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Comment on "Some exact solutions of convection-diffusion and diffusion equations" by J. R. Philip

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In a recent paper, *Philip* [1994] proposed the solutions of the advective-dispersion equation (ADE) in which the dispersion coefficient (or diffusivity) is proportional to P^n, with P the Peclet number, for divergently steady radial flow in porous media. Exact solutions for instantaneous and continuous point source were developed when $n = 2$ in two- and three-dimensional radial flow fields. With the case of $n = 1$ in two-dimensional radial coordinates, *Philip* [1994] pointed out that the ADE used by *Hoopes and Harleman* [1967] is not correct. Consequently, the equation used by *Tang and Babu* [1979] and *Hsieh* [1986] is also erroneous. This assertion aroused our curiosity. After a careful check and mathematical derivations, we find that when the molecular diffusion is neglected the ADE used by *Hoopes and Harleman* [1967] is exactly the same as the one used by *Philip* [1994].

If the effect of molecular diffusion is neglected, the ADE in the radially diverging flow field given by *Hoopes and Harleman* [1967, p. 3599, equation (15)] may be expressed as

$$\frac{\partial C}{\partial t} + \frac{A}{r}\frac{\partial C}{\partial r} = \alpha \frac{A}{r}\frac{\partial^2 C}{\partial r^2}, \tag{1}$$

where C is the concentration of tracer; t is the time; r is the radial distance; A is a constant equal to $Q/(2\pi\phi b)$; Q is the injection rate of tracer; ϕ is the porosity; b is the aquifer thickness; and α is called dispersivity and is a constant.

The ADE in the two-dimensional radial flow field given by *Philip* [1994, p. 3545, equation (1)]

$$\frac{\partial \theta_*}{\partial t_*} + \frac{v_1}{r_*}\frac{\partial \theta_*}{\partial r_*} = \frac{1}{r_*}\frac{\partial}{\partial r_*}\left(Dr_* \frac{\partial \theta_*}{\partial r_*}\right), \tag{2}$$

where θ_* is the dimensionless concentration; t_* is the physical time; r_* is the physical radial coordinate; and D is the diffusivity. The term v_1/r_*, where v_1 is a constant, represents the velocity V. The diffusivity is usually considered to vary with the first power of velocity; thus D can be written as $D = \alpha V$. Note that the term Dr_* on the right side of (2) may be expressed as $Dr_* = \alpha V r_* = \alpha v_1$ and, thus, is a constant. Therefore Dr_* can be moved out of the parentheses on the right side of (2). Accordingly, (2) can be expressed as

$$\frac{\partial \theta_*}{\partial t_*} + \frac{v_1}{r_*}\frac{\partial \theta_*}{\partial r_*} = \frac{\alpha v_1}{r_*}\frac{\partial^2 \theta_*}{\partial r_*^2}. \tag{3}$$

If one lets $v_1 = A$, drops the asterisk subscripts in (3), and normalizes the concentration of (1), then equations (1) and (3) are essentially identical. Let C_0 represent the concentration of the injected tracer. Three dimensionless variables are introduced as

$$\theta = C/C_0 \tag{4}$$

$$\tau = \tfrac{1}{2}(At/\alpha^2) \tag{5}$$

$$\sigma = \tfrac{1}{4}(r/\alpha)^2 \tag{6}$$

On the basis of (5) and (6), the following partial derivatives may be derived:

$$\frac{\partial}{\partial t} = \frac{A}{2\alpha^2}\frac{\partial}{\partial \tau} \tag{7}$$

$$\frac{\partial}{\partial r} = \frac{r}{2\alpha^2}\frac{\partial}{\partial \sigma} \tag{8}$$

$$\frac{\partial^2}{\partial r^2} = \frac{1}{2\alpha^2}\frac{\partial}{\partial \sigma} + \frac{r^2}{4\alpha^4}\frac{\partial^2}{\partial \sigma^2} \tag{9}$$

After substituting (7)–(9) into (1), the result divided by C_0 gives the dimensionless form of (1) as

$$\frac{A}{2\alpha^2}\frac{\partial \theta}{\partial \tau} + \frac{A}{2\alpha^2}\frac{\partial \theta}{\partial \sigma} = \frac{A}{2\alpha r}\frac{\partial \theta}{\partial \sigma} + \frac{Ar}{4\alpha^3}\frac{\partial^2 \theta}{\partial \sigma^2}. \tag{10}$$

Dividing (10) by $A/(2\alpha^2)$ and letting $r/\alpha = 2\sigma^{1/2}$ from (6) yields

$$\frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial \sigma} = \frac{1}{2}\sigma^{-1/2}\frac{\partial \theta}{\partial \sigma} + \sigma^{1/2}\frac{\partial^2 \theta}{\partial \sigma^2} = \frac{\partial}{\partial \sigma}\left(\sigma^{1/2}\frac{\partial \theta}{\partial \sigma}\right). \tag{11}$$

It is worth noting that (11) is equal to *Philip*'s [1994] ADE in dimensionless form when $n = 1$ [*Philip*, 1994, p. 3546, equation (6)].

In conclusion, we have shown that the ADE of *Hoopes and Harleman* [1967], either in dimensional form when the molecular diffusion is neglected or in dimensionless form, is equivalent to the one given in *Philip*'s [1994] paper when $n = 1$. Accordingly, the ADE in radial two-dimensional flow used by both *Tang and Babu* [1979] and *Hsieh* [1986] is not in error.

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