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Hydrodynamic limits of the nonlinear Klein–Gordon equation

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Abstract

We perform the mathematical derivation of the compressible and incompressible Euler equations from the modulated nonlinear Klein–Gordon equation. Before the formation of singularities in the limit system, the nonrelativistic-semiclassical limit is shown to be the compressible Euler equations. If we further rescale the time variable, then in the semiclassical limit (the light speed kept fixed), the incompressible Euler equations are recovered. The proof involves the modulated energy introduced by Brenier (2000) [1].
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Résumé

On obtient une dérivation mathématique des équations d'Euler compressibles et incompressibles à partir de l'équation de Klein–Gordon non linéaire modulée. Avant la formation de singularités pour le système limite, on démontre dans la limite non relativiste et semi-classique la convergence vers les équations d'Euler compressibles. Au moyen d'un changement d'échelle supplémentaire en temps, on démontre la limite semi-classique la vitesse de la lumière restant fixée, la limite vers les équations d'Euler incompressibles. La démonstration utilise l'énergie modulée introduite par Brenier (2000) [1].
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1. Introduction

In this paper, we study the nonlinear Klein–Gordon equation

$$\frac{\hbar^2}{2mc^2}\partial_t^2\Psi - \frac{\hbar^2}{2m}\Delta\Psi + \frac{mc^2}{2}\Psi + V'\big(|\Psi|^2\big)\Psi = 0, \tag{1.1}$$

where m is mass, c is the speed of light, $\hbar$ is the Planck constant and $\Psi(x,t)$ is a complex-valued vector field over a spatial domain $\Omega \subset \mathbb{R}^n$. The nonlinear function V' is the first derivative of a twice differentiable nonlinear real-valued function over $\mathbb{R}^+$. Thus, V' is the potential energy and V is the potential energy density of the fields.

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The Klein–Gordon equation for the complex scalar field is the relativistic version of the Schrödinger equation, which is used to describe spinless particles. The reader should refer to [23,25] for the physical background and [27] for a general introduction to nonlinear wave equations. Since the Planck constant $\hbar$ has the dimension of action, $[\hbar] =$ [energy] $\times$ [time] $=$ [action], it is easy to check that (1.1) is dimensionally balanced. Furthermore, mc^2t and the Planck constant $\hbar$ have the same dimensions, $[mc^2t] = [\hbar] =$ [action], so we may consider the modulated wave function

$$\psi(x,t) = \Psi(x,t)\exp\bigl(imc^2t/\hbar\bigr),$$

where the factor $\exp(imc^2t/\hbar)$ describes the oscillations of the wave function, then ψ satisfies the modulated nonlinear Klein–Gordon equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi - V'\bigl(|\psi|^2\bigr)\psi = \frac{\hbar^2}{2mc^2}\partial_t^2\psi. \tag{1.2}$$

The relations between different terms in (1.2) are best seen when the equation is written in terms of dimensionless variables, which will be adorned with carets. The dimensionless independent variables are given by

$$x = L\hat{x}, \qquad t = T\hat{t},$$

where L and T denote the reference length and time respectively. We also define the reference velocity by $U = L/T$ and rescale the potential energy as

$$V' = mU^2\hat{V}'.$$

Substituting all of these rescaled quantities into the original equation (1.2), and dropping all carets, yields

$$i\varepsilon\partial_t\psi + \frac{1}{2}\varepsilon^2\Delta\psi - V'\bigl(|\psi|^2\bigr)\psi = \frac{1}{2}\varepsilon^2\nu^2\partial_t^2\psi. \tag{1.3}$$

Note that the first important dimensionless parameter ν is given by the ratio of reference velocity and speed of light, $\nu = U/c$, and the scaled Planck constant $\varepsilon = \frac{\hbar}{mU^2T}$ is the second important dimensionless parameter. The two dimensionless parameters ν and ε show the relativistic and quantum effects respectively. Formally letting $\nu \to 0$ or $c \to \infty$ (more precisely $U \ll c$), i.e., the so-called nonrelativistic limit, the modulated nonlinear Klein–Gordon equation (1.3) will reduce to the nonlinear Schrödinger equation

$$i\varepsilon\partial_t\psi + \frac{1}{2}\varepsilon^2\Delta\psi - V'\bigl(|\psi|^2\bigr)\psi = 0, \tag{1.4}$$

though one has to be extremely careful with heuristics due to the double time derivative on the right-hand side of (1.3).

Over the last twenty years, there has been a vast amount of research concerning the nonrelativistic limit of the Cauchy problem for the nonlinear Klein–Gordon equation. In particular, in [18] Machihara–Nakanishi–Ozawa proved that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after infinite oscillations in time are removed. The Strichartz estimate plays the most important role to obtain the uniform bound in space and time (see also [22,24] and references therein). However, to the best of our knowledge, the semiclassical limit $\varepsilon \to 0$ is not well studied and is not clear from (1.3). On the other hand, based on the hydrodynamical structure, the semiclassical limit, $\varepsilon \to 0$, of the defocusing nonlinear Schrödinger equation is quite well understood (see [4] for the review). In [6], Jin–Levermore–McLaughlin applied the inverse scattering to establish the semiclassical limit of the defocusing cubic nonlinear Schrödinger equation; the complete integrability was exploited to obtain the global characterization of the weak limits of the entire cubic NLS hierarchy. Therefore to study the various singular (hydrodynamics) limits of the nonlinear Klein–Gordon equation (1.1), it is better to start from (1.3) because of its analogue to the nonlinear Schrödinger equation (1.4).

For the defocusing nonlinear Schrödinger equation, the semiclassical limit for initial data with Sobolev regularity in short time has been studied by Grenier [5]. In this limit, the Euler equations for an isentropic compressible flow are recovered. The basic idea is introducing the modified Madelung transform and rewrite the hydrodynamical equations as a linear dispersive perturbation of the quasilinear symmetric hyperbolic system. The same method also works well to other Schrödinger type equations [3,8,9,11]. But unfortunately, this method cannot be applied to the modulated nonlinear Klein–Gordon equation (1.3). The main difficulty is that we are not able to rewrite it as a quasilinear symmetric hyperbolic system and thus the standard energy estimate cannot be applied.

We proved in [12,28] that the semiclassical limit of the modulated cubic nonlinear Klein–Gordon equation is a relativistic wave map and the associated phase function satisfies a linear relativistic wave equation. The nonrelativistic-semiclassical limit is the classical wave map for the limit wave function and the typical linear wave equation for the associated phase function. In this paper, we consider the singular limits from the point of view of hydrodynamics (see [13] for the fluid approximation). As has been pointed out by Masmoudi and Nakanishi in [21] (see also [12]), the Klein–Gordon equation behaves like the Schrödinger equation in the nonrelativistic region, and like the wave equation in the relativistic region. Thus, we consider the nonrelativistic-semiclassical limit, i.e. the two parameters $\varepsilon, \nu \to 0$ simultaneously, of the modulated nonlinear Klein–Gordon equation (1.3). To avoid carrying out a double limit, we restrict the case when the two parameters ν and ε are related. In this situation, the compressible Euler equations are recovered as the nonrelativistic-semiclassical limit. On the other hand, if we rescale the time variable, then the extra degree of the parameter ε enables us to discuss the semiclassical limit no matter when ν is of order $O(1)$ or tends to zero as $\varepsilon \to 0$ and the limit of the current is shown to satisfy the incompressible Euler equations.

The modulated energy method was introduced by Brenier [1] to prove the convergence of the Vlasov–Poisson system to the incompressible Euler equations. It was immediately extended by Masmoudi in [20] to general initial data allowing the presence of high oscillations in time (see also [10] for the quantum hydrodynamic model of semiconductor). The same idea is also applied to study various singular limits of other equations, for example the Schrödinger–Poisson equation [26], the Gross–Pitaevskii equation [14] and the coupled nonlinear Schrödinger equation [7,15]. In fact, we will employ this method to study the hydrodynamic limits of the modulated nonlinear Klein–Gordon equation (1.3). Similar to [1], we limit ourselves in this paper to the case when the initial data is well prepared. For general initial condition, the method used in [10,20] should be applicable and it will be our next research project.

The modulated energy is designed to control the propagation of the charge and current (or momentum) for Vlasov–Poisson, Schrödinger and the related nonrelativistic type equations. Since the charge (current) of the Klein–Gordon equation is constituted by the Schrödinger and relativistic parts, thus, the main idea is to show that the relativistic charge and current are small and the main contribution of the nonrelativistic-semiclassical limit comes from the Schrödinger part. In contrast with the Schrödinger equation and its variants, we have to introduce one correction term of the modulated energy which controls the propagation of the relativistic charge and current. In fact, the relativistic parts vanishes as ε tends to zero. Thus we prove the convergence of the charge and the current defined by the modulated nonlinear Klein–Gordon equation towards the solution of the γ-law compressible Euler equations. The range of the adiabatic exponent γ is different: $\gamma > 1$ for even spatial dimension n and $\gamma \geqslant \frac{n}{n-1}$ for odd n.

Turning to the incompressible limit, we have to rescale the time variable and consider the potential energy designed to represent in the form of pressure instead of the charge (or density). In this case, we show that the current converges to the incompressible Euler equations in the semiclassical limit. For the potential energy given in terms of the power of density, we have a similar result. However, the associated adiabatic exponent γ is different because of the convexity of the potential energy discussed. The difference between the compressible and incompressible flow is typified by consideration of the way in which an acoustic wave travels in the fluid. Besides the correction term of the modulated energy as discussed in the compressible Euler limit, we have to introduce one more correction term which describes the propagation of the density fluctuation in order to obtain the incompressible limit. This is similar to the zero Mach number limit of the compressible fluid [2,17,19]. The convergent result can be improved for $n = 2$ by the standard bootstrap process.

The rest of the paper is organized as follows. In Section 2, we derive the hydrodynamical structure of the modulated nonlinear Klein–Gordon equation and discuss their relation to the compressible and incompressible Euler equations. The proof of the convergence of the modulated nonlinear Klein–Gordon equation to the compressible Euler equations is established in Section 3. In Section 4, we prove the convergence of the time-scaled modulated nonlinear Klein–Gordon equation to the incompressible Euler equations.

Notation. In this paper, $L^p(\Omega)$ $(p \geqslant 1)$ denotes the classical Lebesgue space with norm $\|f\|_p = (\int_\Omega |f|^p\,dx)^{1/p}$, the Sobolev space of functions with all its k-th partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$. We abbreviate "$\leqslant C$ " to "$\lesssim$", where C is a positive constant depending only on a fixed parameter.

2. Hydrodynamical structure

A fluid mechanical interpretation for the linear Schrödinger equation was put forth by Madelung in 1927, and applies to nonlinear Schrödinger equations. Indeed, as shown in [5], the same idea also applied to the modulated nonlinear Klein–Gordon equation (1.3). We introduce the complex wave function, the so-called Madelung transformation,

$$\psi = A\exp(iS/\varepsilon), \tag{2.1}$$

in which both A, the amplitude, and S, the action function, are real-valued functions. It is important that the amplitude is assumed to be non-negative at every point: $A \geqslant 0$. Plugging (2.1) into modulated nonlinear Klein–Gordon equation (1.3) and separating the real and imagine parts, we obtain

$$\partial_t A + \frac{A}{2}\big(\Delta S - \nu^2\partial_t^2 S\big) + \nabla A\cdot\nabla S - \nu^2\partial_t A\,\partial_t S = 0, \tag{2.2}$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 - \frac{1}{2}\nu^2(\partial_t S)^2 + V'\big(A^2\big) = \frac{\varepsilon^2}{2}\frac{\Box_\nu A}{A}, \tag{2.3}$$

where the d'Alerbertian $\Box_\nu$ is defined by $\Box_\nu \equiv \Delta - \nu^2\partial_t^2$. Eqs. (2.2) and (2.3) are equivalent to the modulated nonlinear Klein–Gordon equation (1.3) for smooth functions A and S. Eq. (2.2) turns out to be the continuity equation for the *relativistic quantum fluid* and Eq. (2.3) is the relativistic quantum Hamilton–Jacobi equation. Introducing the new functions

$$\rho = A^2 = \psi\overline{\psi} = |\psi|^2,$$
$$u = \nabla S = \frac{i\varepsilon}{2}\frac{1}{|\psi|^2}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi),$$
$$\rho_K = \nu^2A^2\partial_t S = \frac{i\varepsilon\nu^2}{2}(\psi\partial_t\overline{\psi} - \overline{\psi}\partial_t\psi),$$

we can rewrite (2.2)–(2.3) as the dispersive perturbation of the compressible Euler type equations

$$\partial_t(\rho - \rho_K) + \nabla\cdot(\rho u) = 0, \qquad \nu^2\partial_t u = \nabla\left(\frac{\rho_K}{\rho}\right), \tag{2.4}$$

$$\partial_t(\rho u - \rho_K u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) = \frac{\varepsilon^2}{4}\nabla\cdot\big(\rho\nabla^2\log\rho\big) - \frac{\varepsilon^2\nu^2}{4}\partial_t(\rho\nabla\partial_t\log\rho), \tag{2.5}$$

where $P(\rho) = \rho V'(\rho) - V(\rho)$ is the pressure and ∇^2 denotes the Hessian. Eqs. (2.4)–(2.5) are constituted by the Euler, relativistic and quantum parts. If the "Euler part" of these equations is to be hyperbolic, then the pressure $P(\rho)$ must be a strictly increasing function of ρ; in that case, $P'(\rho) = \rho V''(\rho) > 0$. This means that V must be a strictly convex function of ρ and corresponds to a *defocusing* nonlinear Klein–Gordon equation. Defining the Schrödinger part energy density E_S and relativistic part energy density E_K respectively by

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho) = \frac{\varepsilon^2}{2}|\nabla\psi|^2 + V\big(|\psi|^2\big),$$
$$E_K = \frac{1}{2\nu^2}\frac{\rho_K^2}{\rho} + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} = \frac{\varepsilon^2\nu^2}{2}|\partial_t\psi|^2,$$

we obtain from (2.4)–(2.5) the conservation of energy

$$\partial_t(E_S + E_K) + \nabla\cdot\big(\big(E_S + P(\rho)\big)u\big) = \frac{\varepsilon^2}{4}\nabla\cdot\left[u\Delta\rho - \nabla\cdot(\rho u)\frac{\nabla\rho}{\rho}\right].$$

In the formal nonrelativistic limit $\nu \to 0$, the relativistic part energy E_K gives $\rho_K \to 0$, and (2.4)–(2.5) reduce to the quantum hydrodynamical equations

$$\partial_t\rho + \nabla\cdot(\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) = \frac{\varepsilon^2}{4}\nabla\cdot\big[\rho\nabla^2\log\rho\big],$$

which are exactly the fluid formulation of the defocusing nonlinear Schrödinger equation (1.4). In this case the relativistic part energy density E_K vanishes and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^2}{8}\frac{|\nabla\rho|^2}{\rho} + V(\rho),$$

and will satisfy

$$\partial_t E + \nabla \cdot \big(\big(E + P(\rho)\big)u\big) = \frac{\varepsilon^2}{4}\nabla \cdot \left[(\rho u)\frac{\Delta\rho}{\rho} - \nabla \cdot (\rho u)\frac{\nabla\rho}{\rho}\right].$$

Next letting $\nu \to 0$ and $\varepsilon \to 0$ simultaneously, both the relativistic and quantum correction terms in (2.4)–(2.5) vanish and the limit densities ρ, u and P will satisfy the compressible Euler equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = 0,$$

and the limit energy density E will be given by

$$E = \frac{1}{2}\rho|u|^2 + V(\rho),$$

and will satisfy

$$\partial_t E + \nabla \cdot \big(\big(E + P(\rho)\big)u\big) = 0$$

hence playing the role of a Lax entropy for the Euler system.

In order to investigate the incompressible limit, we introduce the scaling

$$\tilde{t} = \varepsilon^\alpha t, \qquad \tilde{x} = x, \quad \alpha > 0.$$

After dropping the tilde, the modulated nonlinear Klein–Gordon equation (1.3) becomes

$$i\varepsilon^{1+\alpha}\partial_t\psi - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi + \frac{\varepsilon^2}{2}\Delta\psi - V'\big(|\psi|^2\big)\psi = 0.$$

For this model the corresponding fluid dynamics equations (2.4)–(2.5) turn out to be

$$\partial_t(\rho - \rho_K) + \nabla \cdot (\rho u) = 0, \tag{2.6}$$

$$\partial_t\left(\rho u - \rho_K u + \frac{\varepsilon^2\nu^2}{4}\rho\nabla\partial_t \log\rho\right) + \nabla \cdot (\rho u \otimes u) + \frac{1}{\varepsilon^{2\alpha}}\nabla P(\rho) = \frac{\varepsilon^{2-2\alpha}}{4}\nabla \cdot \big(\rho\nabla^2 \log\rho\big), \tag{2.7}$$

and the associated energy equation becomes

$$\partial_t(E_S + E_K) + \nabla \cdot \left(\left(E_S + \frac{P(\rho)}{\varepsilon^{2\alpha}}\right)u\right) = \frac{\varepsilon^{2-2\alpha}}{4}\nabla \cdot \left[(\rho u)\frac{\Delta\rho}{\rho} - \nabla \cdot (\rho u)\frac{\nabla\rho}{\rho}\right],$$

where the Schrödinger part energy density E_S and relativistic part energy density E_K are given respectively by

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{\varepsilon^{2\alpha}}V(\rho),$$
$$E_K = \frac{1}{2\nu^2\varepsilon^{2\alpha}}\frac{\rho_K^2}{\rho} + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho}.$$

It follows immediately from the energy equation that

$$\int \frac{1}{2\nu^2\varepsilon^{2\alpha}}\frac{\rho_K^2}{\rho} + \frac{\varepsilon^2\nu^2}{8}\frac{|\partial_t\rho|^2}{\rho} + \frac{1}{2}\rho|u|^2 + \frac{\varepsilon^{2-2\alpha}}{8}\frac{|\nabla\rho|^2}{\rho} + \frac{V(\rho)}{\varepsilon^{2\alpha}}dx \leqslant C \tag{2.8}$$

for all $0 < t < \infty$ if the initial energy is bounded. Assuming the minimum of the convex function $V(\rho)$ occurs at $\rho = 1$ then the energy bound (2.8) implies $\rho \to 1$ and $\rho_K \to 0$ as $\varepsilon \to 0$. Since the density ρ goes to 1, we expect that Eq. (2.6) yields the limit: $\nabla \cdot u = 0$. And writing $\nabla P(\rho) = \nabla(P(\rho) - P(1))$, we deduce from (2.7) that

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla \widetilde{P} = 0,$$

where $\widetilde{P}$ is the limit of $\frac{P(\rho)-P(1)}{\varepsilon^{2\alpha}}$. In other words, we recover the incompressible Euler equations, for which we refer to [16].

3. Compressible Euler equations

The first result we shall prove rigorously in this paper is the convergence towards the compressible Euler equations. In fact, we consider the so-called nonrelativistic-semiclassical limit, i.e. $\nu \to 0$ and $\varepsilon \to 0$ simultaneously. In order to avoid carrying out a double limits the parameters ν and ε must be related. For convenience we set $\nu = \varepsilon^\kappa$ for some $\kappa > 0$, $0 < \varepsilon \ll 1$ and assume the potential energy $V'(|\psi^\varepsilon|^2) = |\psi^\varepsilon|^{2(\gamma-1)}$. Indeed we consider the modulated nonlinear Klein–Gordon equation

$$i\varepsilon\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{2+2\kappa}\partial_t^2\psi^\varepsilon + \frac{1}{2}\varepsilon^2\Delta\psi^\varepsilon - \left|\psi^\varepsilon\right|^{2(\gamma-1)}\psi^\varepsilon = 0, \tag{3.1}$$

supplemented with the initial conditions:

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x), \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x), \quad x \in \Omega, \tag{3.2}$$

satisfying

$$\int_{\mathbb{T}^n} \frac{1}{2}\varepsilon^{2+2\kappa}\left|\psi_1^\varepsilon\right|^2 + \frac{1}{2}\varepsilon^2\left|\nabla\psi_0^\varepsilon\right|^2 + \frac{1}{\gamma}\left|\psi_0^\varepsilon\right|^{2\gamma}dx \leqslant C. \tag{3.3}$$

Here and below, C denotes various positive constants independent of ε. To avoid the complications at the boundary, we concentrate below on the case where $x \in \Omega = \mathbb{T}^n$, the n-dimensional torus.

Associated with (3.1) are the local conservation laws corresponding to charge, momentum (current) and energy conservation. In fact, we have the hydrodynamical variables: Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum (current) J_K^ε and energy e^ε given as follows:

$$\begin{aligned}
&\rho_S^\varepsilon = \left|\psi^\varepsilon\right|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\varepsilon^{1+2\kappa}\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right),\\
&J_S^\varepsilon = \left(J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, \ldots, J_{S,n}^\varepsilon\right) = \frac{i}{2}\varepsilon\left(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right),\\
&J_K^\varepsilon = \left(J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, \ldots, J_{K,n}^\varepsilon\right) = \frac{1}{2}\varepsilon^{2+2\kappa}\left(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right),\\
&e^\varepsilon = \frac{1}{2}\varepsilon^{2+2\kappa}\left|\partial_t\psi^\varepsilon\right|^2 + \frac{1}{2}\varepsilon^2\left|\nabla\psi^\varepsilon\right|^2 + \frac{1}{\gamma}\left|\psi^\varepsilon\right|^{2\gamma}.
\end{aligned} \tag{3.4}$$

The local conservation laws of the modulated Klein–Gordon equation (3.1) are the charge, momentum (current) and energy given below:

(A) Conservation of charge

$$\frac{\partial}{\partial t}\left(\rho_S^\varepsilon - \rho_K^\varepsilon\right) + \nabla \cdot J_S^\varepsilon = 0, \tag{3.5}$$

(B) Conservation of momentum (current)

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(J_S^\varepsilon - J_K^\varepsilon\right) + \frac{1}{4}\varepsilon^2\nabla\cdot\left[2\left(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\right) - \nabla^2\left(\left|\psi^\varepsilon\right|^2\right)\right]\\
&\quad + \frac{1}{4}\varepsilon^{2+2\kappa}\nabla\partial_t\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right) + \frac{\gamma-1}{\gamma}\nabla\left|\psi^\varepsilon\right|^{2\gamma} = 0,
\end{aligned} \tag{3.6}$$

(C) Conservation of energy

$$\frac{\partial}{\partial t}e^\varepsilon - \nabla\cdot\left[\frac{1}{2}\varepsilon^2\left(\nabla\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right)\right] = 0. \tag{3.7}$$

They play the crucial role of the hydrodynamics limits. Leu us state the main theorem of this section first.

Theorem 3.1. *Let $\gamma > 1$ for even spatial dimension n and $\gamma \geqslant \frac{n}{n-1}$ for odd n. Let ψ^ε be the solution of the modulated nonlinear Klein–Gordon equations (3.1)–(3.2) and the initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2}+1$, satisfying (3.3) and (3.11). Then there exists $T_* > 0$ such that*

$$\big\| \big(\rho_S^\varepsilon - \rho\big)(\cdot,t)\big\|_{L^\gamma(\mathbb{T}^n)} \to 0, \qquad \big\|\rho_K^\varepsilon(\cdot,t)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0,$$
$$\big\| \big(J_S^\varepsilon - \rho u\big)(\cdot,t)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0, \qquad \big\|J_K^\varepsilon(\cdot,t)\big\|_{L^1(\mathbb{T}^n)} \to 0,$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$, where $(\rho, u) \in C([0,T_*); H^s(\mathbb{T}^n))$ is the unique local smooth solution of the γ-law compressible Euler equations

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \qquad x\in\mathbb{T}^n,\ t\in[0,T_*), \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) = 0, \\ \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x),\ x\in\mathbb{T}^n, \end{cases} \tag{3.8}$$

where $0 < \rho_0 \in H^s(\mathbb{T}^n)$, $u_0 \in H^s(\mathbb{T}^n)$ and the equation of states is given by $P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$.

Motivated by Brenier's pioneer work [1], we will prove this theorem by modulated energy. It is easy to see that when the parameter ε is small, the wave function ψ^ε and hydrodynamic variables ρ, u are related according to

$$\big|\psi^\varepsilon\big|^2 = \rho_S^\varepsilon \approx \rho, \qquad \frac{i\varepsilon}{2}\frac{1}{|\psi^\varepsilon|^2}\big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\big) \approx u.$$

The symbol "$A \approx B$" means that A almost equals B. Moreover, as ε tends to zero, the limiting energy will be $\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma}\rho^\gamma$. Keeping this term in mind and comparing with the energy of the modulated nonlinear Klein–Gordon equation (3.1), we have:

$$\frac{1}{2}\varepsilon^2\big|\nabla\psi^\varepsilon\big|^2 \approx \frac{1}{2}\rho|u|^2, \qquad \frac{1}{2}\varepsilon^{2+2\kappa}\big|\partial_t\psi^\varepsilon\big|^2 \approx 0, \qquad \frac{1}{\gamma}\big(\rho_S^\varepsilon\big)^\gamma \approx \frac{1}{\gamma}\rho^\gamma + \rho^{\gamma-1}\big(\rho_S^\varepsilon - \rho\big).$$

Thus, we have the relation

$$\begin{aligned} \frac{1}{2}\varepsilon^2\big|\nabla\psi^\varepsilon\big|^2 - \frac{1}{2}\rho|u|^2 &\approx \frac{\varepsilon^2}{2}\big(\big|\nabla\psi^\varepsilon\big|^2 - 2\varepsilon^{-2}\big|\psi^\varepsilon\big|^2|u|^2 + \varepsilon^{-2}\big|\psi^\varepsilon\big|^2|u|^2\big) \\ &\approx \frac{\varepsilon^2}{2}\big(\big|\nabla\psi^\varepsilon\big|^2 - i\varepsilon^{-1}\big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\big)\cdot u + \varepsilon^{-2}\big|\psi^\varepsilon\big|^2|u|^2\big) \\ &= \frac{1}{2}\big|(\varepsilon\nabla - iu)\psi^\varepsilon\big|^2. \end{aligned}$$

Therefore we can define the modulated energy of (3.1) as

$$H^\varepsilon(t) = \frac{1}{2}\int\limits_{\mathbb{T}^n}\big|(\varepsilon\nabla - iu)\psi^\varepsilon\big|^2\,dx + \frac{1}{2}\int\limits_{\mathbb{T}^n}\big|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\big|^2\,dx + \int\limits_{\mathbb{T}^n}\Theta\big(\rho_S^\varepsilon,\rho\big)\,dx \tag{3.9}$$

where

$$\Theta\big(\rho_S^\varepsilon,\rho\big) = \frac{1}{\gamma}\big(\big(\rho_S^\varepsilon\big)^\gamma - \rho^\gamma\big) - \rho^{\gamma-1}\big(\rho_S^\varepsilon - \rho\big) \tag{3.10}$$

is a convex function, minimum occurs at $\rho_S^\varepsilon = \rho$ and satisfies $\Theta(\rho_S^\varepsilon) \geqslant 0$. We also assume

$$\begin{aligned} H^\varepsilon(0) = \frac{1}{2}\int\limits_{\mathbb{T}^n}\big|(\varepsilon\nabla - iu_0)\psi_0^\varepsilon\big|^2\,dx + \frac{1}{2}\int\limits_{\mathbb{T}^n}\big|\varepsilon^{1+\kappa}\psi_1^\varepsilon\big|^2\,dx \\ + \int\limits_{\mathbb{T}^n}\Theta\big(\big|\psi_0^\varepsilon\big|^2,\rho_0\big)\,dx = O\big(\varepsilon^\beta\big), \quad \text{for some } \beta > 0, \end{aligned} \tag{3.11}$$

i.e., we consider the well-prepared initial data. We can rewrite the modulated energy (3.9) in terms of hydrodynamical variables only as

$$H^\varepsilon(t) = \int_{\mathbb{T}^n} e^\varepsilon\,dx - \int_{\mathbb{T}^n} u\cdot J_S^\varepsilon\,dx + \frac{1}{2}\int_{\mathbb{T}^n} \rho_S^\varepsilon|u|^2\,dx + \int_{\mathbb{T}^n}\left(\frac{\gamma-1}{\gamma}\rho - \rho_S^\varepsilon\right)\rho^{\gamma-1}\,dx. \tag{3.12}$$

Therefore to obtain the hydrodynamic limit we have to show that the modulated energy $H^\varepsilon(t)$ tends to zero as $\varepsilon\to 0$. Indeed, we have the following estimate.

Lemma 3.2. *Under the hypothesis of Theorem 3.1 and let $\lambda = \min\{1,\kappa,\beta\}$, we have*

$$H^\varepsilon(t) \leqslant O\big(\varepsilon^\lambda\big) \quad \text{uniformly in } t\in[0,T].$$

Proof. We have to check the evolution of the modulated energy $H^\varepsilon(t)$ given by (3.12). Differentiating the modulate energy H^ε with respect to time variable t and using the conservation of energy (3.7), we obtain

$$\frac{d}{dt}H^\varepsilon(t) = -\frac{d}{dt}\int_{\mathbb{T}^n} u\cdot J_S^\varepsilon\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}\rho_S^\varepsilon|u|^2\,dx + \frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{\gamma-1}{\gamma}\rho - \rho_S^\varepsilon\right)\rho^{\gamma-1}\,dx. \tag{3.13}$$

We discuss the right-hand side of (3.13) separately. Integration by parts and using conservation of momentum (3.6), the first term of the right-hand side of (3.13) becomes

$$\begin{aligned}
-\frac{d}{dt}\int_{\mathbb{T}^n} u\cdot J_S^\varepsilon\,dx ={}& -\int_{\mathbb{T}^n}\partial_t u\cdot J_S^\varepsilon\,dx - \int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}\big(\rho_S^\varepsilon\big)^\gamma\nabla\cdot u\,dx\\
&- \frac{\varepsilon^2}{4}\int_{\mathbb{T}^n} 2\big(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\big):\nabla u + \nabla\big|\psi^\varepsilon\big|^2\cdot(\nabla\nabla\cdot u)\,dx\\
&- \frac{1}{4}\varepsilon^{2+2\kappa}\frac{d}{dt}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^\varepsilon\big|^2\big)\nabla\cdot u\,dx - \frac{d}{dt}\int_{\mathbb{T}^n} u\cdot J_K^\varepsilon\,dx\\
&+ \frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^\varepsilon\big|^2\big)\nabla\cdot\partial_t u\,dx + \int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon\,dx.
\end{aligned} \tag{3.14}$$

Next, by conservation of charge (3.5) and integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}\rho_S^\varepsilon|u|^2\,dx ={}& \int_{\mathbb{T}^n}\rho_S^\varepsilon u\cdot\partial_t u\,dx + \frac{1}{2}\int_{\mathbb{T}^n}\nabla|u|^2\cdot J_S^\varepsilon\,dx\\
&+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}\rho_K^\varepsilon|u|^2\,dx - \int_{\mathbb{T}^n}\rho_K^\varepsilon u\cdot\partial_t u\,dx.
\end{aligned} \tag{3.15}$$

The third term of the right-hand side of (3.13) becomes

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^n}\left(\frac{\gamma-1}{\gamma}\rho - \rho_S^\varepsilon\right)\rho^{\gamma-1}\,dx ={}& \int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-2}\big(\rho-\rho_S^\varepsilon\big)\partial_t\rho\,dx - \frac{d}{dt}\int_{\mathbb{T}^n}\rho^{\gamma-1}\rho_K^\varepsilon\,dx\\
&+ \int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon - \nabla\rho^{\gamma-1}\cdot J_S^\varepsilon\,dx.
\end{aligned} \tag{3.16}$$

From (3.14)–(3.16) we define the correction term of the modulated energy H^ε as

$$G^\varepsilon(t) = -\frac{1}{2}\int_{\mathbb{T}^n}|u|^2\rho_K^\varepsilon\,dx + \int_{\mathbb{T}^n}\rho^{\gamma-1}\rho_K^\varepsilon\,dx + \frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^\varepsilon\big|^2\big)\nabla\cdot u\,dx + \int_{\mathbb{T}^n} u\cdot J_K^\varepsilon\,dx.$$

It is designed to control the propagation of the relativistic charge and current and will be proved to be small as $\varepsilon \to 0$. Using crucially the limit compressible Euler equations (3.8), we have

$$\begin{aligned}\frac{d}{dt}\big(H^\varepsilon(t)+G^\varepsilon(t)\big) &= \int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot(u\cdot\nabla u)\,dx+\frac{1}{2}\int_{\mathbb{T}^n} J_S^\varepsilon\cdot\nabla|u|^2\,dx\\ &\quad-\int_{\mathbb{T}^n}\bigg[\frac{\gamma-1}{\gamma}\big(\rho_S^\varepsilon\big)^\gamma-(\gamma-1)\rho^{\gamma-1}\rho_S^\varepsilon\bigg]\nabla\cdot u\,dx\\ &\quad-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n}\nabla\big|\psi^\varepsilon\big|^2\cdot(\nabla\nabla\cdot u)\,dx+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^\varepsilon\big|^2\big)\nabla\cdot\partial_t u\,dx\\ &\quad-\int_{\mathbb{T}^n}u\cdot\partial_t u\rho_K^\varepsilon\,dx+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^\varepsilon\,dx+\int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon\,dx+R_1+R_2,\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}R_1&=-\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n}\big(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon}+\nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\big):\nabla u\,dx,\\ R_2&=-\int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)\,dx.\end{aligned}$$

To deal with R_1, we can rewrite R_1 as

$$\begin{aligned}R_1=-\frac{1}{2}\int_{\mathbb{T}^n}&\big((\varepsilon\nabla-iu)\psi^\varepsilon\otimes\overline{(\varepsilon\nabla-iu)\psi^\varepsilon}\\ &+\overline{(\varepsilon\nabla-iu)\psi^\varepsilon}\otimes(\varepsilon\nabla-iu)\psi^\varepsilon\big):\nabla u\,dx+K_1+K_2+K_3,\end{aligned}$$

where

$$\begin{aligned}K_1&=\frac{\varepsilon}{2}\int_{\mathbb{T}^n}\big((-iu)\psi^\varepsilon\otimes\overline{\nabla\psi^\varepsilon}+\overline{(-iu)\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\big):\nabla u\,dx,\\ K_2&=\frac{1}{2}\int_{\mathbb{T}^n}\big((-iu)\psi^\varepsilon\otimes\overline{(-iu)\psi^\varepsilon}+\overline{(-iu)\psi^\varepsilon}\otimes(-iu)\psi^\varepsilon\big):\nabla u\,dx,\\ K_3&=\frac{\varepsilon}{2}\int_{\mathbb{T}^n}\big(\nabla\psi^\varepsilon\otimes\overline{(-iu)\psi^\varepsilon}+\overline{\nabla\psi^\varepsilon}\otimes(-iu)\psi^\varepsilon\big):\nabla u\,dx.\end{aligned}$$

Using integration by part and go back to the hydrodynamic variables (3.4), one can calculate that

$$K_1=-\int_{\mathbb{T}^n}\big(u\otimes J_S^\varepsilon\big):\nabla u\,dx=\int_{\mathbb{T}^n}\frac{1}{2}|u|^2\nabla\cdot J_S^\varepsilon\,dx,$$

and

$$K_2+K_3=\int_{\mathbb{T}^n}\big(\rho_S^\varepsilon u\otimes u\big):\nabla u\,dx-\int_{\mathbb{T}^n}\big(J_S^\varepsilon\otimes u\big):\nabla u\,dx=\int_{\mathbb{T}^n}\big[(u\cdot\nabla)u\big]\cdot\big(\rho_S^\varepsilon u-J_S^\varepsilon\big)\,dx,$$

i.e.

$$\begin{aligned}R_1&=-\frac{1}{2}\int_{\mathbb{T}^n}\big((\varepsilon\nabla-iu)\psi^\varepsilon\otimes\overline{(\varepsilon\nabla-iu)\psi^\varepsilon}+\overline{(\varepsilon\nabla-iu)\psi^\varepsilon}\otimes(\varepsilon\nabla-iu)\psi^\varepsilon\big):\nabla u\,dx\\ &\quad+\int_{\mathbb{T}^n}\frac{1}{2}|u|^2\nabla\cdot J_S^\varepsilon\,dx+\int_{\mathbb{T}^n}\big[(u\cdot\nabla)u\big]\cdot\big(\rho_S^\varepsilon u-J_S^\varepsilon\big)\,dx.\end{aligned}\tag{3.18}$$

Also using the identity

$$(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)=\frac{\gamma-1}{\gamma}\big(\nabla\rho^{\gamma}\big)\cdot u+(\gamma-1)\rho^{\gamma}\nabla\cdot u,$$

we have

$$R_2=-\int_{\mathbb{T}^n}(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)\,dx=-\frac{(\gamma-1)^2}{\gamma}\int_{\mathbb{T}^n}\rho^{\gamma}\nabla\cdot u\,dx. \tag{3.19}$$

Employing (3.18) and (3.19), we can rewrite (3.17) as

$$\begin{aligned}\frac{d}{dt}\big(H^{\varepsilon}(t)+G^{\varepsilon}(t)\big)={}&-\frac{1}{2}\int_{\mathbb{T}^n}\big((\varepsilon\nabla-iu)\psi^{\varepsilon}\otimes\overline{(\varepsilon\nabla-iu)\psi^{\varepsilon}}+\overline{(\varepsilon\nabla-iu)\psi^{\varepsilon}}\otimes(\varepsilon\nabla-iu)\psi^{\varepsilon}\big):\nabla u\,dx\\&-(\gamma-1)\int_{\mathbb{T}^n}\bigg[\frac{1}{\gamma}\big(\big(\rho_S^{\varepsilon}\big)^{\gamma}-\rho^{\gamma}\big)-\rho^{\gamma-1}\big(\rho_S^{\varepsilon}-\rho\big)\bigg]\nabla\cdot u\,dx\\&-\frac{\varepsilon^2}{4}\int_{\mathbb{T}^n}\nabla\big|\psi^{\varepsilon}\big|^2\cdot(\nabla\nabla\cdot u)\,dx-\int_{\mathbb{T}^n}u\cdot\partial_t u\rho_K^{\varepsilon}\,dx+\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^{\varepsilon}\,dx\\&+\frac{1}{4}\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^{\varepsilon}\big|^2\big)\nabla\cdot\partial_t u\,dx+\int_{\mathbb{T}^n}\partial_t u\cdot J_K^{\varepsilon}\,dx.\end{aligned} \tag{3.20}$$

One can estimate the first term of the right-hand side of (3.20) as follows

$$\begin{aligned}\big|\nabla u:(\varepsilon\nabla-iu)\psi^{\varepsilon}\otimes\overline{(\varepsilon\nabla-iu)\psi^{\varepsilon}}\big|&\leqslant\|\nabla u\|_{L^{\infty}(\mathbb{T}^n)}\sum_{j,\ell=1}^{n}\big|(\varepsilon\partial_j-iu_j)\psi^{\varepsilon}\overline{(\varepsilon\partial_{\ell}-iu_{\ell})\psi^{\varepsilon}}\big|\\&\leqslant n\|\nabla u\|_{L^{\infty}(\mathbb{T}^n)}\big|(\varepsilon\nabla-iu)\psi^{\varepsilon}\big|^2.\end{aligned}$$

Furthermore, for $t\in[0,T_*)$, by (3.3), (3.5) and (3.7) we have

$$\big\|\varepsilon\nabla\psi^{\varepsilon}\big\|_{L^2(\mathbb{T}^n)}=\big\|\varepsilon^{1+\kappa}\partial_t\psi^{\varepsilon}\big\|_{L^2(\mathbb{T}^n)}=O(1) \tag{3.21}$$

and

$$\big\|\psi^{\varepsilon}\big\|_{L^q(\mathbb{T}^n)}=O(1),\quad 2\leqslant q\leqslant 2\gamma. \tag{3.22}$$

Then by Hölder inequality we have the following estimates

$$\varepsilon^2\int_{\mathbb{T}^n}\nabla\big|\psi^{\varepsilon}\big|^2\cdot(\nabla\nabla\cdot u)\,dx\leqslant\varepsilon\big\|\varepsilon\nabla\psi^{\varepsilon}\big\|_{L^2(\mathbb{T}^n)}\big\|\psi^{\varepsilon}\big\|_{L^{2\gamma}(\mathbb{T}^n)}\|\nabla\nabla\cdot u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)}\lesssim\varepsilon\|u\|_{H^s(\mathbb{T}^n)}, \tag{3.23}$$

and

$$\begin{aligned}\int_{\mathbb{T}^n}\rho_K^{\varepsilon}u\cdot\partial_t u\,dx&\leqslant\varepsilon^{\kappa}\|u\cdot\partial_t u\|_{L^{\infty}(\mathbb{T}^n)}\big\|\psi^{\varepsilon}\big\|_{L^2(\mathbb{T}^n)}\big\|\varepsilon^{1+\kappa}\partial_t\psi^{\varepsilon}\big\|_{L^2(\mathbb{T}^n)}\\&\lesssim\varepsilon^{\kappa}\|u\cdot\partial_t u\|_{L^{\infty}(\mathbb{T}^n)}\lesssim\varepsilon^{\kappa}\|u\|_{H^s(\mathbb{T}^n)}^2.\end{aligned} \tag{3.24}$$

Similar to (3.23)–(3.24), we also have

$$\int_{\mathbb{T}^n}\partial_t\rho^{\gamma-1}\rho_K^{\varepsilon}\,dx\lesssim\varepsilon^{\kappa}\|\rho\|_{H^s(\mathbb{T}^n)}^{\gamma-1}, \tag{3.25}$$

$$\varepsilon^{2+2\kappa}\int_{\mathbb{T}^n}\big(\partial_t\big|\psi^{\varepsilon}\big|^2\big)\nabla\cdot\partial_t u\,dx\lesssim\varepsilon^{1+\kappa}\|u\|_{H^s(\mathbb{T}^n)}, \tag{3.26}$$

$$\int_{\mathbb{T}^n}\partial_t u\cdot J_K^{\varepsilon}\,dx\lesssim\varepsilon^{\kappa}\|u\|_{H^s(\mathbb{T}^n)}. \tag{3.27}$$

Combing the above estimates we obtain the inequality

$$\frac{d}{dt}\big(H^\varepsilon(t)+G^\varepsilon(t)\big)\lesssim \|\nabla u\|_{L^\infty(\mathbb{T}^n)}H^\varepsilon(t)+\varepsilon^\delta\big(\|u\|_{H^s(\mathbb{T}^n)}+\|u\|^2_{H^s(\mathbb{T}^n)}+\|\rho\|^{\gamma-1}_{H^s(\mathbb{T}^n)}\big) \tag{3.28}$$

for $t\in[0,T_*)$ and $\delta=\min\{1,\kappa\}$. Integrating (3.28) with respect to time variable t yields

$$H^\varepsilon(t)\leqslant H^\varepsilon(0)+G^\varepsilon(0)-G^\varepsilon(t)+C_1\int_0^t H^\varepsilon(\tau)\,d\tau+C_2\varepsilon^\delta t.$$

Similar to (3.23)–(3.27) one can show that $G^\varepsilon(0)-G^\varepsilon(t)=O(\varepsilon^\kappa)$; and hence

$$H^\varepsilon(t)\leqslant C_1\int_0^t H^\varepsilon(\tau)\,d\tau+H^\varepsilon(0)+C_2\varepsilon^\delta t+C_3\varepsilon^\kappa.$$

Employing the initial condition $H^\varepsilon(0)$ and the Gronwall inequality we derive

$$H^\varepsilon(t)\leqslant\big(C_4\varepsilon^\beta+C_2\varepsilon^\delta t+C_3\varepsilon^\kappa\big)\big(1+C_1te^{C_1t}\big).$$

This shows $H^\varepsilon(t)\leqslant O(\varepsilon^\lambda)$ for $t\in[0,T_*)$, where $\lambda=\min\{1,\kappa,\beta\}$. □

It is easy to check that the modulated energy can be rewritten as

$$H^\varepsilon(t)=\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n}\big|\nabla\sqrt{\rho_S^\varepsilon}\big|^2\,dx+\frac12\int_{\mathbb{T}^n}\bigg|\frac{1}{\sqrt{\rho_S^\varepsilon}}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\bigg|^2\,dx+\frac12\int_{\mathbb{T}^n}\big|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\big|^2\,dx+\int_{\mathbb{T}^n}\Theta\big(\rho_S^\varepsilon,\rho\big)\,dx. \tag{3.29}$$

Using (3.29) and Lemma 3.2, we have

$$\int_{\mathbb{T}^n}\bigg|\frac{1}{\sqrt{\rho_S^\varepsilon}}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\bigg|^2\,dx\to0,\qquad\int_{\mathbb{T}^n}\Theta\big(\rho_S^\varepsilon,\rho\big)\,dx\to0 \tag{3.30}$$

as $\varepsilon\to0$. Also the elementary computation shows that ([17])

$$\frac1\gamma\big|\rho_S^\varepsilon-\rho\big|^\gamma\leqslant\Theta\big(\rho_S^\varepsilon,\rho\big)\quad\text{if }\gamma\geqslant2; \tag{3.31}$$

$$\big|\rho_S^\varepsilon-\rho\big|^2\lesssim\rho^{2-\gamma}\Theta\big(\rho_S^\varepsilon,\rho\big)\quad\text{if }1<\gamma<2\text{ and }\rho_S^\varepsilon\leqslant2\rho; \tag{3.32}$$

$$\big|\rho_S^\varepsilon-\rho\big|^\gamma\lesssim\Theta\big(\rho_S^\varepsilon,\rho\big)\quad\text{if }1<\gamma<2\text{ and }\rho_S^\varepsilon\geqslant2\rho; \tag{3.33}$$

and hence $\|\rho_S^\varepsilon-\rho\|_{L^\gamma(\mathbb{T}^n)}\to0$ as $\varepsilon\to0$. On the other hand, applying the triangle and Hölder inequalities we have

$$\begin{aligned}\big\|\big(J_S^\varepsilon-\rho u\big)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)}&\leqslant\big\|\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)}+\big\|\big(\rho_S^\varepsilon-\rho\big)u\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)}\\&\leqslant\big\|\sqrt{\rho_S^\varepsilon}\big\|_{L^{2\gamma}(\mathbb{T}^n)}\bigg\|\frac{1}{\sqrt{\rho_S^\varepsilon}}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\bigg\|_{L^2(\mathbb{T}^n)}+\big\|\rho_S^\varepsilon-\rho\big\|_{L^\gamma(\mathbb{T}^n)}\|u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)}\end{aligned}$$

which converges to zero as $\varepsilon\to0$ by (3.30)–(3.33). Combing (3.21) and (3.22) we have

$$\big\|\rho_K^\varepsilon(\cdot,t)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)}\lesssim\varepsilon^\kappa\big\|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\big\|_{L^2(\mathbb{T}^n)}\big\|\psi^\varepsilon\big\|_{L^{2\gamma}(\mathbb{T}^n)}\to0,$$

and

$$\big\|J_K^\varepsilon(\cdot,t)\big\|_{L^1(\mathbb{T}^n)}\lesssim\varepsilon^\kappa\big\|\varepsilon^{1+\kappa}\partial_t\psi^\varepsilon\big\|_{L^2(\mathbb{T}^n)}\big\|\varepsilon\nabla\psi^\varepsilon\big\|_{L^2(\mathbb{T}^n)}\to0$$

as $\varepsilon\to0$. This completes the proof of Theorem 3.1. □

Remark. The main reason why the adiabatic exponent γ depending on either spacial dimension is even or odd comes from the estimates (3.23) and (3.26) given respectively by,

$$\|\nabla\nabla\cdot u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} \lesssim \|u\|_{H^s(\mathbb{T}^n)}, \qquad \|\nabla\cdot\partial_t u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)} \lesssim \|u\|_{H^s(\mathbb{T}^n)}.$$

Since $\nabla\nabla\cdot u \in H^{s-2}(\mathbb{T}^n)$, $s-2 > \frac{n}{2}-1$, we have $\nabla\nabla\cdot u \in H^{n/2}(\mathbb{T}^n)$ for even spatial dimension n, thus $\nabla\nabla\cdot u \in L^p(\mathbb{T}^n)$ for any $1 \leqslant p < \infty$ by the Sobolev inequality, so we conclude $\gamma > 1$ for even spatial dimensions. On the other hand, for odd n, $\nabla\nabla\cdot u \in H^{(n-1)/2}(\mathbb{T}^n)$, the Sobolev inequality implies $\nabla\nabla\cdot u \in L^p(\mathbb{T}^n)$, $1 \leqslant p \leqslant 2n$, so we need $\frac{2\gamma}{\gamma-1} \leqslant 2n$, and hence $\gamma \geqslant \frac{n}{n-1}$ for odd spatial dimensions. The argument for the estimate of $\nabla\cdot\partial_t u$ is similar.

4. Incompressible Euler equations

The second result we want to address in this paper concerns the convergence towards the incompressible Euler equations. We still consider only the n-dimensional torus $\mathbb{T}^n$ as discussed in the previous section. To obtain the incompressible limit, the time variable need to be rescaled, $t \to \varepsilon^\alpha t$, $\alpha > 0$, and potential energy is given by $V'(|\psi^\varepsilon|^2) = (|\psi^\varepsilon|^\gamma - 1)|\psi^\varepsilon|^{\gamma-2}$, $\gamma \geqslant 2$. More precisely, we will investigate the time-scaled modulated nonlinear Klein–Gordon equation

$$i\varepsilon^{1+\alpha}\partial_t\psi^\varepsilon - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - \left(\left|\psi^\varepsilon\right|^\gamma - 1\right)\left|\psi^\varepsilon\right|^{\gamma-2}\psi^\varepsilon = 0, \tag{4.1}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x), \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x), \quad x \in \mathbb{T}^n,$$

satisfying the uniform bound

$$\int_{\mathbb{T}^n} \frac{1}{2}\nu^2\varepsilon^2\left|\psi_1^\varepsilon\right|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}\left|\nabla\psi_0^\varepsilon\right|^2 + \frac{1}{\gamma\varepsilon^{2\alpha}}\left(\left|\psi_0^\varepsilon\right|^\gamma - 1\right)^2 dx < C. \tag{4.2}$$

We will consider the limit as the scaled Planck constant $\varepsilon \to 0$ and the parameter ν is kept fixed. To prove the incompressible limit of (4.1) we have to define the hydrodynamical variables; Schrödinger part charge ρ_S^ε, relativistic part charge ρ_K^ε, Schrödinger part momentum (current) J_S^ε, relativistic part momentum J_K^ε and energy e^ε as follows:

$$\begin{aligned}
&\rho_S^\varepsilon = \left|\psi^\varepsilon\right|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\nu^2\varepsilon^{1+\alpha}\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right),\\
&J_S^\varepsilon = \left(J_{S,1}^\varepsilon, J_{S,2}^\varepsilon, \ldots, J_{S,n}^\varepsilon\right) = \frac{i}{2}\varepsilon^{1-\alpha}\left(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right),\\
&J_K^\varepsilon = \left(J_{K,1}^\varepsilon, J_{K,2}^\varepsilon, \ldots, J_{K,n}^\varepsilon\right) = \frac{\nu^2\varepsilon^2}{2}\left(\partial_t\psi^\varepsilon\nabla\overline{\psi^\varepsilon} + \partial_t\overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right),\\
&e^\varepsilon = \frac{1}{2}\nu^2\varepsilon^2\left|\partial_t\psi^\varepsilon\right|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}\left|\nabla\psi^\varepsilon\right|^2 + \frac{1}{\gamma\varepsilon^{2\alpha}}\left(\left|\psi^\varepsilon\right|^\gamma - 1\right)^2.
\end{aligned}$$

The local conservation laws associated with the rescaled modulated nonlinear Klein–Gordon equation (4.1) are the charge, momentum and energy given respectively by:

(A) Conservation of charge

$$\frac{\partial}{\partial t}\left(\rho_S^\varepsilon - \rho_K^\varepsilon\right) + \nabla\cdot J_S^\varepsilon = 0, \tag{4.3}$$

(B) Conservation of momentum

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(J_S^\varepsilon - J_K^\varepsilon\right) + \frac{1}{4}\varepsilon^{2-2\alpha}\nabla\cdot\left[2\left(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon} + \nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\right) - \nabla^2\left(\left|\psi^\varepsilon\right|^2\right)\right]\\
&\quad + \frac{1}{4}\nu^2\varepsilon^2\nabla\partial_t\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} + \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right) + \frac{1}{\gamma\varepsilon^{2\alpha}}\nabla\left((\gamma-1)\left|\psi^\varepsilon\right|^{2\gamma} - (\gamma-2)\left|\psi^\varepsilon\right|^\gamma\right) = 0,
\end{aligned} \tag{4.4}$$

(C) Conservation of energy

$$\frac{\partial}{\partial t}e^{\varepsilon} - \nabla \cdot \left[\frac{1}{2}\varepsilon^{2-2\alpha}\left(\nabla\psi^{\varepsilon}\partial_t\overline{\psi^{\varepsilon}} + \nabla\overline{\psi^{\varepsilon}}\partial_t\psi^{\varepsilon}\right)\right] = 0. \tag{4.5}$$

We now state the main theorem of this section.

Theorem 4.1. *Let $\alpha > 0$, $\gamma \geqslant 2$ and ψ^{ε} be the solution of the time scale modulated nonlinear Klein–Gordon equation (4.1) with initial condition $(\psi_0^{\varepsilon}, \psi_1^{\varepsilon}) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 1$, satisfying (4.2) and (4.8). Then there exists $T_* > 0$ such that*

$$\left\|\left(\rho_S^{\varepsilon} - 1\right)(\cdot, t)\right\|_{L^{\gamma}(\mathbb{T}^n)} \to 0, \qquad \left\|\rho_K^{\varepsilon}(\cdot, t)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0,$$
$$\left\|\left(J_S^{\varepsilon} - \rho_S^{\varepsilon}u\right)(\cdot, t)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0, \qquad \left\|J_K^{\varepsilon}(\cdot, t)\right\|_{L^1(\mathbb{T}^n)} \to 0,$$

for $t \in [0, T_*)$ as $\varepsilon \downarrow 0$, where $u \in C([0, T_*); H^s(\mathbb{T}^n))$ is the unique local smooth solution of the incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = 0, & \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), & \nabla \cdot u_0 = 0. \end{cases} \tag{4.6}$$

Similar to the previous section, we define the modulated energy

$$H^{\varepsilon}(t) = \frac{1}{2}\int_{\mathbb{T}^n}\left|\left(\varepsilon^{1-\alpha}\nabla - iu\right)\psi^{\varepsilon}\right|^2 dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}\left|\partial_t\psi^{\varepsilon}\right|^2 dx \frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left(\left(\rho_S^{\varepsilon}\right)^{\frac{\gamma}{2}} - 1\right)^2 dx \tag{4.7}$$

which satisfies the well-prepared initial condition

$$H^{\varepsilon}(0) = \frac{1}{2}\int_{\mathbb{T}^n}\left|\left(\varepsilon^{1-\alpha}\nabla - iu_0\right)\psi_0^{\varepsilon}\right|^2 dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}\left|\psi_1^{\varepsilon}\right|^2 dx + \frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left(\left|\psi_0^{\varepsilon}\right|^{\gamma} - 1\right)^2 dx = O\left(\varepsilon^{\beta}\right), \tag{4.8}$$

for some $\beta > 0$. The modulated energy can be further rewritten in terms of the hydrodynamic variables as

$$H^{\varepsilon}(t) = \int_{\mathbb{T}^n} e^{\varepsilon}\, dx - \int_{\mathbb{T}^n} u \cdot J_S^{\varepsilon}\, dx + \frac{1}{2}\int_{\mathbb{T}^n} \rho_S^{\varepsilon}|u|^2\, dx. \tag{4.9}$$

Lemma 4.2. *Under the hypothesis of Theorem 4.1 and let $\lambda = \min\{\beta, \delta\}$, where $\delta = 2\alpha/\gamma$, we have*

$$H^{\varepsilon}(t) \leqslant O\left(\varepsilon^{\lambda}\right), \quad t \in [0, T].$$

Proof. Differentiating the modulated energy (4.9) with respect to t and using conservation of energy (4.5), we obtain

$$\frac{d}{dt}H^{\varepsilon}(t) = -\frac{d}{dt}\int_{\mathbb{T}^n} u \cdot J_S^{\varepsilon}\, dx + \frac{d}{dt}\int_{\mathbb{T}^n}\frac{1}{2}\rho_S^{\varepsilon}|u|^2\, dx \equiv I_1 + I_2.$$

By conservation of momentum (4.4), integration by part and using the fact that u is divergence free, we obtain

$$I_1 = -\int_{\mathbb{T}^n}\partial_t u \cdot \left(J_S^{\varepsilon} - J_K^{\varepsilon}\right) dx - \frac{d}{dt}\int_{\mathbb{T}^n} u \cdot J_K^{\varepsilon}\, dx - \frac{\varepsilon^{2-2\alpha}}{4}\int_{\mathbb{T}^n} 2\left(\nabla\psi^{\varepsilon} \otimes \nabla\overline{\psi^{\varepsilon}} + \nabla\overline{\psi^{\varepsilon}} \otimes \nabla\psi^{\varepsilon}\right) : \nabla u\, dx.$$

Next employing conservation of charge (4.3) and integration by part, we have

$$I_2 = \int_{\mathbb{T}^n}\rho_S^{\varepsilon}u \cdot \partial_t u\, dx + \int_{\mathbb{T}^n}\frac{1}{2}\nabla|u|^2 \cdot J_S^{\varepsilon}\, dx + \frac{d}{dt}\int_{\mathbb{T}^n}\frac{1}{2}\rho_K^{\varepsilon}|u|^2\, dx - \int_{\mathbb{T}^n}\rho_K^{\varepsilon}u \cdot \partial_t u\, dx.$$

As before we define the relativistic correction term of the modulation energy by

$$G^\varepsilon(t) = -\frac{1}{2}\int_{\mathbb{T}^n} \rho_K^\varepsilon |u|^2\,dx + \int_{\mathbb{T}^n} u\cdot J_K^\varepsilon\,dx,$$

then using crucially the incompressible Euler system (4.6), we have

$$\begin{aligned}\frac{d}{dt}\big(H^\varepsilon(t)+G^\varepsilon(t)\big) = {}& -\frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\big(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon}+\nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\big):\nabla u\,dx \\ & + \int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot(u\cdot\nabla u)\,dx + \int_{\mathbb{T}^n}\frac{1}{2}J_S^\varepsilon\cdot\nabla|u|^2\,dx \\ & + \int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot\nabla\pi\,dx - \int_{\mathbb{T}^n}\rho_K^\varepsilon u\cdot\partial_t u\,dx + \int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon\,dx. \end{aligned} \tag{4.10}$$

To deal with the first integral of the right-hand side of (4.10), we need the following equality

$$\begin{aligned} & -\frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\big(\nabla\psi^\varepsilon\otimes\nabla\overline{\psi^\varepsilon}+\nabla\overline{\psi^\varepsilon}\otimes\nabla\psi^\varepsilon\big):\nabla u\,dx \\ & = -\frac{1}{2}\int_{\mathbb{T}^n}\Big(\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon\otimes\overline{\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon} \\ & \quad + \overline{\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon}\otimes\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon\Big):\nabla u\,dx \\ & \quad + \int_{\mathbb{T}^n}\frac{1}{2}|u|^2\nabla\cdot J_S^\varepsilon\,dx + \int_{\mathbb{T}^n}\big[(u\cdot\nabla)u\big]\cdot\big(\rho_S^\varepsilon u - J_S^\varepsilon\big)\,dx. \end{aligned} \tag{4.11}$$

This equality similar to (3.18) as discussed in previous section. Combining (4.10) and (4.11), we have the equality

$$\begin{aligned}\frac{d}{dt}\big(H^\varepsilon(t)+G^\varepsilon(t)\big) = {}& -\frac{1}{2}\int_{\mathbb{T}^n}\Big(\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon\otimes\overline{\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon} \\ & + \overline{\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon}\otimes\big(\varepsilon^{1-\alpha}\nabla-iu\big)\psi^\varepsilon\Big):\nabla u\,dx \\ & + \int_{\mathbb{T}^n}\big(J_S^\varepsilon-\rho_S^\varepsilon u\big)\cdot\nabla\pi\,dx - \int_{\mathbb{T}^n}\rho_K^\varepsilon u\cdot\partial_t u\,dx + \int_{\mathbb{T}^n}\partial_t u\cdot J_K^\varepsilon\,dx. \end{aligned} \tag{4.12}$$

Now we will estimate the second, third and fourth integral of right side of (4.12) separately. By (4.2) and (4.5), we have for $t\in[0,T_*)$

$$\big\|\varepsilon^{1-\alpha}\nabla\psi^\varepsilon\big\|_{L^2(\mathbb{T}^n)} = \big\|\varepsilon\partial_t\psi^\varepsilon\big\|_{L^2(\mathbb{T}^n)} = O(1) \tag{4.13}$$

and

$$\big\|\big(\rho_S^\varepsilon\big)^{\frac{\gamma}{2}}-1\big\|_{L^2(\mathbb{T}^n)} = O\big(\varepsilon^\alpha\big). \tag{4.14}$$

We deduce from the inequality

$$\big|\rho_S^\varepsilon-1\big|^{\frac{\gamma}{2}} \leqslant \big|\big(\rho_S^\varepsilon\big)^{\frac{\gamma}{2}}-1\big|,$$

and (4.14) that

$$\big\|\rho_S^\varepsilon-1\big\|_{L^\gamma(\mathbb{T}^n)} = O\big(\varepsilon^{\frac{2\alpha}{\gamma}}\big). \tag{4.15}$$

Hence by (4.13), (4.14) and Hölder inequality, we arrive at the inequality

$$\int_{\mathbb{T}^n} \rho_S^\varepsilon (u\cdot\nabla\pi)\,dx = \int_{\mathbb{T}^n} \big(\rho_S^\varepsilon - 1\big)(u\cdot\nabla\pi) + u\cdot\nabla\pi\,dx$$
$$= \int_{\mathbb{T}^n} \big(\rho_S^\varepsilon - 1\big)(u\cdot\nabla\pi)\,dx \lesssim \varepsilon^{\frac{2\alpha}{\gamma}} \|u\cdot\nabla\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)}.$$

To go further, we need the relation

$$\int_{\mathbb{T}^n} J_S^\varepsilon\cdot\nabla\pi\,dx = \int_{\mathbb{T}^n} \pi\,\partial_t\big(\rho_S^\varepsilon - 1\big) - \pi\,\partial_t\rho_K^\varepsilon\,dx$$
$$= \frac{d}{dt}\int_{\mathbb{T}^n} \pi\big(\rho_S^\varepsilon - 1\big) - \rho_K^\varepsilon\pi\,dx - \int_{\mathbb{T}^n} \partial_t\pi\big(\rho_S^\varepsilon - 1\big) - \rho_K^\varepsilon\partial_t\pi\,dx. \tag{4.16}$$

The last integral of (4.16) can be estimated by Hölder inequality

$$\int_{\mathbb{T}^n} \partial_t\pi\big[\big(\rho_S^\varepsilon - 1\big) - \rho_K^\varepsilon\big]\,dx \lesssim \varepsilon^{\frac{2\alpha}{\gamma}} \|\partial_t\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \varepsilon^\alpha\|\partial_t\pi\|_{L^\infty(\mathbb{T}^n)},$$

and the estimates of the third and fourth integrals of the right-hand side of (4.12) are given respectively by

$$\int_{\mathbb{T}^n} \rho_K^\varepsilon u\cdot\partial_t u\,dx \lesssim \varepsilon^\alpha\|u\cdot\partial_t u\|_{L^\infty(\mathbb{T}^n)},$$

and

$$\int_{\mathbb{T}^n} \partial_t u\cdot J_K^\varepsilon\,dx \lesssim \varepsilon^\alpha\|\partial_t u\|_{L^\infty(\mathbb{T}^n)}.$$

To obtain the incompressible limit we have to introduce one more correction term of the modulated energy defined by

$$W^\varepsilon(t) = \int_{\mathbb{T}^n} \big[\rho_K^\varepsilon - \big(\rho_S^\varepsilon - 1\big)\big]\pi\,dx.$$

The correction term $W^\varepsilon(t)$ can be served as the acoustic part (density fluctuation) of the modulated energy $H^\varepsilon(t)$. It is designed to control the propagation of the acoustic wave. Hence for $t\in[0,T_*)$ we have

$$\frac{d}{dt}\big(H^\varepsilon(t) + G^\varepsilon(t) + W^\varepsilon(t)\big) \lesssim \|\nabla u\|_{L^\infty(\mathbb{T}^n)}H^\varepsilon(t)$$
$$+ \varepsilon^\delta\big(\|u\cdot\nabla\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \|\partial_t\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^n)} + \|\partial_t\pi\|_{L^\infty(\mathbb{T}^n)}$$
$$+ \|u\cdot\partial_t u\|_{L^\infty(\mathbb{T}^n)} + \|\partial_t u\|_{L^\infty(\mathbb{T}^n)}\big) \tag{4.17}$$

where $\delta = 2\alpha/\gamma$. Integrating (4.17) yields

$$H^\varepsilon(t) \leqslant H^\varepsilon(0) + G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) + C_1\int_0^t H^\varepsilon(\tau)\,d\tau + C_2\varepsilon^\delta t.$$

One can show that $G^\varepsilon(0) + W^\varepsilon(0) - G^\varepsilon(t) - W^\varepsilon(t) = O(\varepsilon^\delta)$, and hence

$$H^\varepsilon(t) \leqslant C_1\int_0^t H^\varepsilon(\tau)\,d\tau + H^\varepsilon(0) + C_2\varepsilon^\delta t + C_3\varepsilon^\delta.$$

Applying the Gronwall inequality and the decay rate of $H^\varepsilon(0)$ we derive the inequality

$$H^\varepsilon(t) \leqslant \big(C_4\varepsilon^\beta + C_2\varepsilon^\delta t + C_3\varepsilon^\delta\big)\big(1 + C_1te^{C_1t}\big).$$

Thus $H^\varepsilon(t) \leqslant O(\varepsilon^\lambda)$ for $t\in[0,T_*)$, where $\lambda = \min\{\beta,\delta\}$. □

It is easy to rewrite the modulated energy (4.7) as

$$H^\varepsilon(t) = \frac{\varepsilon^{2-2\alpha}}{2}\int_{\mathbb{T}^n}\left|\nabla\sqrt{\rho_S^\varepsilon}\right|^2 dx + \frac{1}{2}\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho_S^\varepsilon}}\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)\right|^2 dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}\left|\partial_t\psi^\varepsilon\right|^2 dx + \int_{\mathbb{T}^n}\frac{1}{\gamma\varepsilon^{2\alpha}}\left(\left(\rho_S^\varepsilon\right)^{\frac{\gamma}{2}} - 1\right)^2 dx, \tag{4.18}$$

then from Lemma 4.2 and (4.18) we have

$$\int_{\mathbb{T}^n}\left|\frac{1}{\sqrt{\rho_S^\varepsilon}}\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)\right|^2 dx \to 0 \tag{4.19}$$

as $\varepsilon \to 0$. We deduce from (4.19) and Hölder inequality that

$$\left\|\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \leqslant \left\|\sqrt{\rho_S^\varepsilon}\right\|_{L^{2\gamma}(\mathbb{T}^n)}\left\|\frac{1}{\sqrt{\rho_S^\varepsilon}}\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)\right\|_{L^2(\mathbb{T}^n)}$$

which converges to zero as $\varepsilon \to 0$. Finally, combing (4.13) and (4.15), we have

$$\left\|\rho_K^\varepsilon(\cdot,t)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \lesssim \varepsilon^\alpha\left\|\varepsilon\partial_t\psi^\varepsilon\right\|_{L^2(\mathbb{T}^n)}\left\|\psi^\varepsilon\right\|_{L^{2\gamma}(\mathbb{T}^n)} \to 0$$

and

$$\left\|J_K^\varepsilon(\cdot,t)\right\|_{L^1(\mathbb{T}^n)} \lesssim \varepsilon^\alpha\left\|\varepsilon\partial_t\psi^\varepsilon\right\|_{L^2(\mathbb{T}^n)}\left\|\varepsilon^{1-\alpha}\nabla\psi^\varepsilon\right\|_{L^2(\mathbb{T}^n)} \to 0$$

as $\varepsilon \to 0$. This completes the proof of Theorem 4.1. □

When $\alpha > 1 - \frac{\lambda}{2}$, we deduce from (4.18) that

$$\int_{\mathbb{T}^n}\left|\nabla\sqrt{\rho_S^\varepsilon}\right|^2 dx = \frac{1}{2}\int_{\mathbb{T}^n}\left|\frac{\nabla\rho_S^\varepsilon}{\sqrt{\rho_S^\varepsilon}}\right|^2 dx \to 0$$

as $\varepsilon \to 0$, and

$$\left\|\nabla\left(\rho_S^\varepsilon - 1\right)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \leqslant \left\|\frac{\nabla\rho_S^\varepsilon}{\sqrt{\rho_S^\varepsilon}}\right\|_{L^2(\mathbb{T}^n)}\left\|\sqrt{\rho_S^\varepsilon}\right\|_{L^{2\gamma}(\mathbb{T}^n)}, \tag{4.20}$$

by Hölder inequality. Thus, $\rho_S^\varepsilon \to 1$ strongly in $W^{1,\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)$. Furthermore, by Sobolev inequality we can show that $\rho_S^\varepsilon \to 1$ strongly in $L^{\frac{2n\gamma}{n(\gamma+1)-2\gamma}}(\mathbb{T}^n)$ for $n \geqslant 2$. In particular $n = 2$, iterating the estimate (4.20) by the so-called "*bootstrap process*", we have $\rho_S^\varepsilon \to 1$ in $L^p(\mathbb{T}^2)$ for any $1 \leqslant p < \infty$, and hence we have the following improvement of Theorem 4.1.

Theorem 4.3. *Assume the same hypothesis of Theorem 4.1 and Lemma 4.2. Let $\alpha > 1 - \frac{\lambda}{2}$ and $n = 2$ then there exists $T_* > 0$ such that for any $\eta > 0$,*

$$\left\|\left(\rho_S^\varepsilon - 1\right)(\cdot,t)\right\|_{L^{\frac{1}{\eta}}(\mathbb{T}^2)} \to 0, \qquad \left\|\rho_K^\varepsilon(\cdot,t)\right\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0,$$
$$\left\|\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)(\cdot,t)\right\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0, \qquad \left\|J_K^\varepsilon(\cdot,t)\right\|_{L^1(\mathbb{T}^2)} \to 0,$$

for $t \in [0, T_*)$ as $\varepsilon \to 0$, where u is the unique local smooth solution of the incompressible Euler equations (4.6).

If we replace the potential energy $(|\psi^\varepsilon|^\gamma - 1)|\psi^\varepsilon|^{\gamma-2}$ by $(|\psi^\varepsilon|^{2(\gamma-1)} - 1)$ then all the previous analysis still works for this model with small modification. Indeed, we investigate the time-scaled modulated nonlinear Klein–Gordon equation

$$i\varepsilon^{1+\alpha}\partial_t\psi^\varepsilon - \frac{\varepsilon^{2+2\alpha}\nu^2}{2}\partial_t^2\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - \left(\left|\psi^\varepsilon\right|^{2(\gamma-1)} - 1\right)\psi^\varepsilon = 0, \tag{4.21}$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x), \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x), \quad x \in \mathbb{T}^n.$$

It follows from the energy conservation law that we may assume the initial condition satisfies the uniform bound

$$\int_{\mathbb{T}^n} \frac{1}{2}\nu^2\varepsilon^2\left|\psi_1^\varepsilon\right|^2 + \frac{1}{2}\varepsilon^{2-2\alpha}\left|\nabla\psi_0^\varepsilon\right|^2 + \frac{1}{\gamma\varepsilon^{2\alpha}}\left(\left|\psi_0^\varepsilon\right|^{2\gamma} + (\gamma-1) - \gamma\left|\psi_0^\varepsilon\right|^2\right) dx < C. \tag{4.22}$$

In this case the associated modulated energy is defined by

$$H^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^n}\left|\left(\varepsilon^{1-\alpha}\nabla - iu\right)\psi^\varepsilon\right|^2 dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}\left|\partial_t\psi^\varepsilon\right|^2 dx + \frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left(\left(\rho_S^\varepsilon\right)^\gamma + (\gamma-1) - \gamma\rho_S^\varepsilon\right) dx$$

and is well-prepared, i.e.,

$$H^\varepsilon(0) = \frac{1}{2}\int_{\mathbb{T}^n}\left|\left(\varepsilon^{1-\alpha}\nabla - iu_0\right)\psi_0^\varepsilon\right|^2 dx + \frac{\nu^2\varepsilon^2}{2}\int_{\mathbb{T}^n}\left|\psi_1^\varepsilon\right|^2 dx + \frac{1}{\gamma\varepsilon^{2\alpha}}\int_{\mathbb{T}^n}\left(\left|\psi_0^\varepsilon\right|^{2\gamma} + (\gamma-1) - \gamma\left|\psi_0^\varepsilon\right|^2\right) dx = O\left(\varepsilon^\beta\right), \tag{4.23}$$

for some $\beta > 0$. Then employing the same argument as previous discussion, we can show that

$$H^\varepsilon(t) \leqslant O\left(\varepsilon^\lambda\right), \quad t \in [0,T],$$

where $\delta = 2\alpha/\gamma$ and $\lambda = \min\{\beta,\delta\}$.

Theorem 4.4. *Let $\gamma > 1$ for even spatial dimension n, $\gamma \geqslant \frac{n}{n-1}$ for odd spacial dimension n. Let $\alpha > 0$ and ψ^ε be the solution of the time scale modulated nonlinear Klein–Gordon equation (4.21) with initial condition $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^{s+1}(\mathbb{T}^n) \oplus H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 1$, satisfying (4.22) and (4.23). Then there exists $T_* > 0$ such that*

$$\left\|\left(\rho_S^\varepsilon - 1\right)(\cdot,t)\right\|_{L^\gamma(\mathbb{T}^n)} \to 0, \qquad \left\|\rho_K^\varepsilon(\cdot,t)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0,$$
$$\left\|\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)(\cdot,t)\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \to 0, \qquad \left\|J_K^\varepsilon(\cdot,t)\right\|_{L^1(\mathbb{T}^n)} \to 0,$$

for $t \in [0,T_*)$ as $\varepsilon \downarrow 0$, where $u \in C([0,T_*); H^s(\mathbb{T}^n))$ is the unique local smooth solution of the incompressible Euler equations (4.6). In particular, when $n = 2$ and $\alpha > 1 - \frac{\lambda}{2}$, the convergent results can be improved as follows:

$$\left\|\left(\rho_S^\varepsilon - 1\right)(\cdot,t)\right\|_{L^{\frac{1}{\eta}}(\mathbb{T}^2)} \to 0, \qquad \left\|\rho_K^\varepsilon(\cdot,t)\right\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0,$$
$$\left\|\left(J_S^\varepsilon - \rho_S^\varepsilon u\right)(\cdot,t)\right\|_{L^{2-\eta}(\mathbb{T}^2)} \to 0, \qquad \left\|J_K^\varepsilon(\cdot,t)\right\|_{L^1(\mathbb{T}^2)} \to 0,$$

for any $\eta > 0$ and $t \in [0,T_*)$ as $\varepsilon \to 0$.

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References

[1] Y. Brenier, Convergence of the Vlasov–Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 25 (2000) 737–754.
[2] D. Bresch, B. Desjardins, E. Grenier, C.K. Lin, Low Mach number limit of viscous polytropic flows: Formal asymptotics in the periodic case, Stud. Appl. Math. 109 (2002) 125–149.
[3] B. Desjardins, C.K. Lin, T.C. Tso, Semiclassical limit of the derivative nonlinear Schrödinger equation, Math. Models Methods Appl. Sci. 10 (2000) 261–285.
[4] I. Gasser, C.K. Lin, P. Markowich, A review of dispersive limit of the (non)linear Schrödinger-type equation, Taiwanese J. Math. 4 (2000) 501–529.
[5] E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, Proc. Amer. Math. Soc. 126 (1998) 523–530.
[6] S. Jin, C.D. Levermore, D.W. McLaughlin, The semiclassical limit of the defocusing NLS hierarchy, Comm. Pure Appl. Math. 52 (1999) 613–654.
[7] C.C. Lee, T.C. Lin, Incompressible and compressible limits of two-component Gross–Pitaevskii equations with rotating fields and trap potentials, J. Math. Phys. 49 (2008) 043517.
[8] J.H. Lee, C.K. Lin, The behavior of solutions of NLS equation of derivative type in the semiclassical limit, Chaos, Solitons & Fractals 13 (2002) 1475–1492.
[9] J.H. Lee, C.K. Lin, O.K. Pashaev, Shock waves, chiral solitons and semiclassical limit of one-dimensional anyons, Chaos, Solitons & Fractals 19 (2004) 109–128.
[10] H.L. Li, C.K. Lin, Zero Debye length asymptotic of the quantum hydrodynamic model of semiconductors, Comm. Math. Phys. 256 (2005) 195–212.
[11] C.K. Lin, Y.S. Wong, Zero-dispersion limit of the short-wave-long-wave interaction equation, J. Differential Equations 228 (2006) 87–110.
[12] C.K. Lin, K.C. Wu, Singular limits of the Klein–Gordon equation, Arch. Ration. Mech. Anal. 197 (2010) 689–711.
[13] C.K. Lin, K.C. Wu, On the fluid approximation to the Klein–Gordon equation, Discrete Contin. Dyn. Syst. Ser. A 32 (2012) 2233–2251.
[14] F.H. Lin, P. Zhang, Semiclassical limit of the Gross–Pitaevskii equation in an exterior domain, Arch. Ration. Mech. Anal. 179 (2005) 79–107.
[15] T.C. Lin, P. Zhang, Incompressible and compressible limit of coupled systems of nonlinear Schrödinger equations, Comm. Math. Phys. 266 (2006) 547–569.
[16] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1, Incompressible Models, The Clarendon Press, Oxford University Press, New York, 1996.
[17] P.L. Lions, N. Masmoudi, Incompressible limit for a viscous compressible fluid, J. Math. Pures Appl. 77 (1998) 585–627.
[18] S. Machihara, K. Nakanishi, T. Ozawa, Nonrelativistic limit in the energy space for nonlinear Klein–Gordon equations, Math. Ann. 322 (2002) 603–621.
[19] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Applied Mathematical Sciences, vol. 53, Springer-Verlag, 1984.
[20] N. Masmoudi, From Vlasov–Poisson system to the incompressible Euler system, Comm. Partial Differential Equations 26 (2001) 1913–1928.
[21] N. Masmoudi, K. Nakanishi, From nonlinear Klein–Gordon equation to a system of coupled nonlinear Schrödinger equations, Math. Ann. 324 (2002) 359–389.
[22] N. Masmoudi, K. Nakanishi, Nonrelativistic limit from Maxwell–Klein–Gordon and Maxwell–Dirac to Poisson–Schrödinger, Int. Math. Res. Not. 13 (2003) 697–734.
[23] A. Messiah, Quantum Mechanics, vols. 1 & 2 (two volumes bound as one), Dover Publications Inc., 1999.
[24] K. Nakanishi, Nonrelativistic limit of scattering theory for nonlinear Klein–Gordon equations, J. Differential Equations 180 (2002) 453–470.
[25] L.M. Pismen, Vortices in Nonlinear Fields, International Series on Monograph on Physics, vol. 100, Clarendon Press, Oxford, 1999.
[26] M. Puel, Convergence of the Schrödinger–Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 27 (2002) 2311–2331.
[27] W. Strauss, Nonlinear Wave Equations, CBMS, vol. 73, American Math. Soc., 1989.
[28] K.C. Wu, Convergence of the Klein–Gordon equation to the wave map equation with magnetic field, J. Math. Anal. Appl. 365 (2010) 584–589.