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n+1個變數的廣義超幾何函數

On A Generalized Hypergeometric Function of (n+1) Variables

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Abstract — It is the purpose of this paper to study a generalized hypergeometric function $F_P^{(n+1)}$ of (n+1) variables defined by (1) below. Specifically, we are concerned with the extensions of transformation, reduction, multiplication and summation formulas in [9], [14], [17] and [21] to $F_P^{(n+1)}$.

I. Introduction

In this paper we study a generalized hypergeometric function of (n+1) variables defined by

$$F_P^{(n+1)}\left(\begin{matrix} a:b;d_1,\ldots,d_n;d; \\ c:-;e_1,\ldots,e_n;-;z_1,\ldots,z_n,z \end{matrix}\right)$$

$$= \sum_{s_1,\ldots,s_n,r=0}^{\infty} \frac{(a,r+s_1+\ldots+s_n)(b,s_1+\ldots+s_n)(d,r)\ \prod_{k=1}^{n}(d_k,s_k)\ z_k^{s_k}\ z^r}{(c,r+s_1+\ldots+s_n)\ \prod_{k=1}^{n}(e_k,s_k)\ s_k!\ r!} \tag{1}$$

where $|z|$, $|z_k|<1$, $k \in \{1,\ldots,n\}$, $(a,n)=\Gamma(a+n)/\Gamma(a)$ and by analytic continuation, none of the quantities $c, e_1,\ldots, e_k$ are zero or a negative integer.

The following reducible cases of $F_P^{(n+1)}$ are obvious:

(i) For n=3, (1) reduces to a hypergeometric function $F_P^{(4)}$ of four variables which was considered recently by the author in [13].

(ii) For n=2, it reduces to a special form of a general triple hypergeometric series $F^{(3)}$ of Srivastava [19, p. 428] and in his notation it is

$$F^{(3)}\left[\begin{matrix} a::b;-;-:d_1;d_2;d_3; \\ c::-;-;-:e_1;e_2;-; \end{matrix}\quad z_1,z_2,z\right]$$

(iii) For $z \to 0$ and a=c, (1) becomes Lauricella's function F_A of n variables [6, p. 113].

(iv) The function $F_P^{(n+1)}$ for n=2 and $z \to 0$ is Kampé de Fériet's double series

$$F^{(2)}\left[\begin{matrix} a,b:d_1;d_2; \\ c:e_1;e_2; \end{matrix}\quad z_1,z_2\right]$$

in the contracted notation of Burchnall and Chaundy [1, p. 112].

Definitions of generalized Kampé de Fériet's function $F^{1:2}_{1:1}$ of Karlsson [7, p. 265], generalized Lauricella function of n variables of Srivastava and Daoust [23, p. 454] [see, also [22]], $H_C^{(N)}$ of Karlsson [8, p. 37] which itself is a generalisation of Gaussian hypergeometric function ${}_2F_1$ with variables z_1+z_2, $H_C^{(3)}$ of Srivastava [20, p. 100] and K_{16} of Exton [5] are readily obtained form (1) by deletion and or addition of parameters and variables.

In a recent series of papers [10] to [14] we made extensive use of an integral for the product of Whittaker functions $W_{k,m}(x)$ and $M_{k,m}(x)$ evaluated by the author [10, p. 378]

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{1}{2}p)t} W_{k,\mu}(pt) M_{k_1, m_1-\frac{1}{2}}(x_1 t) M_{k_2, m_2-\frac{1}{2}}(x_2 t)\, dt$$

$$= \frac{\Gamma(a+\mu)\,\Gamma(a-\mu)\, x_1^{m_1} x_2^{m_2} p^{\mu+\frac{1}{2}}}{\Gamma(a-k+\frac{1}{2})\,(\sigma)^{a+\mu}}$$

$$F^{(3)}\left[\begin{matrix} a+\mu::a-\mu;-;-:m_1-k_1;\, m_2-k_2;\ \mu-k+\frac{1}{2}; \\ a-k+\frac{1}{2}::-;-;-:\ 2m_1;\ 2m_2;\ \ -; \end{matrix}\ \frac{x_1}{\sigma}, \frac{x_2}{\sigma}, \frac{\sigma-p}{\sigma}\right] \tag{2}$$

where $a = \frac{1}{2} + \lambda + m_1 + m_2$, $\sigma = z + p + \frac{1}{2}(x_1 + x_2)$, $\mathrm{Re}(a + \mu) > 0$, $\mathrm{Re}(2z + p - x_1 - x_2 \pm p \pm x_1 \pm x_2) > 0$,

$$M_{k,m}(x) = x^{\frac{1}{2}+m} e^{-\frac{1}{2}x} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+m-k)_r}{(2m+1)_r} \frac{x^r}{r!} \tag{3}$$

and

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} M_{k,-m}(x), \tag{4}$$

to obtain a number of transformation and reduction formulas of $F^{(3)}$. This paper was motivated by the observation that as an immediate consequence of the integral (2) and (3) if we attempt to generalize (2) in the form

$$\int_0^\infty t^{\lambda-1} e^{-(z+\frac{1}{2}p)t} W_{k,\mu}(pt) \prod_{i=1}^{n} M_{k_i, m_i-\frac{1}{2}}(x_i t)\, dt$$

$$= \frac{\Gamma(A+\mu)\,\Gamma(A-\mu)\, p^{\mu+\frac{1}{2}}}{\Gamma(A-K+\frac{1}{2})\ \delta^{A+\mu}} \prod_{i=1}^{n} x_i^{m_i}$$

$$F_P^{(n+1)}\left(\begin{matrix} A+\mu : A-\mu;\, m_1-k_1; \ldots; m_n-k_n;\ \mu-k+\frac{1}{2}; \\ A-k+\frac{1}{2} :\quad -;\ 2m_1;\ldots;\ 2m_n;\quad -; \end{matrix}\ \frac{x_1}{\delta},\ldots,\frac{x_n}{\delta}, \frac{\delta-p}{\delta}\right) \tag{5}$$

where $A = \frac{1}{2} + \lambda + \sum_{i=1}^{n} m_i$, $\delta = z + p + \frac{1}{2}\sum_{i=1}^{n} x_i$, $\mathrm{Re}(a+\mu) > 0$, $\mathrm{Re}(2z + p - \sum_{i=1}^{n} x_i + p + \sum_{i=1}^{n} x_i) > 0$, then many new and general transformation, reduction, multiplication and summation formulas can be established in terms of $F_P^{(n+1)}$. When these results are specialized, it is found that several known and unknown formulas associated with various classes of hypergeometric functions of one and more variables readily follow. We mention a few known results and mainly recent ones.

This work consists of four sections. In section 2, we give a transformation of $F_P^{(n+1)}$ into a Lauricella's series $F_C^{(n)}$ [6, p. 113]. Section 3 will develop a multiplication formula involving $F_A^{(n+1)}$ and $F^{(2)}$ and section 4 will be devoted to certain summation formulas of $F_P^{(n+1)}$.

In this paper we have omitted many of the more obvious connections or relations of our $F_P^{(n+1)}$ to Lauricella, Kampé de Fériet, Appell functions or recent generalizations of these functions. [cf., e.g., [5], [7], [8], [19], [20], [22] and [23]]. Their inclusion would have made this paper unreasonably long. For example by using (4), (5) and [4, p. 216 (14)] it is easily seen that $F_P^{(n+1)}$ is also connected to F_A by a relation

$$F_P^{(n+1)}\left(\begin{matrix} a+\mu & : a-\mu & ; d_1, \ldots, d_n; c+\mu & ; \\ a+c & : & -; e_1, \ldots, e_n; & -; \end{matrix} \; x_1, \ldots, x_n, 1-y\right)$$

$$= \frac{\Gamma(a+c)}{\Gamma(a+\mu)\,\Gamma(a-\mu)}\, y^{-\mu} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)\,\Gamma(a+\mu)}{\Gamma(c-\mu)}\, y^{\mu}$$

$$F_A^{(n+1)}(a+\mu, d_1, \ldots, d_n, c+\mu\, ; e_1, \ldots, e_n, 2\mu+1; x_1, \ldots, x_n, y) \tag{6}$$

II. Transformation Formulas

Setting $k_1=k_2=\ldots=k_n=k=0$ in (3) and using

$$W_{0,\mu}(x) = \sqrt{\frac{x}{\pi}}\, k_\mu\left(\tfrac{1}{2}x\right),$$

$$M_{0,\mu}(x) = 2^{2\mu}\,\Gamma(\mu+1)\, x^{1/2}\, I_\mu\left(\tfrac{1}{2}x\right)$$

where $K_\upsilon(x)$ and $I_\upsilon(x)$ are modified Bessel functions and [16, p. 162 (6)], we get a transformation of $F_P^{(n+1)}$ into Lauricella's F_C [6, p. 113] in the form

$$F_P^{(n+1)}\left(\begin{matrix} \alpha+\mu+\frac{1}{2}: & \alpha-\mu+\frac{1}{2}; & \gamma_1-\frac{1}{2}, \ldots, \gamma_n-\frac{1}{2}; & \mu+\frac{1}{2}; \\ \alpha+1: & -\,; & 2\gamma_1-1, \ldots, 2\gamma_n-1\,; & -; \end{matrix} \; \frac{2x_1}{1+X}, \ldots, \frac{2x_n}{1+X}, \frac{X-1}{1+X}\right)$$

$$= \frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\,\Gamma(\frac{\alpha}{2}+1)}{2^{\mu+\frac{1}{2}}\,\Gamma\frac{1}{2}(\alpha+\mu+3/2)\;\Gamma\frac{1}{2}(\alpha-\mu+3/2)}\,(1+x)^{\alpha+\mu+\frac{1}{2}}$$

$$F_C^{(n)}\left(\tfrac{1}{2}(\alpha-\mu+\tfrac{1}{2}), \tfrac{1}{2}(\alpha+\mu+\tfrac{1}{2}), \gamma_1, \ldots, \gamma_n; x_1^2, \ldots, x_n^2\right) \tag{7}$$

where $X=x_1+\ldots+x_n$.

By a simple change of variables, (7) can be written in its equivalent form

$$F_P^{(n+1)}\left(\begin{matrix} \alpha+\mu+\frac{1}{2}: & \alpha-\mu+\frac{1}{2}; & \gamma_1-\frac{1}{2}, \ldots, \gamma_n-\frac{1}{2}\,; & \mu+\frac{1}{2}\,; \\ \alpha+1: & -\,; & 2\gamma_1-1, \ldots, 2\gamma_n-1\,; & -; \end{matrix} \; 2x_1, \ldots, 2x_n, 2X-1\right)$$

$$= \frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\,\Gamma(\frac{\alpha}{2}+1)}{2^{\mu+\frac{1}{2}}\,\Gamma\frac{1}{2}(\alpha+\mu+3/2)\,\Gamma\frac{1}{2}(\alpha-\mu+3/2)}\,(1-X)^{-(\alpha+\mu+\frac{1}{2})}$$

$$F_C^{(n)}\left(\tfrac{1}{2}(\alpha-\mu+\tfrac{1}{2}), \tfrac{1}{2}(\alpha+\mu+\tfrac{1}{2}); \gamma_1,\dots,\gamma_n; \frac{x_1^2}{(1-X)^2},\dots,\frac{x_n^2}{(1-X)^2}\right), \tag{8}$$

where $X=x_1+\dots+x_n$.

For $\mu=-\tfrac{1}{2}$, (7) yields a linear connection of two Lauricella's functions F_C and F_A:

$$F_A^{(n)}\left(\alpha, \gamma_1-\tfrac{1}{2},\dots\gamma_n-\tfrac{1}{2}; 2\gamma_1-1,\dots,2\gamma_n-1; \frac{2x_1}{1+x_1+\dots+x_n},\dots,\frac{2x_n}{1+x_1+\dots+x_n}\right)$$

$$=(1+x_1+\dots+x_n)^{\alpha} F_C^{(n)}\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \gamma_1,\dots,\gamma_n; x_1^2,\dots,x_n^2\right) \tag{9}$$

which was given recently by Srivastava [21, p.39].

For n=3, (7) reduces to

$$F_P^{(4)}\left(\begin{matrix}\alpha+\mu+\tfrac{1}{2}: \alpha-\mu+\tfrac{1}{2}; \gamma_1-\tfrac{1}{2}, \gamma_2-\tfrac{1}{2}, \gamma_3-\tfrac{1}{2}; \mu+\tfrac{1}{2}; \\ \alpha+1 \quad : \quad -; 2\gamma_1-1, 2\gamma_2-1, 2\gamma_3-1; -;\end{matrix} \frac{2x_1}{1+y}, \frac{2x_2}{1+y}, \frac{2x_3}{1+y}, \frac{y-1}{1+y}\right)$$

$$=\frac{\Gamma(\frac{\alpha+1}{2})\,\Gamma(\frac{\alpha}{2}+1)}{2^{\mu+\frac{1}{2}}\,\Gamma\tfrac{1}{2}(\alpha+\mu+3/2)\,\Gamma\tfrac{1}{2}(\alpha-\mu+3/2)}(1+y)^{\alpha+\mu+\frac{1}{2}}$$

$$F_C\left(\tfrac{1}{2}(\alpha-\mu+\tfrac{1}{2}), \tfrac{1}{2}(\alpha+\mu+\tfrac{1}{2}); \gamma_1, \gamma_2, \gamma_3; x_1^2, x_2^2, x_3^2\right), \tag{10}$$

where $Y=x_1+x_2+x_3$.

Now in a recent result of the author [13]

$$F_P^{(4)}\left(\begin{matrix}2a+2b:2b+1;a,b,b;a; \\ a+2b+1: -;2a,2b,2b;-;\end{matrix} \frac{2x_2}{\rho}, \frac{2x_3}{\rho}, \frac{2x_4}{\rho}, \frac{\rho-2x_1}{\rho}\right)$$

$$=\frac{\Gamma(b+\tfrac{1}{2})\,\Gamma(a+2b+1)\,\rho^{2a+2b}\,\theta^{-a-b}}{2^{2a}\,\Gamma(2b+1)\,\Gamma(a+b+\tfrac{1}{2})}$$

$$F_4\left(\frac{a+b}{2}, \frac{a+b+1}{2}, a+\tfrac{1}{2}, b+\tfrac{1}{2}; \frac{4x_1^2x_2^2}{\theta^2}, \frac{4x_3^2x_4^2}{\theta^2}\right), \tag{11}$$

where $\rho=x_1+x_2+x_3+x_4$, $\theta=x_1^2+x_2^2-x_3^2-x_4^2$ and F_4 is Appell's function of fourth kind (3, p.224), if we set $a=\gamma_1-\tfrac{1}{2}=\mu+\tfrac{1}{2}$, $b=\gamma_2-\tfrac{1}{2}=\gamma_3-\tfrac{1}{2}$, $x_1=1$, and compare it with (10), we get a reduction formula of Manocha and Sharma [9, p. 432]

$$F_C(1+\alpha+\beta, 1+\alpha; 1+\beta, 1+\alpha, 1+\alpha; x, y, z) = (1+x-y-z)^{-1-\alpha-\beta}$$

$$F_4\left(\frac{1+\alpha+\beta}{2}, 1+\frac{\alpha+\beta}{2}, 1+\beta, 1+\alpha; \frac{4x}{(1+x-y-z)^2}, \frac{4yz}{(1+x-y-z)^2}\right). \tag{12}$$

One more special case of (7) is worthy of note. For n=1 it yields

$$F^{(2)}\left[\begin{matrix}\alpha+\mu+\frac{1}{2}: \alpha-\mu+\frac{1}{2},\ \beta-\frac{1}{2};\ \mu+\frac{1}{2}; \\ \alpha+1\ :\ 2\beta-1\ ;\ -;\end{matrix}\ \frac{2x}{1+X},\frac{x-1}{1+X}\right]$$

$$=\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\alpha}{2}+1)(1+x)^{\alpha+\mu+\frac{1}{2}}}{2^{\mu+\frac{1}{2}}\ \Gamma\frac{1}{2}(\alpha+\mu+3/2)\Gamma\frac{1}{2}(\alpha-\mu+3/2)}\ {}_2F_1\left(\begin{matrix}\frac{1}{2}(\alpha-\mu+\frac{1}{2}),\frac{1}{2}(\alpha+\mu+\frac{1}{2}) \\ \beta\end{matrix};x^2\right) \tag{13}$$

which for $\mu=-\frac{1}{2}$ gives a known quadratic transformation ot ${}_2F_1$ [3, p. 111 (4)].

III. Multiplication Formalae

Next we consider [2, p. 156]

$$L_{m_1}^{(\alpha_1)}(x_1t)\dots L_{m_n}^{(\alpha_n)}(x_nt)=\sum_{r=0}^{m_1+\dots+m_n} C_r L_r^{(\alpha)}(t) \tag{14}$$

where

$$C_r=\binom{\alpha_1+m_1}{m_1}\dots\binom{\alpha_n+m_n}{m_n}$$

$$F_A^{(n+1)}(\alpha+1,-m_1,\dots,-m_n,-r;\ \alpha_1+1,\dots,\alpha_n+1,\alpha+1;x_1,\dots,x_n,1), \tag{15}$$

$r\geqslant 0$, $L_n^{(\alpha)}$ is Laguerre polynomial [24] and

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_m}{m!}x^{-\frac{1}{2}(\alpha+1)}\ e^{\frac{1}{2}x}\ M_{\frac{\alpha}{2}+m+\frac{1}{2},\frac{\alpha}{2}}(x). \tag{16}$$

Replacing t by yt in (14), multiplying both the sides by

$$t^{\lambda-1}\ e^{-(z+\frac{1}{2}p)t}\ W_{k,\mu}(pt)$$

and integrating with respect to t between the limits 0 and ∞ by using (5), (17) and (2), we get a multiplication formula in the form

$$F_p^{(n+1)}\left(\begin{matrix}a:b;-m_1,\dots,-m_n;c-b; \\ c:-;\alpha_1+1,\dots,\alpha_n+1;-;\end{matrix}\ \frac{x_1y}{z+1},\dots,\frac{x_ny}{z+1},\frac{z}{z+1}\right)$$

$$=\sum_{r=0}^{m_1+\dots+m_n}\frac{(\alpha+1)_r}{r!}F_A^{(n+1)}(\alpha+1,-m_1,\dots,-m_n,-r;\alpha_1+1,\dots,\alpha_n+1,\alpha+1;x_1,\dots,x_n,1)$$

$$F^{(2)}\left[\begin{matrix}a,b:-r;c-b; \\ C:\alpha+1;-;\end{matrix}\ \frac{y}{z+1},\frac{z}{z+1}\right] \tag{18}$$

For c=b and $z\to 0$, (18) reduces to a multiplication formula associated with the Lauricella's hypergeometric function F_A which was obtained earlier by Srivastava [17, p. 67 (6)].

A special case of (18) for n=1 is of interest. We have

$$F^{(2)}\left[\begin{matrix}a:b,-m;c-b; \\ c:\alpha_1+1;-;\end{matrix}\ xy,z\right]$$

$$=\sum_{r=0}^{m}\frac{(\alpha+1)_r}{r!}F_2(\alpha+1,-m,-r,\alpha_1+1,\alpha+1;x,1)F^{(2)}\left[\begin{matrix}a,b:-r;c-b; \\ c:\alpha+1;-;\end{matrix}\ y,z\right], \tag{19}$$

where F_2 is Appell's function of second kind [3, p. 224].

For the generalized Rice polynomial

$$H_n^{(\alpha,\beta)}(k,p,v) = \binom{\alpha+n}{n}\, {}_3F_2\left(\begin{matrix}-n, \alpha+\beta+n+1, k\\ \alpha+1, \quad p\end{matrix}; v\right), \quad n=0, 1, 2, \ldots$$

which, when $\alpha = \beta = 0$ reduces to the original form [15, p. 108]

$$H_n(k, p, v) = {}_3F_2\left(\begin{matrix}-n, n+1, k\\ 1, p\end{matrix}; v\right),$$

(19) yields when $z \to 0$, a multiplication formula

$$H_m^{(\delta,\beta)}(a, c, xy) = \sum_{r=0}^{m} \binom{m}{r}\binom{\delta+m}{m}\binom{\delta+r}{r}^{-1} x^r$$

$${}_2F_1\left(\begin{matrix}r-m, \alpha+r+1\\ \delta+r+1\end{matrix}; x\right) H_r^{(\alpha,\beta+m-r)}(a, c, y) \tag{20}$$

with the help of a result of Srivastava [17, p. 67]

$$F_2(\beta+1, -m, -n, \alpha+1, \beta+1; \lambda, 1)$$

$$= \frac{m!\ \lambda^n}{(m-n)!(\alpha+1)_n}\ {}_2F_1\left(\begin{matrix}n-m, \quad \beta+n+1\\ \alpha+n+1\end{matrix}; \lambda\right). \tag{21}$$

Since

$$H_n^{(\alpha,\beta)}(v,v,x) = P_n^{(\alpha,\beta)}(1-2x)$$

where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial [24, p. 68], formula (20) can be further specialized by taking a=c to get a multiplication formula for Jacobi polynomial.

IV. Summation Formulae

In a result associated with the product of two Laguerre polynomials with different arguments which was given by Srivastava [18, p. 6]

$$L_m^{(\alpha)}(x)\, L_m^{(\beta)}(y) = \binom{m+\alpha}{m}\binom{m+\beta}{m}\binom{m+\gamma}{m}^{-1}$$

$$\sum_{r+s\leq m} \frac{(\gamma+1)_r\,(\beta-\gamma)_s}{r!s!\,(\alpha+1)_r(\beta+1)_{r+s}}\, x^r y^{r+s}\, \xi_{rs}\, L_{m-r-s}^{(\gamma+2r+s)}(x+y), \tag{22}$$

where

$$\xi_{rs} = {}_3F_2\left(\begin{matrix}-s, \alpha-\gamma, \beta-r-s\\ \alpha+r+1, \gamma-\beta-s+1\end{matrix}; -\frac{x}{y}\right), \tag{23}$$

replacing x by xt, y by yt, multiplying both the sides by

$$t^{\lambda}\, e^{-zt}\, W_{k,\mu}(pt) \prod_{i=1}^{3} M_{k_i, m_i - \frac{1}{2}}(x_i t),$$

integrating with respect to t between the limits 0 and ∞ by using (5) and making suitable adjustment in parameters, we obtain a summation formula for the product of hypergeometric functions in the form

$$F_p^{(n+1)}\left(\begin{matrix}a: b; -m, -m, d_3,\dots, d_n; c\text{-}b; \\ c: -; \alpha+1, \beta+1, e_3,\dots,e_n; -;\end{matrix}\ x, y, x_3,\dots,x_n, z\right)$$

$$= \binom{m+\gamma}{m}^{-1} \sum_{r+s\leqslant m} \frac{(\gamma+1)_r(\beta-\gamma)_r(\gamma+2r+s+1)_{m-r-s}(a)_{2r+s}(b)_{2r+s}\, x^r y^{r+s}}{r!\, s!\, (\alpha+1)_r (\beta+1)_{r+s}(m-r-s)!\,(c)_{2r+s}}$$

$$\xi_{rs} F_p^{(n)}\left(\begin{matrix}a+2r+s: b+2r+s; r+s-m, d_3,\dots,d_n; c\text{-}b; \\ c+2r+s: \quad -; \gamma+2r+s+1, e_3,\dots,e_n; -;\end{matrix}\ x+y, x_3,\dots,x_n, z\right) \tag{24}$$

where ξ_{rs} is given by (23).

For b=c, (24) reduces to

$$F_A^{(n)}(a, -m, -m, d_3,\dots, d_n;\ \alpha+1,\ \beta+1, e_3,\dots,e_n; x, y, x_3,\dots,x_n)$$

$$= \binom{m+\gamma}{m}^{-1} \sum_{r+s\leqslant m} \frac{(\gamma+1)_r(\beta-\gamma)_r(\gamma+2r+s+1)_{m-r-s}(a)_{2r+s}}{r!\, s!\, (m-r-s)!\,(\alpha+1)_r(\beta+1)_{r+s}}\, x^r y^{r+s}\ \xi_{rs}$$

$$F_A^{(n-1)}(a+2r+s, r+s-m, d_3,\dots,d_n; \gamma+2r+s+1, e_3,\dots,e_n; x+y, x_3,\dots, x_n) \tag{25}$$

On setting $x_3,\dots,x_n=0$, our formula (24) would assume the form

$$F^{(3)}\left[\begin{matrix}a:: b; -; -: -m; -m; c\text{-}b; \\ c:: -; -; -: \alpha+1; \beta+1; -;\end{matrix}\ x,y,z\right] = \binom{m+\gamma}{m}^{-1} \sum_{r+s\leqslant m}$$

$$\frac{(\gamma+1)_r(\beta-\gamma)_r(\gamma+2r+s+1)_{m-r-s}(a)_{2r+s}(b)_{2r+s}}{r!\, s!\, (m-r-s)!\,(\alpha+1)_r(\beta+1)_{r+s}(c)_{2r+s}}\, x^r y^{r+s}\ \xi_{rs}$$

$$F^{(2)}\left[\begin{matrix}a+2r+s: b+2r+s, r+s-m; c\text{-}b; \\ c+2r+s: \gamma+2r+s+1 \quad ; -;\end{matrix}\ x+y, z\right]. \tag{26}$$

Relationships [10, p.378(2) and (3)]

$$F^{(3)}\left[\begin{matrix}a:: b; -; -: g\text{-}f; d; c\text{-}b; \\ c:: -; -: -: g \ ; e; -;\end{matrix}\ \frac{x}{x+1}, \frac{y}{x+1}, \frac{x}{x+1}\right]$$

$$= (1+x)^a\ F^{(2)}\left[\begin{matrix}a, b: d; f; \\ c: e; g;\end{matrix}\ y, -x\right], \tag{27}$$

$$F^{(2)}\left[\begin{matrix}a: b; e; c\text{-}b; \\ c: d \ ; -;\end{matrix}\ \frac{x}{z+1}, \frac{z}{z+1}\right] = (z+1)^a F^{(2)}\left[\begin{matrix}a: b: -; d\text{-}e; \\ c: -; d;\end{matrix}\ x\text{-}y, -x\right] \tag{28}$$

would enable us to write a special case of (26), when z=x, in the form

$$F^{(2)}\left[\begin{matrix}a, b: -m; \ \alpha+m+1; \\ c: \beta+1; \ \alpha+1 \quad ;\end{matrix}\ y, -x\right] = \binom{m+\gamma}{m}^{-1} \sum_{r+s\leqslant m}$$

$$\frac{(\gamma+1)_r(\beta-\gamma)_r(\gamma+2r+s+1)_{m-r-s}(a)_{2r+s}(b)_{2r+s}}{r!s!(m-r-s)!(\alpha+1)_r(\beta+1)_{r+s}(c)_{2r+s}}\ x^r y^{r+s}\ \xi_{rs}$$

$$F^{(2)}\left[\begin{matrix} a+2r+s,\ b+2r+s: -;\ \gamma+r+m+1; \\ c+2r+s\quad : -;\ \gamma+2r+s+1\ ; \end{matrix}\ y, -(x+y)\right] \tag{29}$$

which is obtained by the author in [14].

Similarly on using a result [18, p. 7]

$$L_m^{(\alpha)}(x)\,L_m^{(\beta-m)}(y) = \binom{m+\alpha}{m} \sum_{r+s\leq m} \frac{(-\beta)_r\, x^{r+s}\, y^s}{r!\,s!\,(\alpha+1)_{r+s}}\ L_{m-r-s}^{(\beta-m+s)}(y) \tag{30}$$

in place of (22) would yield another summation formula

$$F_p^{(n+1)}\left(\begin{matrix} a:\ b;\ -m,\ -m,\ d_3,\dots,d_n;\ c-b; \\ c:\ -;\ \alpha+1,\ \beta-m+1,\ e_3,\dots,e_n;\ -\ ; \end{matrix}\ x, y, x_3,\dots, x_n, z\right)$$

$$= \binom{\beta}{m}^{-1} \sum_{r+s\leq m} \frac{(-\beta)_r\,(a)_{2r+s}\,(b)_{2r+s}}{r!\,s!\,(\alpha+1)_{r+s}\,(c)_{2r+s}}\ \binom{\beta-r}{m-r-s}\ x^{r+s}\,y^s$$

$$F_p^{(n)}\left(\begin{matrix} a+2r+s:\ b+2r+s;\ r+s-m,\ d_3,\dots,d_n;\ c-b; \\ c+2r+s\ :\ -\ ;\ \beta-m+s+1,\ e_3,\dots,e_n;\ -\ ; \end{matrix}\ y, x_3,\dots,x_n, z\right) \tag{31}$$

Formula (31) contains as a special case a recent result of the author [14]

$$F^{(2)}\left[\begin{matrix} a,b:\ -m;\ \alpha+m+1; \\ c\ :\ \beta+1;\ \alpha+1\ ; \end{matrix}\ y, -x\right] = \binom{\beta}{m}^{-1} \sum_{r+s\leq m}$$

$$\frac{(-\beta-m)_r\,(\beta+s+1)_{m-r-s}\,(a)_{r+2s}\,(b)_{r+2s}}{r!\,s!\,(m-r-s)!\,(\alpha+1)_{r+s}\,(c)_{r+2s}}\ x^{r+s}\,y^s$$

$$F^{(2)}\left[\begin{matrix} a+2s+r,\ b+2s+r: -;\ \beta+1+n-r; \\ c+2s+r\quad : -;\ \beta+s+1\ ; \end{matrix}\ y-x, -y\right], \tag{32}$$

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