion presented in this section is somehow heuristic in character.

Theorem 2: The family of distributions defined in (10) indexed by the scale parameter $\theta \epsilon (0, \infty)$ is complete.

Proof: We first note $E_\theta[\varphi(Y)]=0$ implies $E_\theta[|\varphi(Y)|]<\infty$. Therefore for any $\theta>0$, we have

$$\infty > \int_0^\infty |\varphi(t)| \operatorname{sech}\theta t dt \geqslant \int_0^R |\varphi(t)| \operatorname{sech}\theta t dt \geqslant \operatorname{sech}R\theta \int_0^R |\varphi(t)| dt, R>0, \text{i.e.}$$

$\varphi(t)$ belongs to L_1 in $(0,R)$ for every $R>0$.

It is sufficient to show that $\int_0^\infty \varphi(t)\operatorname{sech}\theta t dt=0$ implies $\varphi(t)=0$ a.e.

By omitting analytical details we consider

$$2h(\theta)= \int_0^\infty \varphi(t)\operatorname{sech}\theta t dt, 0 < \theta < \infty$$

$$= 2\int_0^\infty \{e^{-\theta t}[1+e^{-2\theta t}]^{-1}\}\varphi(t) dt$$

$$= 2\sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(2n+1)\theta t}\varphi(t)dt. \tag{10}$$

Let $\varphi^*(\theta)$ denote the Laplace transform of $\varphi(t)$. Now it is evident

$$h(\theta)= \sum_{n=0}^\infty (-1)^n \varphi^*((2n+1)\theta). \tag{11}$$

Our next step is to find a way to express $\varphi^*(\theta)$ in terms of $h(\theta)$. We shall begin by writting down (11) term-by-term in the manner

$$h(\theta)= \varphi^*(\theta)-\varphi^*(3\theta)+\varphi^*(5\theta)-\varphi^*(7\theta)+\ldots\ldots$$

$$h(3\theta)= \varphi^*(3\theta) \qquad -\varphi^*(7\theta) \qquad +\varphi^*(15\theta)-\ldots\ldots$$

$$h(5\theta)= \varphi^*(5\theta) \qquad -\varphi^*(15\theta)+\ldots\ldots$$

$$\ldots\ldots \qquad \ldots\ldots \text{ etc.}$$

We can verify by means of a careful observation that the above system of equation reduces to

$$\varphi^*(\theta)= \sum_{n=0}^\infty (-1)^n \mu(2n+1)h((2n+1)\theta), \tag{12}$$

where $\mu(.)$ is the Möbius function which can be defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n=1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

For a detailed discussion on Möbius function the reader may consult Van Der Waerden [7, p.114].

Equation (12) shows that $\varphi^*(\theta)=0$ if $h(\theta)=0$, i.e. if $E_\theta[\varphi(Y)] = \frac{4\theta}{\pi} h(\theta)=0$.

As we have established earlier, $\varphi(t)$ belongs to L_1 in $(0, R)$ for every $R>0$ if $E_\theta[\varphi(Y)]=0$.

Upon applying Lerch Theorem [8, p.62] we deduce at once that $\varphi(t)=0$ a.e., that is to say $\{ g_\theta(y), 0 \leq y < \infty, 0 < \theta < \infty\}$ is complete.

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