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Production yield measure for multiple characteristics processes based on S_{pk}^T under multiple samples

W. L. Pearn · Ching-Ho Yen · Dong-Yuh Yang

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Abstract Process yield is an important criterion used in the manufacturing industry for measuring process performance. Methods for measuring yield for processes with single characteristic have been investigated extensively. However, methods for measuring yield for processes with multiple characteristics have been comparatively neglected. Chen et al. (Qual Reliab Eng Int 19:101–110, 2003) proposed a measurement formula called S_{pk}^T, which provides an exact measure of the overall process yield, for processes with multiple characteristics. In this paper, we considered the natural estimator of S_{pk}^T under multiple samples, and derived the asymptotic distribution for the estimator. In addition, a comparison between the SB (standard bootstrap) and the proposed method based on the lower confidence bound is executed. Generally, the result indicates that the proposed approach is more reliable than the standard bootstrap method.

Keywords Asymptotic distribution · Multiple characteristics · Process yield · Standard bootstrap

W. L. Pearn
Department of Industrial Engineering and Management,
National Chiao Tung University, Hsinchu, Taiwan

C.-H. Yen
Department of Industrial Engineering and Management Information,
Huafan University, Taipei, Taiwan

D.-Y. Yang (✉)
Institute of Information Science and Management,
National Taipei College of Business, Taipei, Taiwan
e-mail: yangdy@webmail.ntcb.edu.tw

1 Introduction

Process capability indices, which establish the relationship between the actual process performance and the manufacturing specifications, have been the focus in quality assurance and capability analysis for the past fifteen years. The capability indices, C_p, C_{pk} and C_{pm}, are widely used in the manufacturing industry to evaluate process performance for cases with single quality characteristic. On the other hand, the index S_{pk} Boyles (1994) is introduced to establish the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on the process yield. Capability calculations for processes with single characteristic have been investigated extensively. Kotz and Johnson (2002) presented a thorough review for the development of process capability indices from 1992 to 2000.

Process yield has been the most basic and common criterion used in the manufacturing industry for measuring process performance. Process yield is currently defined as the percentage of processed product unit passing inspection. That is, the product characteristic must fall within the manufacturing tolerance. In many cases, a benchmark of minimum 99.73% for assessing the process is suggested. The relationships between the process yield and the process capability indices have been discussed extensively for processes with single characteristics, but comparatively neglected for processes with multiple characteristics. In this paper, the asymptotic distribution for the estimator $\hat{S}_{pk}^{T*}$ of S_{pk}^{T} under multiple samples is derived. According to this result, hypothesis testing and confidence interval can be executed. The proposed method can be used to convince whether a process meets the yield requirement. Also a comparison between the SB and the proposed method based on the lower confidence bound is executed. The coverage percentage is used to evaluate the performance of these two methods. Finally, an application example is chosen to illustrate the proposed methodology.

2 The yield index S_{pk}^{T} for multiple characteristics

For normally distributed processes with a single characteristic, the index S_{pk} (see Boyles 1994) is used to establish the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on the process yield, defined as

$$\begin{aligned} S_{pk} &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu}{\sigma}\right)+\frac{1}{2}\Phi\left(\frac{\mu-LSL}{\sigma}\right)\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\} \end{aligned} \tag{1}$$

where $C_{dr} = (\mu - m)/d$, $C_{dp} = \sigma/d$, $m = (USL + LSL)/2$ and $d = (USL - LSL)/2$. It provides an exact measure of process yield. If $S_{pk} = c$, then the process yield can be expressed as Yield $= 2\Phi(3c) - 1$. Obviously, there is a one-to-one correspondence between S_{pk} and the process yield. Considering processes with multiple characteristics (assuming characteristics are mutually independent), Chen et al. (2003)

Table 1 Some S_{pk}^T values and the corresponding nonconformities

S_{pk}^T	Yield	PPM
1.00	0.9973002039	2699.796
1.33	0.9999339267	66.073
1.50	0.9999932047	6.795
1.67	0.9999994557	0.544
2.00	0.9999999980	0.002

defined the yield index as

$$S_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3S_{pkj}) - 1\right) + 1\right]/2\right\}, \tag{2}$$

where S_{pkj} denote the S_{pk} value of the jth characteristic for $j = 1, 2, \ldots, v$, and v is the number of characteristics. This index provides an exact measure of the overall process yield when the characteristics are mutually independent. A one-to one correspondence relationship between the index S_{pk}^T and the overall process yield P can be established as

$$P = \prod_{j=1}^{v} P_j = \prod_{j=1}^{v}\left[2\Phi(3S_{pkj}) - 1\right] = 2\Phi\left(3S_{pk}^T\right) - 1. \tag{3}$$

Hence, the new index S_{pk}^T provides an exact measure of the overall process yield. For example, if $S_{pk}^T = 1$, then the entire process yield would be exactly 99.73%. Table 1 display various commonly used capability requirements and the corresponding overall process yield.

For a process with v characteristics, if the requirement for the overall process capability is $S_{pk}^T \geq c_0$, a sufficient condition for the requirement to each single characteristic can be obtained by

$$S_{pkj} \geq \frac{1}{3}\Phi^{-1}\left(\frac{\sqrt[v]{2\Phi(3c_0) - 1} + 1}{2}\right), \quad j = 1, 2, \ldots, v. \tag{4}$$

Table 2a–e display the low bound on S_{pkj} for different S_{pk}^T, respectively. In order to calculate the index value, sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments due to sampling errors. The approach is unreliable by simply looking at the calculated values of the estimated index and then making a conclusion on whether the given process is capable since the sampling errors have been ignored. In next section, the statistical properties of the estimator of this index S_{pk}^T under multiple samples will be investigated.

Table 2 Capability zones for multiple characteristics

Characteristics number (v)	Lower bound for S_{pkj}
a ($S_{pk}^T = 1$)	
1	1.000
2	1.068
3	1.107
4	1.133
5	1.153
6	1.170
7	1.183
8	1.195
9	1.205
10	1.214
b ($S_{pk}^T = 1.33$)	
1	1.333
2	1.384
3	1.414
4	1.436
5	1.452
6	1.465
7	1.476
8	1.486
9	1.495
10	1.502
c ($S_{pk}^T = 1.5$)	
1	1.500
2	1.548
3	1.576
4	1.595
5	1.610
6	1.622
7	1.632
8	1.641
9	1.649
10	1.656
d ($S_{pk}^T = 1.67$)	
1	1.677
2	1.714
3	1.739
4	1.757
5	1.770
6	1.781
7	1.791

Table 2 continued

Characteristics number (v)	Lower bound for S_{pkj}
8	1.799
9	1.806
10	1.812
e=(S_{pk}^T = 2)	
1	2.000
2	2.037
3	2.059
4	2.074
5	2.085
6	2.095
7	2.103
8	2.110
9	2.116
10	2.121

3 Estimating S_{pk}^T under multiple samples

For the case when the studied v characteristics of the process are normally distributed and we have m multiple samples with the sample size of n. Let x_{ijk}, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, v$; $k = 1, 2, \ldots, n$, be the j characteristic value of $m \times n$ samples with mean μ_j and variance σ_j^2. Assume that the process is in statistical control during the time period that the multiple samples are taken. Consider the process is monitored using a $\overline{X}$-chart together with a S-chart. Then, for each multiple sample, $\overline{x}_{ij}$ and s_{ij}^2 denote the sample mean and sample variance, respectively, of the ith sample and jth variable, and N denote the total number of observations of variable j, i.e.

$$\overline{x}_{ij} = \frac{1}{n}\sum_{k=1}^{n} x_{ijk}, \quad s_{ij}^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(x_{ijk} - \overline{x}_{ij}\right)^2, \quad N = \sum_{i=1}^{m} n = mn.$$

As an estimator of μ_j, we use the sample mean of jth variable, i.e.

$$\hat{\mu}_j = \overline{x}_j = \frac{1}{m}\sum_{i=1}^{m}\overline{x}_{ij} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} x_{ijk}.$$

Also the pooled variance estimator of σ_j^2 is defined as

$$\hat{\sigma}_j^2 = s_j^2 = \frac{1}{mn}\sum_{i=1}^{m}(n-1)s_{ij}^2 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(x_{ijk} - \overline{x}_{ij}\right)^2.$$

To estimate the yield measurement index S_{pk}^T under multiple samples, we consider the following natural estimator $\hat{S}_{pk}^{T*}$, expressed as

$$\hat{S}_{pk}^{T*} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi\left(3\hat{S}_{pkj}^{*}\right)-1\right)+1\right]\Big/ 2\right\} \tag{5}$$

where $\hat{S}_{pkj}^{*}$ denotes the estimator of S_{pkj} under multiple samples, defined as

$$\begin{aligned}\hat{S}_{pkj}^{*} &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL_j-\overline{x}_j}{s_j}\right)+\frac{1}{2}\Phi\left(\frac{\overline{x}_j-LSL_j}{s_j}\right)\right\}\\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-\hat{C}_{drj}}{\hat{C}_{dpj}}\right)+\frac{1}{2}\Phi\left(\frac{1+\hat{C}_{drj}}{\hat{C}_{dpj}}\right)\right\}\end{aligned} \tag{6}$$

$$\hat{a}_j = \frac{1}{\sqrt{2}}\left\{\frac{\left(1-\hat{C}_{drj}\right)}{\hat{C}_{dpj}}\phi\left(\frac{1-\hat{C}_{drj}}{\hat{C}_{dpj}}\right)+\frac{\left(1+\hat{C}_{drj}\right)}{\hat{C}_{dpj}}\phi\left(\frac{1+\hat{C}_{drj}}{\hat{C}_{dpj}}\right)\right\}, \tag{7}$$

$$\hat{b}_j = \phi\left(\frac{1-\hat{C}_{drj}}{\hat{C}_{dpj}}\right)-\phi\left(\frac{1+\hat{C}_{drj}}{\hat{C}_{dpj}}\right). \tag{8}$$

From the result of Pearn and Cheng (2007), the distribution of $\hat{S}_{pkj}^{*}$ can be written as

$$\hat{S}_{pkj}^{*} \sim N\left(S_{pkj},\ \left(a_j^2+b_j^2\right)\Big/ 36mn(\phi(3S_{pkj}))^2\right),\ j=1,2,\ldots,v.$$

Applying the first-order expansion of v-variate Taylor,

$$f(X) = f(X_0)+\sum_{j=1}^{v}\frac{\partial f(X_0)}{\partial x_i}(x_i-x_{i0}),$$

where $X=(x_1,x_2,\ldots,x_v)$.

The natural estimator $\hat{S}_{pk}^{T*}$ can expressed as

$$\begin{aligned}\hat{S}_{pk}^{T*} &= \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi\left(3\hat{S}_{pkj}^{*}\right)-1\right)+1\right]\Big/ 2\right\}\\ &= f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*},\ldots,\hat{S}_{pkv}^{*}\right)\\ &= f\left(S_{pk1}\ , S_{pk2},\ldots,S_{pkv}\right)+\frac{\partial f\left(S_{pk1}\ S_{pk2},\ldots,S_{pkv}\right)}{\partial\hat{S}_{pk1}^{*}}\left(\hat{S}_{pk1}^{*}-S_{pk1}\right)\\ &\quad+\frac{\partial f\left(S_{pk1}\ , S_{pk2},\ldots,S_{pkv}\right)}{\partial\hat{S}_{pk2}^{*}}\left(\hat{S}_{pk2}^{*}-S_{pk2}\right)\\ &\quad+\cdots\frac{\partial f\left(S_{pk1}\ , S_{pk2},\ldots,S_{pkv}\right)}{\partial\hat{S}_{pkv}^{*}}\left(\hat{S}_{pkv}^{*}-S_{pkv}\right)\end{aligned}$$

We can obtain

$$E(\hat{S}_{pk}^{T*}) = S_{pk}^T,$$

$$\text{Var}\left(\hat{S}_{pk}^{T*}\right) = \frac{1}{36mn\left(\phi\left(3S_{pk}^T\right)\right)^2}\left(\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}\left(2\Phi\left(3S_{pki}\right)-1\right)^2}{\left(2\Phi\left(3S_{pkj}\right)-1\right)^2}\right]\right\}\right).$$

(See Corollary 1 in Appendix).

Since $\hat{S}_{pk}^{T*}$ is linear combination of $\hat{S}_{pk1}^*$, $\hat{S}_{pk2}^*$, ... and $\hat{S}_{pkv}^*$, we know $\hat{S}_{pk}^{T*}$ is from normal distribution. So $\hat{S}_{pk}^{T*}$ has the asymptotic normal distribution with the mean S_{pk}^T and variance

$$\frac{1}{36mn\left(\phi\left(3S_{pk}^T\right)\right)^2}\left(\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi\left(3S_{pkj}\right)-1\right)^2}\right]\right\}\right).$$

That is, the asymptotic distribution of $\hat{S}_{pk}^{T*}$ is

$$\hat{S}_{pk}^{T*} \sim N\left(S_{pk}^T, \frac{1}{36mn\left(\phi\left(3S_{pk}^T\right)\right)^2}\left(\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right). \tag{9}$$

When $v = 1$, the asymptotic distribution of $\hat{S}_{pk}^{T*}$ can be expressed as

$$\hat{S}_{pk}^* \sim N\left(S_{pk}, \frac{1}{36mn\left(\phi\left(3S_{pk}\right)\right)^2}\left(a^2+b^2\right)\right),$$

which is equivalent to that derived from Pearn and Cheng (2007).

4 Inference based on $\hat{S}_{pk}^{T*}$

4.1 Hypothesis testing and confidence interval for S_{pk}^T

From Eq. (9), it shows that $\hat{S}_{pk}^{T*}$ is an asymptotic unbiased estimator of S_{pk}^T. Also, according the asymptotic distribution of $\hat{S}_{pk}^{T*}$ given in Eq. (9), hypothesis testing and a confidence interval for S_{pk}^T can be constructed. To test whether a given process is capable, we may consider the following statistical hypothesis testing:

$$H_0 : S_{pk}^T \le c_0 \text{(Process is not capable.)}$$
$$H_1 : S_{pk}^T > c_0 \text{(Process is capable.)}$$

where c_0 is the standard minimal criteria for S_{pk}^T. The test can be executed by using the testing statistic,

$$T = \frac{6\left(\hat{S}_{pk}^{T*} - c_0\right)\sqrt{mn}\phi\left(3\hat{S}_{pk}^{T*}\right)}{k(v)}. \tag{10}$$

where $k(v) = \sqrt{\sum_{j=1}^{v}\left\{\hat{a}_j^2 + \hat{b}_j^2\left[\frac{\prod_{i=1}^{v}\left(2\Phi\left(3\hat{S}_{pki}^{*}\right)-1\right)^2}{\left(2\Phi\left(3\hat{S}_{pkj}^{*}\right)-1\right)^2}\right]\right\}}$

The null hypothesis H_0 is rejected at α level if $T > Z_\alpha$, where Z_α is the upper $100\alpha\%$ point of the standard normal distribution.

We are able to obtain the approximate $100(1-\alpha)\%$ lower confidence bound for S_{pk}^T which can be expressed as

$$\hat{S}_{pk}^{T*} - \frac{k(v)}{6\sqrt{mn}\phi\left(3\hat{S}_{pk}^{T*}\right)}Z_\alpha, \tag{11}$$

where $\hat{S}_{pk}^{T*}$, $\hat{S}_{pki}^{*}$, $\hat{a}_j^2$ and $\hat{b}_j^2$ are given in (5), (6), (7), and (8), respectively.

4.2 Sample size required for the normal approximation to converge

We further consider how many sample size n should be taken to ensure that the sampling estimator $\hat{S}_{pk}^{T*}$ is closed to the real S_{pk}^T with a designated accuracy ε. Tables 3 and 4 display the sample size required for the normal approximation to converge to the real S_{pk}^T with a designated accuracy ε less than 0.1, 0.09,…, 0.01. The derivation is shown as follows:

$$P\left\{\left|\hat{S}_{pk}^{T*} - S_{pk}^T\right| \le \varepsilon\right\} \ge 1-\alpha \Rightarrow P\left\{\frac{\hat{S}_{pk}^{T*} - S_{pk}^T}{\sqrt{\mathrm{Var}\left(\hat{S}_{pk}^{T*}\right)}} \le \frac{\varepsilon}{\sqrt{\mathrm{Var}\left(\hat{S}_{pk}^{T*}\right)}}\right\}$$

$$\ge 1-\frac{\alpha}{2} \Rightarrow \frac{\varepsilon}{\sqrt{\mathrm{Var}\left(\hat{S}_{pk}^{T*}\right)}} \ge \Phi^{-1}\left(1-\alpha/2\right)$$

$$\Rightarrow \varepsilon^2 \ge \left[\Phi^{-1}(1-\alpha/2)\right]^2 \frac{1}{36mn\left(\phi\left(3S_{pk}^T\right)\right)^2}$$

$$\times\left(\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)$$

$$\Rightarrow mn \ge \frac{\left[\Phi^{-1}(1-\alpha/2)\right]^2}{36\left(\phi\left(3S_{pk}^T\right)\right)^2\varepsilon^2}\left(\sum_{j=1}^{v}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)$$

Table 3 Sample sizes required for the normal approximation to converge with $\alpha = 0.05$ for $\upsilon = 2$

m	S_{pk}^T	ε									
		0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
	1.00	42	51	65	84	115	165	257	457	1,028	4,109
	1.33	66	81	103	134	183	263	410	729	1,640	6,559
3	1.50	83	102	129	168	229	329	514	914	2,056	8,224
	1.67	99	122	154	202	274	394	616	1095	2,463	9,850
	2.00	139	172	217	283	385	555	866	1540	3,464	1,3853
	1.00	21	26	33	42	58	83	129	229	514	2,055
	1.33	33	41	52	67	92	132	205	365	820	3,280
6	1.50	42	51	65	84	115	165	257	457	1,028	4,112
	1.67	50	61	77	101	137	197	308	548	1,232	4,925
	2.00	70	86	109	142	193	278	433	770	1,732	6,927
	1.00	14	17	22	28	39	55	86	153	343	1,370
	1.33	22	27	35	45	61	88	137	243	547	2,187
9	1.50	28	34	43	56	77	110	172	305	686	2,742
	1.67	33	41	52	68	92	132	206	365	821	3,284
	2.00	47	58	73	95	129	185	289	514	1,155	4,618
	1.00	11	13	17	21	29	42	65	115	257	1,028
	1.33	17	21	26	34	46	66	103	183	410	1,640
12	1.50	21	26	33	42	58	83	129	229	514	2,056
	1.67	25	31	39	51	69	99	154	274	616	2,463
	2.00	35	43	55	71	97	139	217	385	866	3,464
	1.00	9	11	13	17	23	33	52	92	206	822
	1.33	14	17	21	27	37	53	82	146	328	1,312
15	1.50	17	21	26	34	46	66	103	183	412	1,645
	1.67	20	25	31	41	55	79	124	219	493	1,970
	2.00	28	35	44	57	77	111	174	308	693	2,771

For example, when $\upsilon = 2$, for $m = 6$, $S_{pk}^T = 1.33$ with risk $\alpha = 0.05$, a sample size of $n \geq 132$ ensures that the difference between the sampling estimator $\hat{S}_{pk}^{T*}$ and the real S_{pk}^T is smaller than 0.05. Thus, we can conclude that the actual performance $S_{pk}^T > 1.28$ with 95% confidence level.

5 A comparison between the standard bootstrap and the proposed method

5.1 The standard bootstrap method

Efron (1979, 1982) introduced a nonparametric, computational intensive but effective estimation method called "Bootstrap", which is a data-based simulation technique for statistical inference. One can use the nonparametric bootstrap method to

Table 4 Sample sizes required for the normal approximation to converge with $\alpha = 0.05$ for $v = 3$

m	S_{pk}^T	ε									
		0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
	1.00	31	39	49	64	86	124	194	344	773	3,089
	1.33	49	60	76	99	135	193	302	537	1,207	4,825
3	1.50	59	72	91	119	162	233	364	647	1,455	5,819
	1.67	70	87	110	143	195	280	438	778	1,749	6,994
	2.00	95	117	148	193	263	378	590	1,049	2,359	9,435
	1.00	16	20	25	32	43	62	97	172	387	1,545
	1.33	25	30	38	50	68	97	151	269	604	2,413
6	1.50	30	36	46	60	81	117	182	324	728	2,910
	1.67	35	44	55	72	98	140	219	389	875	3,497
	2.00	48	59	74	97	132	189	295	525	1,180	4,718
	1.00	11	13	17	22	29	42	65	115	258	1,030
	1.33	17	20	26	33	45	65	101	179	403	1,609
9	1.50	20	24	31	40	54	78	122	216	485	1,940
	1.67	24	29	37	48	65	94	146	260	583	2,332
	2.00	32	39	50	65	88	126	197	350	787	3,145
	1.00	8	10	13	16	22	31	49	86	194	773
	1.33	13	15	19	25	34	49	76	135	302	1,207
12	1.50	15	18	23	30	41	59	91	162	364	1,455
	1.67	18	22	28	36	49	70	110	195	438	1,749
	2.00	24	30	37	49	66	95	148	263	590	2,359
	1.00	7	8	10	13	18	25	39	69	155	618
	1.33	10	12	16	20	27	39	61	108	242	965
15	1.50	12	15	19	24	33	47	73	130	291	1,164
	1.67	14	18	22	29	39	56	88	156	350	1,399
	2.00	19	24	30	39	53	76	118	210	472	1,887

estimate the sampling distribution of a statistic, while assuming only that the sample is a representative of the population from which it is drawn, and that the observations are independent and identically distributed. Efron and Tibshirani (1986) developed three types of bootstrap confidence interval, including the standard bootstrap confidence interval (SB), the percentile bootstrap confidence interval (PB), and the biased corrected percentile bootstrap confidence interval (BCPB). Because of the robustness, high coverage and short length of confidence intervals, the SB is recommended to be used by Franklin and Wasserman (1992).

Efron and Tibshirani (1986) indicated that a rough minimum of 1,000 bootstrap resamples is usually sufficient to compute reasonably accurate confidence interval estimates. Here, we apply the standard bootstrap (SB) to obtain the lower confidence bound of S_{pk}^T. In order to obtain the reliable results, $B = 2,000$ bootstrap resamples are taken and these 2,000 bootstrap estimates of S_{pk}^T are calculated. Bootstrap sampling

Table 5 Four processes under the simulation study and the corresponding values of S_{pk1}, S_{pk2}, S_{pk3} and S_{pk}^T

$(\mu_1,\sigma_1) = (3.0, 0.06)$	$S_{pk1} = 1.107$	
$(\mu_2,\sigma_2) = (25.5, 0.452)$	$S_{pk2} = 1.107$	$S_{pk}^T = 1.00$
$(\mu_3,\sigma_3) = (0.6, 0.03)$	$S_{pk3} = 1.107$	
$(\mu_1,\sigma_1) = (3.0, 0.047)$	$S_{pk1} = 1.414$	
$(\mu_2,\sigma_2) = (25.5, 0.353)$	$S_{pk2} = 1.414$	$S_{pk}^T = 1.33$
$(\mu_3,\sigma_3) = (0.6, 0.024)$	$S_{pk3} = 1.414$	
$(\mu_1,\sigma_1) = (3.0, 0.042)$	$S_{pk1} = 1.576$	
$(\mu_2,\sigma_2) = (25.5, 0.317)$	$S_{pk2} = 1.576$	$S_{pk}^T = 1.50$
$(\mu_3,\sigma_3) = (0.6, 0.021)$	$S_{pk3} = 1.576$	
$(\mu_1,\sigma_1) = (3.0, 0.038)$	$S_{pk1} = 1.739$	
$(\mu_2,\sigma_2) = (25.5, 0.287)$	$S_{pk2} = 1.739$	$S_{pk}^T = 1.67$
$(\mu_3,\sigma_3) = (0.6, 0.019)$	$S_{pk3} = 1.739$	

is equivalent to sampling (with replacement) from the empirical probability distribution function. Thus, the bootstrap distribution of S_{pk}^T is estimator of the distribution of S_{pk}^T.

From the 2,000 bootstrap estimates $\hat{C}^*(i)$, $i = 1, 2, \ldots, 2{,}000$, calculate the sample average and the sample standard deviation, respectively,

$$\overline{C}^* = \frac{1}{2{,}000}\sum_{i=1}^{2{,}000}\hat{C}^*(i),$$

$$S_C^* = \sqrt{\frac{1}{1999}\sum_{i=1}^{2{,}000}\left(\hat{C}^*(i) - \overline{C}^*\right)^2}.$$

The quantity S_C^* is an estimator of the standard deviation of $\hat{S}_{pk}^{T*}$ if the distribution of $\hat{S}_{pk}^{T*}$ is approximately normal. Thus, the $(1-\alpha)100\%$ SB confidence interval for S_{pk}^T can be constructed as

$$\left[\overline{C}^* - Z_{\alpha/2}S_C^*,\ \ \overline{C}^* + Z_{\alpha/2}S_C^*\right],$$

where $Z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution. Thus, the $100(1-\alpha)\%$ SB lower confidence bound for S_{pk}^T is

$$\overline{C}^* - Z_\alpha S_C^*.$$

Table 6 The simulation results for 95% confidence level of proposed and SB method at $B = 2{,}000$ in the parenthesis with $v = 3$, $m = 2(2)12$ and $n = 10(10)100$

$m\backslash n$	10	20	30	40	50	60	70	80	90	100
$S_{pk}^{T} = 1.00$										
2	0.983	0.971	0.968	0.969	0.970	0.971	0.973	0.968	0.970	0.967
	(0.950)	(0.965)	(0.966)	(0.970)	(0.970)	(0.973)	(0.975)	(0.972)	(0.972)	(0.972)
4	0.943	0.947	0.944	0.949	0.951	0.946	0.947	0.954	0.941	0.948
	(0.864)	(0.915)	(0.926)	(0.935)	(0.940)	(0.933)	(0.939)	(0.948)	(0.932)	(0.943)
6	0.904	0.918	0.928	0.921	0.930	0.927	0.929	0.936	0.936	0.938
	(0.746)	(0.849)	(0.871)	(0.891)	(0.895)	(0.899)	(0.900)	(0.914)	(0.918)	(0.916)
8	0.844	0.890	0.895	0.917	0.920	0.922	0.924	0.931	0.934	0.932
	(0.627)	(0.770)	(0.821)	(0.846)	(0.861)	(0.875)	(0.877)	(0.891)	(0.893)	(0.901)
10	0.797	0.868	0.880	0.900	0.896	0.906	0.917	0.922	0.916	0.922
	(0.511)	(0.686)	(0.752)	(0.801)	(0.816)	(0.833)	(0.854)	(0.865)	(0.858)	(0.873)
12	0.772	0.838	0.867	0.887	0.882	0.900	0.902	0.907	0.903	0.911
	(0.410)	(0.631)	(0.694)	(0.752)	(0.781)	(0.808)	(0.826)	(0.837)	(0.840)	(0.857)
$S_{pk}^{T} = 1.33$										
2	0.990	0.984	0.981	0.981	0.982	0.983	0.981	0.975	0.980	0.980
	(0.965)	(0.980)	(0.986)	(0.987)	(0.990)	(0.989)	(0.990)	(0.985)	(0.986)	(0.988)
4	0.967	0.967	0.969	0.966	0.967	0.964	0.964	0.966	0.961	0.962
	(0.918)	(0.956)	(0.966)	(0.968)	(0.968)	(0.969)	(0.966)	(0.968)	(0.964)	(0.967)
6	0.942	0.947	0.954	0.944	0.953	0.946	0.947	0.952	0.953	0.951
	(0.847)	(0.916)	(0.934)	(0.932)	(0.943)	(0.936)	(0.938)	(0.949)	(0.955)	(0.944)
8	0.905	0.931	0.927	0.942	0.940	0.939	0.944	0.944	0.946	0.945
	(0.741)	(0.857)	(0.888)	(0.910)	(0.910)	(0.913)	(0.922)	(0.932)	(0.934)	(0.932)
10	0.866	0.908	0.916	0.922	0.924	0.926	0.942	0.936	0.930	0.938
	(0.639)	(0.794)	(0.848)	(0.873)	(0.883)	(0.886)	(0.900)	(0.906)	(0.901)	(0.914)
12	0.847	0.876	0.900	0.906	0.910	0.920	0.920	0.928	0.922	0.927
	(0.566)	(0.745)	(0.809)	(0.831)	(0.848)	(0.873)	(0.872)	(0.876)	(0.882)	(0.886)
$S_{pk}^{T} = 1.50$										
2	0.992	0.987	0.985	0.986	0.988	0.985	0.984	0.981	0.983	0.984
	(0.924)	(0.986)	(0.990)	(0.990)	(0.993)	(0.992)	(0.991)	(0.988)	(0.991)	(0.994)
4	0.972	0.974	0.975	0.972	0.973	0.977	0.970	0.971	0.966	0.968
	(0.930)	(0.964)	(0.978)	(0.979)	(0.979)	(0.979)	(0.978)	(0.978)	(0.975)	(0.974)
6	0.956	0.961	0.962	0.957	0.963	0.956	0.957	0.960	0.960	0.960
	(0.873)	(0.936)	(0.952)	(0.953)	(0.961)	(0.952)	(0.957)	(0.961)	(0.963)	(0.963)
8	0.921	0.942	0.941	0.952	0.949	0.950	0.951	0.949	0.953	0.954
	(0.785)	(0.899)	(0.913)	(0.936)	(0.934)	(0.941)	(0.944)	(0.944)	(0.946)	(0.948)
10	0.894	0.925	0.929	0.933	0.937	0.937	0.949	0.945	0.938	0.943
	(0.695)	(0.842)	(0.884)	(0.897)	(0.910)	(0.909)	(0.927)	(0.926)	(0.922)	(0.933)
12	0.873	0.895	0.911	0.919	0.922	0.934	0.930	0.939	0.928	0.934
	(0.621)	(0.796)	(0.845)	(0.866)	(0.877)	(0.894)	(0.899)	(0.902)	(0.904)	(0.905)

Table 6 continued

$m\backslash n$	10	20	30	40	50	60	70	80	90	100
$S_{pk}^T = 1.67$										
2	0.993	0.990	0.989	0.990	0.989	0.987	0.988	0.985	0.986	0.987
	(0.994)	(0.989)	(0.992)	(0.993)	(0.994)	(0.994)	(0.995)	(0.995)	(0.995)	(0.997)
4	0.974	0.981	0.980	0.978	0.977	0.980	0.978	0.977	0.973	0.974
	(0.937)	(0.972)	(0.985)	(0.984)	(0.987)	(0.984)	(0.985)	(0.983)	(0.983)	(0.982)
6	0.963	0.967	0.968	0.970	0.971	0.963	0.964	0.967	0.969	0.965
	(0.896)	(0.950)	(0.964)	(0.970)	(0.972)	(0.965)	(0.971)	(0.972)	(0.972)	(0.972)
8	0.935	0.948	0.956	0.957	0.957	0.957	0.961	0.955	0.962	0.960
	(0.813)	(0.920)	(0.937)	(0.954)	(0.951)	(0.957)	(0.957)	(0.956)	(0.961)	(0.958)
10	0.914	0.939	0.940	0.946	0.949	0.948	0.956	0.952	0.945	0.949
	(0.738)	(0.875)	(0.908)	(0.919)	(0.927)	(0.931)	(0.950)	(0.942)	(0.938)	(0.944)
12	0.898	0.913	0.926	0.930	0.932	0.945	0.939	0.945	0.933	0.941
	(0.664)	(0.835)	(0.886)	(0.894)	(0.899)	(0.916)	(0.921)	(0.927)	(0.920)	(0.924)

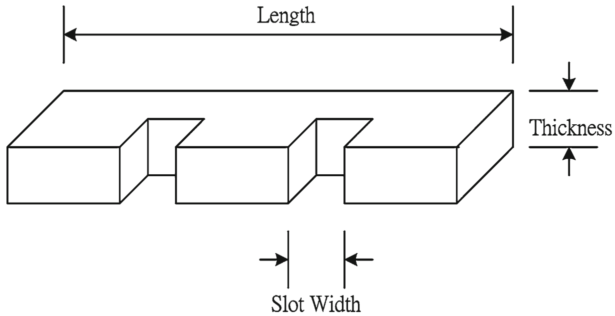

Fig. 1 A metal block produced on a CNC machine

5.2 A simulation study

To compare the coverage percentage of the SB method and our proposed method for constructing the lower confidence bound of S_{pk}^T, four processes with the three characteristics are considered. We choose $(LSL_1, USL_1) = (2.80, 3.20)$, $(LSL_2, USL_2) = (24, 27)$ and $(LSL_3, USL_3) = (0.5, 0.7)$ for each characteristic, respectively. We ran the simulation for various combinations of means (μ_1, μ_2, μ_3) and standard deviations $(\sigma_1, \sigma_2, \sigma_3)$. These combinations are shown in Table 5 and each combination represents a specific process.

For each process, the number of samples $m = 2(2)12$ and the sample sizes $n = 10(10)100$ were used. We construct 95% approximate (11) and SB bootstrap lower confidence bounds. It is then determined if the true S_{pk}^T is larger than the calculated lower confidence bounds. A complete simulation process was replicated $N = 20,00$ times executing the Matlab program. Thus, we are able to calculate the percentage of times that the true S_{pk}^T is larger than our proposed and SB lower confidence

Table 7 The data of 12 samples each of size 50 for three characteristics

sample	1	2	3	4	5	6	7	8	9	10	11	12
$\overline{x}_{i1}$	150.147	149.965	149.997	149.972	150.545	149.644	149.929	150.4	149.99	149.804	150.213	149.981
s_{i1}	1.19812	1.59442	1.57719	1.60336	1.53383	1.39221	1.64296	1.43878	1.29913	1.31108	1.39261	1.63385
$\overline{x}_{i2}$	41.2246	41.0077	41.0235	40.7918	40.9642	41.2576	40.7976	40.8921	41.2553	41.0006	40.9054	40.9453
s_{i2}	1.25991	1.10883	1.11206	1.24488	1.2528	1.13785	1.13814	1.02064	0.97947	0.97271	1.27486	1.10423
$\overline{x}_{i3}$	38.0587	38.1073	37.7488	37.8459	37.8139	38.2993	37.7296	38.0599	38.2387	37.772	37.9365	38.1974
s_{i3}	1.15213	1.14213	1.05379	1.04908	1.23336	1.16323	1.26323	1.28833	1.19945	1.43666	1.23694	1.1247

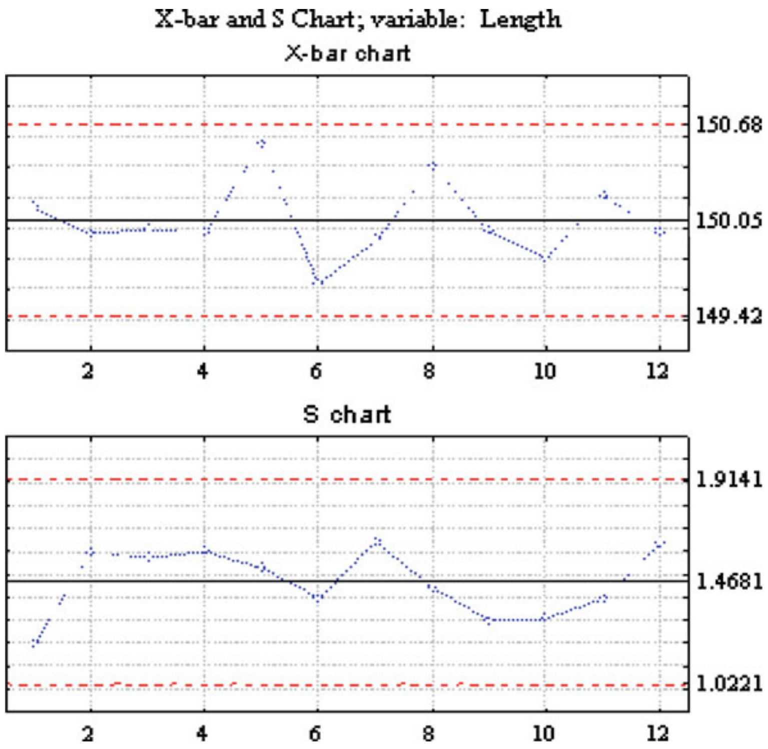

Fig. 2 $\overline{X} - S$ charts for length

bounds, respectively. The percentage of times is also called the "coverage percentage". In general, the coverage percentage is the most important performance to assess a lower confidence bound. A larger coverage percentage corresponds to a better performance. In the simulation study, we would compare the coverage percentage with confidence level 0.95.

The frequency of coverage for the lower confidence bound is binomially distributed with $N = 2{,}000$ and $P = 0.95$. Hence, a 99% confidence interval for the coverage percentage is $0.95 \pm 2.575 \times \sqrt{0.95 * 0.05/2{,}000} = 0.950 \pm 0.125$. Hence, we could be 99% confident that a "true 95% confidence interval" would have a coverage percentage between 0.9375 and 0.9625. Table 6 displays the coverage percentage for 95% confidence interval under particular m, n, and S_{pk}^{T} at $B = 2{,}000$. The coverage percentage of our proposed method is larger than SB method for most of the instances. For the number of coverage percentage exceeds these two limits (0.9375 and 0.9625), our proposed method is much less than the SB method. Numbers of the coverage percentages are close to the interval (0.9375, 0.9625) for the outside instances of the our proposed method. In contrast, SB lower confidence bound has coverage percentage lower than 0.95 for almost all instances. Finally, we can easily see that a larger sample size n produces a better coverage percentage, but a larger sample m provides a worse coverage percentage. Further, each coverage percentage of these two methods decreases as m increases. In summary, our proposed lower confidence bound is more accurate than the SB lower confidence bound.

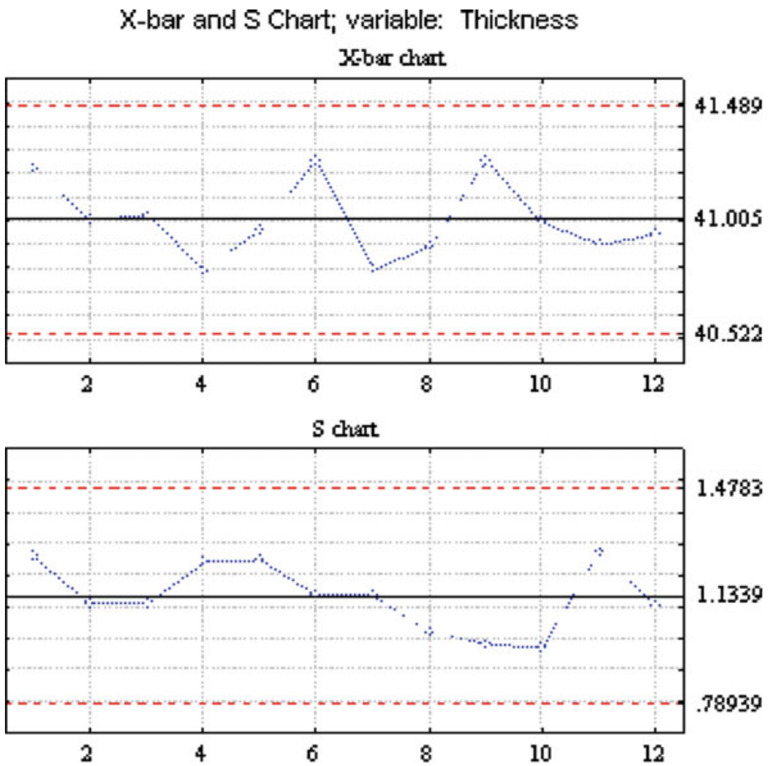

Fig. 3 $\overline{X} - S$ charts for thickness

6 An illustrative example

Conventional capability studies treat products as if they have only one critical characteristic of interest. However, the vast majority of products have many features that are important to the customer. For example, the metal block displayed in Fig. 1 is produced on a CNC machine. Three different key characteristics (length, thickness and slot width) are machined during this one operation. All three must be within their respective print specifications if the product is to be considered acceptable. The specified limits for length, thickness, and slot width are set at [143, 157], [35, 47] and [33, 43], respectively.

The three characteristics data of 12 multiple samples each of size 50 are collected. The sample mean $\overline{x}_{ij}$ and sample deviation s_{ij} for the twelve samples are listed in Table 7. First, we need to ensure the independence among the three characteristics. The Pearson-Correlation test is applied to justify the correlation, which shows the relationship among three characteristics can be regarded as independent. Second, we check if each characteristic data collected from the process is in control and normally distributed. The Shapiro-Wilk test is used to test whether the sample data is normal. For those 12 samples of size 50 each to each characteristic, we found that the p-value is larger than 0.05. Thus, the data can be regarded as taken from a normal process.

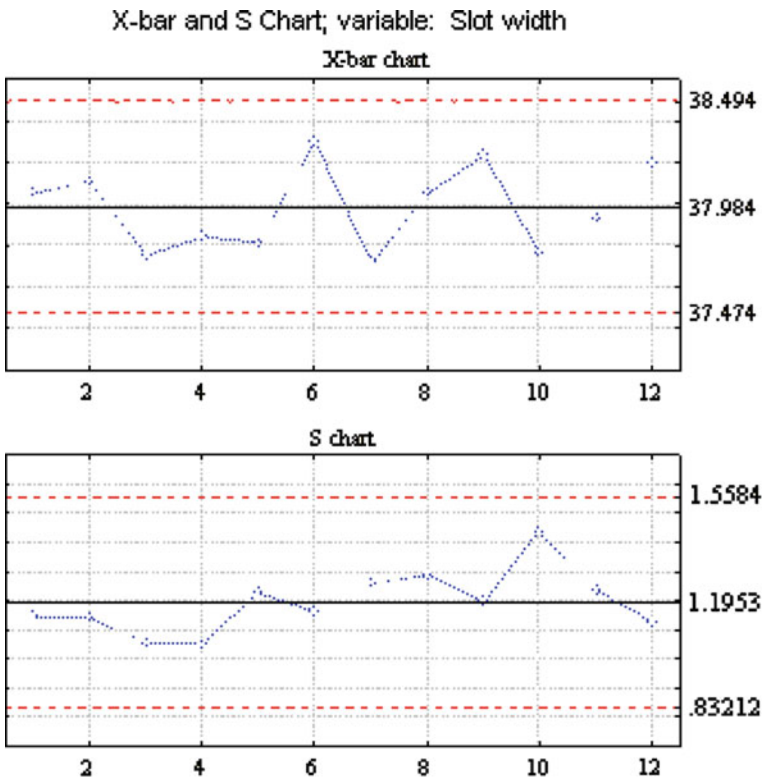

Fig. 4 $\overline{X} - S$ charts for slot width

Then, we conduct the $\overline{X} - S$ charts shown in Figs. 2, 3 and 4 to check if the process is in statistical control. Since all sample points are within the control limits without any special pattern, we can conclude that the process is in control. Finally, we consider the process is stable and proceed the capability measurement.

Suppose the minimal capability requirement for this process is set to S_{pk}^{T}, 1.0. According to the data collected, the sample mean and pooled sample deviation for each characteristic are $\overline{x}_1 = 150.049$ and $s_1 = 1.46029$, $\overline{x}_2 = 41.0055$ and $s_2 = 1.12707$, $\overline{x}_3 = 37.984$ and $s_3 = 1.18761$, respectively. We can get $\hat{S}_{pk1}^{*} = 1.59695$, $\hat{S}_{pk2}^{*} = 1.77448$ and $\hat{S}_{pk3}^{*} = 1.40325$ further. Then, $\hat{S}_{pk}^{T*}$ can be calculated as 1.39823, and the lower bound for S_{pk}^{T} is 1.33547. So we can conclude that this process meets the capability requirement.

7 Conclusions

Process yield is the most common and standard criteria for evaluating the quality of products manufactured. Process yield measure for processes with a single characteristic has been investigated extensively. However, process yield measure for processes with multiple quality characteristics is comparatively neglected. Process yield assurance for processes with multiple characteristics is an important issue. So the proposition of a technique assuring the process yield is necessary. In this paper, we proposed the asymptotic distribution of yield index called $\hat{S}_{pk}^{T*}$ under multiple samples. Applying the asymptotic distribution of $\hat{S}_{pk}^{T*}$, hypothesis testing and lower confidence bound for S_{pk}^{T} can be executed. In addition, a comparison between the SB and the proposed method based on the lower confidence bound is executed. Generally, the result indicates that the proposed approach is more reliable than the standard bootstrap method. The proposed procedure can be used to determine whether their production meets the present yield requirement, and make a reliable decision.

Appendix

Corollary 1 *If $\hat{S}_{pk}^{T*}$ is defined as*

$$\hat{S}_{pk}^{T*} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi\left(3\hat{S}_{pkj}^{*}\right)-1\right)+1\right]\bigg/2\right\},$$

where $\hat{S}_{pkj}^{*}$ denotes the estimator of S_{pkj} under multiple samples and follows a normal distribution with mean S_{pkj} and variance $(a_j^2+b_j^2)/36mn(\phi(3S_{pkj}))^2$ (j =1,2,...,v), then

$$\hat{S}_{pk}^{T*} \sim N\left(S_{pk}^{T}, \frac{1}{36mn\left(\phi\left(3S_{pk}^{T}\right)\right)^2}\left(\sum_{j=1}^{v}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right).$$

Proof Applying the first-order expansion of v-variate Taylor, we can obtain

$$f(X) = f(X_0) + \sum_{j=1}^{v}\frac{\partial f(X_0)}{\partial x_i}(x_i - x_{i0}), \text{ where } X = (x_1, x_2, \ldots, x_v)$$

We take $v = 2$ for example to derive the asymptotic distribution of $\hat{S}_{pk}^{T*}$. Given

$$\hat{S}_{pkj}^{*} \sim N\left(S_{pkj}, \frac{a_j^2+b_j^2}{36mn(\phi(3S_{pkj}))^2}\right), \forall j = 1, 2,$$

where $\hat{S}_{pk1}^{*}$ and $\hat{S}_{pk2}^{*}$ are mutually independent.

Based on the first-order expansion of υ-variate Taylor, it follows that

$$f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*}\right)=f\left(S_{pk1},\ S_{pk2}\right)+\frac{\partial f\left(S_{pk1},\ S_{pk2}\right)}{\partial\hat{S}_{pk1}^{*}}\left(\hat{S}_{pk1}^{*}-S_{pk1}\right)$$
$$+\frac{\partial f\left(S_{pk1},\ S_{pk2}\right)}{\partial\hat{S}_{pk2}^{*}}\left(\hat{S}_{pk2}^{*}-S_{pk2}\right). \quad \text{(A1)}$$

Let $\hat{S}_{pk}^{T*}=f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*}\right)$, we can drive

$$\begin{aligned}E\left(\hat{S}_{pk}^{T*}\right)&=E\left(f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*}\right)\right)\\&=f\left(S_{pk1},S_{pk2}\right)\left(\because E\left(\hat{S}_{pk1}^{*}-S_{pk1}\right)=0\ ,\ E\left(\hat{S}_{pk2}^{*}-S_{pk2}\right)=0\right)\\&=\frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{2}\left(2\Phi(3S_{pkj})-1\right)+1\right]\Big/2\right\}\\&=S_{pk}^{T}.\end{aligned}$$

Form (A1), it is obvious that $\text{Var}(\hat{S}_{pk}^{T*})$ can be written as

$$\text{Var}\left(\hat{S}_{pk}^{T*}\right)=\left(\frac{\partial f(S_{pk1},S_{pk2})}{\partial\hat{S}_{pk1}^{*}}\right)^{2}\text{Var}\left(\hat{S}_{pk1}^{*}\right)+\left(\frac{\partial f\left(S_{pk1},S_{pk2}\right)}{\partial\hat{S}_{pk2}^{*}}\right)^{2}\text{Var}\left(\hat{S}_{pk2}^{*}\right).$$

Since $f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*}\right)=\Phi^{-1}\left\{\left[\left(2\Phi\left(3\hat{S}_{pk1}^{*}-1\right)\right)\left(2\Phi\left(3\hat{S}_{pk2}^{*}-1\right)\right)+1\right]\Big/2\right\}\Big/3,$ we first differentiate $f\left(\hat{S}_{pk1}^{*},\ \hat{S}_{pk2}^{*}\right)$ with respect to $\hat{S}_{pk1}^{*}$. It follows that

$$\begin{aligned}\frac{\partial f\left(\hat{S}_{pk1}^{*},\hat{S}_{pk2}^{*}\right)}{\partial\hat{S}_{pk1}^{*}}&=\frac{1}{3}\frac{\frac{\partial}{\partial\hat{S}_{pk1}^{*}}\left[\frac{\left(2\Phi\left(3\hat{S}_{pk1}^{*}\right)-1\right)\left(2\Phi\left(3\hat{S}_{pk2}^{*}\right)-1\right)+1}{2}\right]}{\phi\left\{\Phi^{-1}\left\{\frac{\left[\left(2\Phi\left(3\hat{S}_{pk1}^{*}\right)-1\right)\left(2\Phi\left(3\hat{S}_{pk2}^{*}\right)-1\right)+1\right]}{2}\right\}\right\}}\\&=\frac{\left(2\Phi\left(3\hat{S}_{pk2}^{*}\right)-1\right)\phi\left(3\hat{S}_{pk1}^{*}\right)}{\phi\left\{\Phi^{-1}\left\{\frac{\left[\left(2\Phi\left(3\hat{S}_{pk1}^{*}\right)-1\right)\left(2\Phi\left(3\hat{S}_{pk2}^{*}\right)-1\right)+1\right]}{2}\right\}\right\}}.\end{aligned} \quad \text{(A2)}$$

Consequently, substituting S_{pk1} for $\hat{S}_{pk1}^{*}$ in (A2), we can obtain

$$\frac{\partial f\left(S_{pk1},S_{pk2}\right)}{\partial\hat{S}_{pk1}^{*}}=\frac{\left(2\Phi\left(3S_{pk2}\right)-1\right)\phi\left(3S_{pk1}\right)}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}.$$

Similarly, by differentiating $f(S_{pk1}, S_{pk2})$ with respect to $\hat{S}^*_{pk2}$ and substituting S_{pk2} for $\hat{S}^*_{pk2}$ in (A2), we have

$$\frac{\partial f\left(S_{pk1}, S_{pk2}\right)}{\partial \hat{S}^*_{pk2}} = \frac{\left(2\Phi\left(3S_{pk1}\right)-1\right)\phi\left(3S_{pk2}\right)}{\phi\left\{\Phi^{-1}\left\{\frac{[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1]}{2}\right\}\right\}}.$$

Hence, it implies that

$$\begin{aligned}
\operatorname{Var}\left(\hat{S}^{T*}_{pk}\right) &= \left(\frac{\left(2\Phi\left(3S_{pk2}\right)-1\right)\phi\left(3S_{pk1}\right)}{\phi\left\{\Phi^{-1}\left\{\frac{[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1]}{2}\right\}\right\}}\right)^2 \operatorname{Var}\left(\hat{S}^*_{pk1}\right) \\
&+ \left(\frac{(2\Phi(3S_{pk1})-1)\phi(3S_{pk2})}{\phi\left\{\Phi^{-1}\left\{\frac{[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1]}{2}\right\}\right\}}\right)^2 \operatorname{Var}\left(\hat{S}^*_{pk2}\right) \\
&= \left(\frac{(2\Phi(3S_{pk2})-1)\phi(3S_{pk1})}{\phi\left\{\Phi^{-1}\left\{\frac{[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1]}{2}\right\}\right\}}\right)^2 \frac{a_2^2+b_2^2}{36mn(\phi(3S_{pk1}))^2} \\
&+ \left(\frac{(2\Phi(3S_{pk1})-1)\phi(3S_{pk2})}{\phi\left\{\Phi^{-1}\left\{\frac{[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1]}{2}\right\}\right\}}\right)^2 \frac{a_2^2+b_2^2}{36mn(\phi(3S_{pk2}))^2} \\
&= \frac{1}{36mn(\phi(3S^T_{pk}))^2}\left[(a_1^2+b_1^2)(2\Phi(3S_{pk2})-1)^2+(a_2^2+b_2^2)(2\Phi(3S_{pk1})-1)^2\right] \\
&= \frac{1}{36mn(\phi(3S^T_{pk}))^2}\left(\sum_{j=1}^{2}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{2}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right).
\end{aligned}$$

It is noted that $\hat{S}^{T*}_{pk}$ is the linear combination of $\hat{S}^*_{pk1}$ and $\hat{S}^*_{pk2}$, so we know that $\hat{S}^{T*}_{pk}$ follows a normal distribution. Consequently, the asymptotic distribution of $\hat{S}^{T*}_{pk}$ can be expressed as

$$\hat{S}^{T*}_{pk} \sim N\left(S^T_{pk}, \frac{1}{36mn\left(\phi\left(3S^T_{pk}\right)\right)^2}\left(\sum_{j=1}^{2}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{2}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right).$$

Consider v variables, the asymptotic distribution of $\hat{S}^{T*}_{pk}$ can be derived as

$$\hat{S}^{T*}_{pk} \sim N\left(S^T_{pk}, \frac{1}{36mn\left(\phi\left(3S^T_{pk}\right)\right)^2}\left(\sum_{j=1}^{v}\left\{\left(a_j^2+b_j^2\right)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right)$$

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