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略談一握德點過程之誤差對時間之曲線

A Note on The Variance Time Curve of A Wold's Point Process

賴振宙 Chin-Diew Lai

Department of Applied Mathematics, N. C. T. U.

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Abstract —— The variance time curve (of counting process) of a Wold point process with exponential marginals is obtained.

I. Introduction

A continuous time Wold's process is a stationary stochastic point process whose succeeding time intervals $\{X_i: i=1, 2,...\}$ between the events formed a Markov process. This class of stochastic models was firstly formulated by Wold [4 & 5] and then studied by others. Lai [2] obtained an integral equation for the renewal density function whence the spectral density is calculated in a special example [3]. In this note, we shall continue the work of [3] and look into another aspect of the second order properties, namely, the variance time curve of a special Wold's point process. Although, in principle, the variance of counts in any stationary point process has a definite form (see p. 75 of [1]), the second order properties of Wold's point processes in general are difficult to obtain as distributions of $\sum_{i=1}^{r} X_i$ ($r=1,2,.....$) are often involved directly. We shall show that in our case, however, the variance function V(t) can be found without actually determining the distributions of $\sum_{i=1}^{r} X_i$.

II. Definitions, Notations and Some Known Results:

In this paper, we shall attempt to adopt the notations and definitions as given in Cox & Lewis ([1], p.67-81).

Let N_t be the accumulative number of events in an interval of length t following the arbitrarily selected point where observation of the process begins.

Define

$$m(t) = \sum_{r=1}^{\infty} f_r(t) \tag{1}$$

where $f_r(t)$ ($r=1,2,...$) are the probability density functions of $\sum_{i=1}^{r} X_i$. $m(t)\Delta t$ is the probability of an event occurs in a small interval $(t, t+\Delta t]$ given that an event has occurred at the origin. m(t) is called the intensity function of the counting process. The variance-time curve can be defined as

$$V(t) = E(N_t^2) - \{E(N_t)\}^2 \tag{2}$$

$$I(t) = \frac{V(t)}{E(N_t)} = \frac{E(X)\,V(t)}{t} \tag{3}$$

is called the index of dispersion. Let

$$\gamma(\tau) = \lim_{\Delta t \to 0} \frac{\operatorname{cov}\{\Delta N_t, \Delta N_{t+\tau}\}}{(\Delta t)^2}, \quad \tau > 0, \text{ (} \Delta N_t \text{, the number of events in } (t, t+\Delta t]$$

$$= m\{m(t) - m\}, m = 1/E(X_i). \tag{4}$$

A complete spectral density function for the counting process is defined as

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m\delta(t)e^{-i\omega t}dt + \frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma(t)e^{-i\omega t}dt, \ (\delta(t), \text{ the Dirac delta function})$$

$$= \frac{m}{2\pi} + \frac{m}{2\pi}\int_{-\infty}^{\infty}\{m(t) - m\}\, e^{-i\omega t}dt,$$

$$= \frac{m}{2\pi}\{1 + m^*(i\omega) + m^*(-i\omega)\}, \quad \omega \geq 0 \tag{5}$$

where $m^*(\theta)$ is the Laplace transform of $m(t)$.

It is known that (see p. 75 Cox & Lewis [1])

$$V(t) = mt + 2\int_0^t\int_0^v \gamma(u)du\, dv, \tag{6}$$

$$V'(t) = m + 2\int_0^t \gamma(u)du, \quad (V'(t) = \frac{dV(t)}{dt}) \tag{7}$$

and

$$V'(\infty) = m + 2\int_0^\infty \gamma(u)du = 2g(0+). \tag{8}$$

The Laplace transform of $V(t)$ is

$$V^*(\theta) = \frac{m}{\theta^2} + 2m\frac{m^*(\theta)}{\theta^2} - 2\frac{m^2}{\theta^3}. \tag{9}$$

III. The Result:

In [3], we assume that the common distribution of $\{X_i\}$ is $f(x)=e^{-x}$ and that probability transition density function is $f(y\,|\,x)=e^{-y}\{1+\rho(2e^{-x}-1)(2e^{-y}-1)\}$, $0 \leq \rho < 1$. We also assume that the 0^{th} event occurs at $t=0$. We found in this case that $m^*(\theta)$ can be determined easily and has the form

$$m^*(\theta) = \sum_{n=0}^{\infty} m_n(\theta)\rho^n = \frac{1}{\theta} + \frac{\rho}{(\theta+2)^2}\sum_{n=0}^{\infty}(c_1(\theta)\rho)^n \tag{10}$$

where $c_1(\theta) = \frac{1}{(\theta+1)(\theta+2)}\{\frac{\theta}{\theta+2} + \frac{\theta^2+\theta+2}{(\theta+3)}\}$.

(See equations (21) & (23) of [3]).

It follows from (10) by inverting the Laplace transform that

$$m(t) = 1 + te^{-2t}\rho * \sum_{n=0}^{\infty} g_n(t)\rho^n, \tag{11}$$

where $g_0(t) \equiv 1$, $g_1(t)=e^{-2t}\{2t-3+4e^{-t}\}$ and $g_n(t)=\{g_1(t)\}^{*n}$.

The $*$ here denotes for convolution:

$(f*g)(x)=\int_0^x f(t)g(x-t)dt$. It is clear that if

$f\in L_2$, $g\in L_2$, then $|(f*g)(x)|\leq(\int_0^x [f(t)]^2dt \int_0^x [g(t)]^2dt)^{1/2}$

In this case, $g_1(t)=e^{-2t}\{2t-3+4e^{-t}\}\geq 0$ for all $t\geq 0$.

A simple calculation shows that

$$\int_0^t [g_1(x)]^2dx=e^{-4t}\{-t^2+t(\frac{5}{2}-\frac{16}{5}e^{-t})+(-\frac{1}{2}+\frac{8}{5}e^{-t}+\frac{8}{3}e^{-2t})\}+\frac{79}{600} \tag{12}$$

Since $g_1(x)\geq 0$, it implies that $\int_0^t [g_1(x)]^2dx\leq\int_0^\infty [g_1(x)]^2dx=\frac{79}{600}$.

Therefore, $g_n(t)\leq(\frac{79}{600})^{n/2}\approx(0.36)^n$. Also, $te^{-2t}*g_n(t)\leq\frac{1}{\sqrt{2}}(0.36)^n$.

$$\therefore m(t)\leq 1+\frac{\rho}{\sqrt{2}}\sum_{n=0}^{\infty}(0.36\rho)^n=1+\frac{\rho}{\sqrt{2}(1-0.36\rho)},\quad 0\leq\rho\leq 1.$$

Note that $\lim_{t\to 0} m(t)=1$, $\lim_{t\to 0} m(t)=1$ and $m(t)$ attains its maximum in the vicinity of $t=\frac{1}{2}$.

From (9) and (10), we have

$$V^*(\theta)=\frac{1}{\theta^2}+\frac{\rho}{2}\{-\frac{1}{\theta}+\frac{1}{\theta^2}+\frac{1}{(\theta+2)}+\frac{1}{(\theta+2)^2}\}\sum_{n=0}^{\infty}[c_1(\theta)\rho]^n, \tag{13}$$

and

$$V'^*(\theta)=\frac{1}{\theta}+\frac{\rho}{2}\{\frac{1}{\theta}-\frac{1}{(\theta+2)}-\frac{2}{(\theta+2)^2}\}\sum_{n=0}^{\infty}[c_1(\theta)\rho]^n. \tag{14}$$

By inverting the Laplace transforms in (13) and (14), we get

$$V(t)=t+\frac{1}{2}\{(t-1)+(t+1)e^{-t}\}*\sum_{n=1}^{\infty}g_{n-1}(t)\rho^n \tag{15}$$

$$V'(t)=1+\frac{1}{2}\{1-e^{-2t}(1+2t)\}*\sum_{n=1}^{\infty}g_{n-1}(t)\rho^n \tag{16}$$

respectively. As $\lim_{\theta\to 0}\theta V'^*(\theta)=1+\frac{\rho}{2}\sum_{n=0}^{\infty}(\frac{\rho}{3})^n=1+2\sum_{n=1}^{\infty}\frac{3}{4}(\frac{\rho}{3})^n$,

it follows that

$$V(t)=-\frac{\rho}{2}+\{1+2\sum_{n=1}^{\infty}\frac{3}{4}(\frac{\rho}{3})^n\}t+o(1),\quad 0\leq\rho<1. \tag{17}$$

$$=-\frac{\rho}{2}+(1+2\sum_{k=1}^{\infty}\rho_k)t+o(1) \tag{18}$$

where $\rho_k=\text{cov}(X_n, X_{n+k})/\sigma^2$, $\sigma^2=\text{var}\,X_i$. (Note: we have shown in [3] that $\rho_k=\frac{3}{4}(\frac{\rho}{3})^k$, k=1,2,...)

Now, for $\rho=0$, $V(t)=t+o(1)$ as expected, because in the case of renewal process, it is well known that

$$V(t)=\frac{\sigma^2 t}{\mu^3}+\left(\frac{5}{4}\frac{\mu_2^{\,2}}{\mu_1^{\,4}}-\frac{2}{3}\frac{\mu_3}{\mu_1^{\,3}}-\frac{\mu_2}{2\mu_1}\right)+o(1),\quad \mu_i=E[X^i].$$

(17) indicates that $V(t)$ increases as ρ increases. This is expected as ρ is the measurement of dependence among $\{X_i\}$ $(i=1,2,\ldots)$.

Since $m_n(\theta)=\frac{c_1(\theta)^{n-1}}{(\theta+2)^2}$ for $n\geq 1$, $m_n(0)=\frac{3}{4}\left(\frac{\rho}{3}\right)^{n-1}$. Therefore, $\rho_k=m_k(0)$. From (18),

$$V(t)=-\frac{\rho}{2}+\left(1+\sum_{n=0}^{\infty} m_n(0)\rho^n\right)t+o(1). \tag{19}$$

After comparing (19) with (8), we see that

$$\int_0^\infty \gamma(u)du=\sum_{n=0}^{\infty} m_n(0)\rho^n=\sum_{n=0}^{\infty}\rho_n,\quad (\rho_0=1). \tag{20}$$

This is not surprising, as, in view of (4),

$$\gamma^*(\theta)=m\{m^*(\theta)-m\}=m\sum_{n=1}^{\infty} m_n(\theta)\rho^n. \text{ Now,}$$

$$\gamma^*(\theta)=\int_0^\infty \gamma(u)du=m\{\sum_{n=1}^{\infty} m_n(0)\rho^n\}.$$

Let $f(\omega)=\frac{1}{2\pi}\{1+2\sum_{k=1}^{\infty}\rho_k\cos(k\omega)\}$, $0\leqslant\omega\leqslant\pi$, (i.e., $f(\omega)$ is the spectral density function for the interval process), we can immediately see that $f(0+)=g(0+)=V'(\infty)$ which is consistent with the result given in [1; p.78].

The index of dispersion is very simple in this case, namely,

$$I(t)=\frac{V(t)}{t}=-\frac{\rho}{2t}+\left(1+\frac{3\rho}{2(3-\rho)}\right)+o(t)\to\left(1+\frac{3\rho}{2(3-\rho)}\right).$$

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