Stability analysis of the Lorentz Chern-Simons expanding solutions

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(Received 19 June 2011; revised manuscript received 24 August 2011; published 26 September 2011)

A class of Bianchi type II expanding solutions in a Lorentz Chern-Simons theory has been known to break the cosmic no-hair theorem. These solutions do not approach the late-time de Sitter Universe. We will show that there are two independent solutions classified by the parameter p for each given cosmological constant Λ. One of them is in the small-p phase; the other solution is in the large-p phase. It can be shown that an unstable mode always exists for the solution in its small-p phase that comes with smaller energy T_{00} and $T_{00} + T/2$. The result indicates that the large-p phase is unlikely to be stable from an observation of a modified version of Wald's theorem.

DOI: 10.1103/PhysRevD.84.063014 PACS numbers: 95.30.Sf, 04.50.Kd, 98.80.Bp, 98.80.Jk

I. INTRODUCTION

The cosmological problems of high-degree homogeneity, isotropy, and flatness of the observed Universe have led to the proposal of the inflationary scenarios as a powerful resolution [1]. Most of these scenarios require a large cosmological constant to induce fast growth of the cosmic scale factor. In the meantime, the cosmic no-hair conjecture proposed by Hawking and Moss asserts that, under certain physically realistic conditions, any expanding universe will evolve asymptotically toward a late-time locally de Sitter geometry. Thus, the no-hair conjecture associated with the inflationary resolution offers a realistic explanation to the homogeneity and isotropy of our current universe.

The Euler-Lagrange equations of any gravitational system with a cosmological constant Λ can always be represented as

$$G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{1.1}$$

The Einstein tensor $G_{\mu\nu}$ on the left-hand side of the above equation stands for the geometric effect of the gravitational interactions derived from $T_{\mu\nu}$ and $\Lambda g_{\mu\nu}$ on the right-hand side of Eq. (1.1).

Gibbons and Hawking [2], Hawking and Moss [3] assert that all physical models with a positive cosmological constant will inevitably approach a late-time de Sitter space. This became known as the cosmic no-hair conjecture for the Einstein theory. Partial proof was given by Robert Wald [4], showing clearly that any model with a positive cosmological constant will drive the late-time evolution toward de Sitter space, at least locally, for all non-type-IX Bianchi spaces once the matter sources obey (a) the dominant energy condition (DEC)

$$T_{\mu\nu}t^\mu t^\nu \geq 0, \tag{1.2}$$

and (b) the strong-energy condition (SEC)

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)t^\mu t^\nu \geq 0 \tag{1.3}$$

for every timelike vector t^μ [4]. Here $T_{\mu\nu}$ and T denote the energy momentum tensor and its trace for all fields coupled to the gravitational system. The type IX Bianchi space behaves similarly if Λ is sufficiently large [4].

Wald proves the no-hair theorem by inspecting the constraint equation and the Raychaudhuri equation

$$\frac{1}{3}K^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}{}^{(3)}R + \Lambda + T_{00}, \tag{1.4}$$

$$\dot{K} + \frac{1}{3}K^2 = \Lambda - \sigma_{ab}\sigma^{ab} - \left(T_{00} + \frac{1}{2}T\right). \tag{1.5}$$

The DEC (1.2) and SEC (1.3) assume that both $\mathcal{H} \equiv T_{00}$ and $\mathcal{E} \equiv T_{00} + \frac{1}{2}T$ are positive. In addition, the 3-space curvature ${}^{(3)}R$ is negative for the Bianchi types I-VIII. Hence, the following inequalities can be derived:

$$\frac{1}{3}K^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}{}^{(3)}R + \Lambda + T_{00} \geq \Lambda + \frac{1}{2}\sigma_{ab}\sigma^{ab}, \tag{1.6}$$

$$\dot{K} + \frac{1}{3}K^2 = \Lambda - \sigma_{ab}\sigma^{ab} - \left(T_{00} + \frac{1}{2}T\right) \leq \Lambda. \tag{1.7}$$

Hence we can obtain:

$$\dot{K} \leq \Lambda - \frac{1}{3}K^2 \leq 0, \tag{1.8}$$

$$0 \leq \sigma_{ab}\sigma^{ab} \leq \frac{1}{3}(K^2 - 3\Lambda). \tag{1.9}$$

As a result, we can write the inequalities (1.8) and (1.9) as a set of inequality of K as

$$\sqrt{3\Lambda} \leq K \leq \sqrt{3\Lambda}\coth\left[\sqrt{\frac{\Lambda}{3}}t\right] \underset{t\to\infty}{\longrightarrow} \sqrt{3\Lambda}. \tag{1.10}$$

Therefore, Eq. (1.9) implies that $\sigma^2 = 0$ at future infinity. Hence, the evolution of the Universe toward the de Sitter space will be inevitable.

The importance of the proof derived by Wald can be made more transparent if we open up a small window of energy conditions. To be more specific, we will assume that the DEC and SEC are both violated. We will also assume the following bounds on the energy variables

$$\Lambda_0 \equiv \Lambda + \mathcal{H} \geq 0, \tag{1.11}$$

$$\Lambda_1 \equiv \Lambda - \mathcal{E} \geq 0. \tag{1.12}$$

It is apparent that $\Lambda_1 \geq \Lambda_0$ unless $\mathcal{H} + \mathcal{E} \geq 0$. The field equations can thus be shown to be

$$\frac{1}{3}K^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}{}^{(3)}R + \Lambda + T_{00} \geq \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}{}^{(3)}R + \Lambda_0 \geq \Lambda_0 + \frac{1}{2}\sigma_{ab}\sigma^{ab}, \tag{1.13}$$

$$\dot{K} + \frac{1}{3}K^2 = \Lambda - \sigma_{ab}\sigma^{ab} - \left(T_{00} + \frac{1}{2}T\right) \leq \Lambda_1 - \sigma_{ab}\sigma^{ab} \leq \Lambda_1. \tag{1.14}$$

As a result, the inequalities (1.8) and (1.9) change from a set of single-bound inequalities to a set of two-bound inequalities (namely from Λ to Λ_1 and Λ_0) as

$$\dot{K} \leq \Lambda_1 - \frac{1}{3}K^2, \tag{1.15}$$

$$0 \leq \sigma_{ab}\sigma^{ab} \leq \frac{1}{3}(K^2 - 3\Lambda_0). \tag{1.16}$$

Equation (1.15) implies

$$K \leq \sqrt{3\Lambda_1}\coth\left[\sqrt{\frac{\Lambda_1}{3}}t\right] \underset{t\to\infty}{\longrightarrow} \sqrt{3\Lambda_1}. \tag{1.17}$$

Hence, we can obtain a set of inequalities

$$\Lambda + \mathcal{H} = \Lambda_0 \leq \frac{1}{3}K^2 \leq \Lambda_1 = \Lambda - \mathcal{E}. \tag{1.18}$$

In contrast to the case with a single-bound Λ, the upper and lower bounds turn on a stability window with a width $\Gamma \equiv \Lambda_1 - \Lambda_0 = -(\mathcal{E} + \mathcal{H}) > 0$.

Once $(\mathcal{E} + \mathcal{H}) \geq 0$, the stability window will be closed, namely, $\Gamma \to 0$. As a result, there is no room for the existence of a stable anisotropically-expanding solution. We will show that the Lorentz Chern-Simons (LCS) theory does allow the existence of two independent sets of expanding solutions with $\Gamma > 0$ satisfying the constraints $(\mathcal{E} + \mathcal{H}) < 0$. Hence, interesting phenomena deserves more attention for the LCS theory. Detailed analysis will be presented in Sec. IV.

Note that a series of cosmic no-hair theorems of varying strengths have also been proved in favor of the existence of certain constraints on the field parameters for its occurrence [4–13]. It is known that counterexamples also exist when these energy conditions do not hold exactly [14–16]. Many of these solutions can be shown to be unstable [9,17–19]. These results appear to support the Hawking's no-hair conjecture. By all means, it is important to verify whether these anisotropically expanding solutions are stable or not. It may further our understanding of the evolutionary process of our physical universe.

In fact, relative studies of higher-order theories can be found in Ref. [20–35]. Most of the papers mentioned above dealt with standard gravitational theory with ordinary matter sources. Superstring theory indicates, however, that the low-energy effective action for gravity also involves curvature terms of higher-order, like the Gauss-Bonnet and the Chern-Simons topological terms. Note that, for certain Bianchi models (type I and type V and diagonal type III and type VI) the LCS terms are shown to obey the energy conditions that secure the validity of the no-hair conjecture [16]. On the other hand, a set of anisotropically expanding Bianchi type II solutions was found in the model with an LCS coupling. These solutions are known to violate the energy conditions and are speculated to be counterexamples of the no-hair conjecture [16].

In this paper, we will show that perturbations along the anisotropic directions indicate that half of the anisotropically expanding solutions are unstable. We can also show that the stable limit of the expanding solutions coincides with the minimal state of the cosmological constant Λ allowed by the expanding solutions. In addition, we will argue that the other half of the expanding solutions are very likely to be unstable from an observation of the energy constraints shown above as Λ_0 and Λ_1. The arguments with a window of energy range presented in this paper will help clarify the useful role of the energy conditions shown in Ref. [4].

This paper will be organized as follows: (i) a brief review is given in Sec. I, (ii) in Sec. II, we will introduce the LCS models with the expanding solutions found in Ref. [16], (iii) the stability constraints associated with the expanding solutions will be discussed in Sec. III, (iv) in Sec. IV, dynamical analysis associated with the Wald's proof on Hawking's conjecture is discussed in detail focusing on the stability property of the unstable modes, (v) for heuristic reasons, a set of exact and anisotropic expanding solutions in the Bianchi type I space is also discussed in Sec. V, (v) finally, conclusions and remarks will be drawn in Sec. VI.

II. EQUATIONS OF MOTION

The action of the Lorentz Chern-Simons theory is given by[16]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \Lambda\right]. \tag{2.1}$$

Here $H_{\mu\nu\lambda}$ is the Kalb-Ramond field strength defined by

$$H = dB + \omega_L \equiv \tilde{H} + \omega_L = (H_{\mu\nu\lambda} + h_{\mu\nu\lambda})(dx^\mu dx^\nu dx^\lambda), \tag{2.2}$$

with B the Kalb-Ramond 2-form, ω_L the LCS coupling, and ω the spin-connection defined by

$$\omega_L = \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right). \tag{2.3}$$

We can show that the variations of the spin-connection and the Riemann curvature tensor equal to

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(D_\alpha \delta g_{\beta\nu} + D_\beta \delta g_{\alpha\nu} - D_\nu \delta g_{\alpha\beta}), \tag{2.4}$$

$$\delta R^\mu{}_{\nu\alpha\beta} = D_\alpha \delta\Gamma^\mu_{\nu\beta} - D_\beta \delta\Gamma^\mu_{\nu\alpha}. \tag{2.5}$$

Therefore, it is clear that

$$\delta\omega_L = \mathrm{Tr}(\delta\omega d\omega + \omega\delta R), \tag{2.6}$$

$$\delta H^2 = -2D_\mu(H^{\alpha\gamma\lambda} R^{\mu\beta}{}_{\gamma\lambda})\delta g_{\alpha\beta}. \tag{2.7}$$

As a result, it can be shown that the variational field equations take the following form [16]:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 6H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} - g_{\mu\nu}H_{\lambda\alpha\rho}H^{\lambda\alpha\rho} + 4D_\sigma(H_{\mu\alpha\lambda}R^\sigma{}_\nu{}^{\alpha\lambda}) - \Lambda g_{\mu\nu}, \tag{2.8}$$

$$\nabla_\lambda H^{\mu\nu\lambda} = 0, \tag{2.9}$$

with $dH = d\omega_L$. Note that we have set $\kappa^2 = 1$ for convenience.

The equations of motion can be recast in a different form by noting that, in four dimensions, an antisymmetric tensor (3-form H) is dual to a vector (1-form V)

$$H = {}^*V. \tag{2.10}$$

As a result, we have

$$dV = 0 \tag{2.11}$$

from Eq. (2.9). If the manifold has vanishing in the first cohomology group, $H_1(M) = 0$, which is true at least locally for all Bianchi models, all closed 1-forms are exact. Consequently, we can write the Kalb-Ramond field strength as a dual to an exact 1-form [16]:

$$H = {}^*db. \tag{2.12}$$

Therefore, the Einstein equations become

$$G_{\mu\nu} = 12\partial_\mu b \partial_\nu b - 6g_{\mu\nu}\partial_\lambda b \partial^\lambda b + 4D_\sigma[\sqrt{-g}\epsilon_{\mu\alpha\lambda\rho}(\partial^\rho b)R^\sigma{}_\nu{}^{\alpha\lambda}] - \Lambda g_{\mu\nu}, \tag{2.13}$$

$$d^* db = d\omega_L. \tag{2.14}$$

Bianchi models represent all possible realizations of spatially homogeneous metrics, which admit isometry groups with three independent transitive spacelike Killing vectors. Note that Kantowski-Sachs space [36] is also a spatially homogeneous space not included in Bianchi's classification table. We will focus on the four-dimensional space with a three-dimensional spatial space classified by Bianchi. These four-dimensional metrics can be written on coset spaces of nine acceptable groups of motions as

$$ds^2 = -dt^2 + h_{ab}(t)\epsilon^a \otimes \epsilon^b, \tag{2.15}$$

with group properties represented by

$$d\epsilon^a = -\frac{1}{2} C^a{}_{bc}\epsilon^b \wedge \epsilon^c. \tag{2.16}$$

Here $C^a{}_{bc}$ are the structure constants of the symmetry group.

In the nonholonomic basis ($\epsilon^0 = dt$, ϵ^a, $a = 1, 2, 3$), the metric can be written as

$$g \equiv h = \begin{pmatrix} -1 & 0 \\ 0 & \exp[2\alpha](e^{2\beta})_{ab} \end{pmatrix}, \tag{2.17}$$

with $\mathrm{Tr}(\beta) = 0$. Accordingly, the Ricci tensor and scalar curvature can be written as

$$\begin{aligned} R^0{}_0 &= \dot{K} + \frac{K^2}{3} + \sigma_{ab}\sigma^{ab}, \\ R^0{}_a &= -C^b{}_{ac}\sigma^c{}_b - C^d{}_{db}\sigma^b{}_a, \\ R^a{}_b &= \frac{1}{\sqrt{h}}\left[\sqrt{h}\left[\sigma^a{}_b + \frac{K}{3}\delta^a{}_b\right]\right]^{\cdot} + {}^{(3)}R^a{}_b, \\ R &= 2\dot{K} + \frac{4}{3}K^2 + \sigma_{ab}\sigma^{ab} + {}^{(3)}R. \end{aligned} \tag{2.18}$$

Overdot denotes time derivative. K and σ_{ab} are the volume expansion factor and the shear, respectively.

The structure of the metric (2.15) and the definition of the LCS form imply that it can be written as

$$\omega_L = B_{abc}(t)\epsilon^a \wedge \epsilon^b \wedge \epsilon^c + A_{ab}(t)\epsilon^0 \wedge \epsilon^a \wedge \epsilon^b, \tag{2.19}$$

with B_{abc} and A_{ab} the antisymmetric time-dependent functions calculated from the metric and the structure constants. From (2.16), we have

$$d\omega_L = [\dot{B}_{abc}(t) + A_{da}(t)C^d{}_{bc}]\epsilon^0 \wedge \epsilon^a \wedge \epsilon^b \wedge \epsilon^c. \tag{2.20}$$

As a result, Eq. (2.14) turns out to be

$$(\sqrt{h}\dot{b})^{\cdot} = \frac{1}{6}[\dot{B}_{abc}(t) + A_{da}(t)C^d{}_{bc}]e^{abc}, \tag{2.21}$$

with e^{abc} the flat 3-space Levi-Civita tensor. In addition, the structure constants can be parametrized as

$$C^a{}_{bc} = M^{ad}e_{dbc} + A_c\delta^a{}_b - A_b\delta^a{}_c, \tag{2.22}$$

with M a symmetric matrix and A_a a 3-vector.

Note that Bianchi models can be classified into two different classes: (i) class A with vanishing vector $A_a = 0$, and (ii) class B with nonvanishing vector $A_a \neq 0$. With the help of Eq. (2.22), Eq. (2.21) can be written as

$$(\sqrt{h}\dot{b}) = \frac{1}{6}[\dot{B}_{abc}(t) + 2A_{ba}(t)A_c]e^{abc}. \tag{2.23}$$

Equation (2.23) can be integrated immediately for class A models, but not for the class B models and not even in the diagonal cases. In fact, it can be shown that, for diagonal Bianchi type IV and type VII$_{h\neq 0}$, the second term in (2.23) is not a total derivative. Concentrating on the class A models, we will have

$$\dot{b} = \frac{C}{\sqrt{h}} + \frac{1}{6\sqrt{h}}B_{abc}(t)\epsilon^{abc}, \tag{2.24}$$

an integration constant. For the definition of the energy momentum tensor given by Eq. (1.1), we can show that

$$T_{\mu\nu} = 12\partial_\mu b\partial_\nu b - 6g_{\mu\nu}b\partial^\lambda b + A_{\mu\nu}, \tag{2.25}$$

with

$$A_{\mu\nu} = 4D_\sigma[\sqrt{-g\epsilon_{\mu\alpha\lambda\rho}}(\partial^\rho b)R^\sigma{}_\nu{}^{\alpha\lambda}], \tag{2.26}$$

Hence, it can be shown that, for example, the T_{00} component of the energy momentum tensor takes the following form:

$$T_{00} = 6\dot{b}^2 + A_{00} = 6\dot{b}^2 + 18\dot{b}^2\dot{\beta}^2e^{2\beta-\alpha}. \tag{2.27}$$

Once the metric is specified, the field equations can be written straightforwardly as a set of ordinary differential equations depending on these field variables.

III. THE BIANCHI TYPE II EXPANDING SOLUTIONS AND ITS STABILITY

A. Bianchi type II models

Bianchi type I, type V and diagonal type III and type VI models with LCS terms have been shown to obey the energy conditions that secure the validity of the no-hair conjecture [16]. On the other hand, for the Bianchi type II metric given by

$$ds^2 = -dt^2 + e^{2\alpha}\{e^{2\beta}(\epsilon^1)^2 + e^{-\beta}[(\epsilon^2)^2 + (\epsilon^3)^2]\}, \tag{3.1}$$

with $\epsilon^1 = dx^2 - x^1dx^3$, $\epsilon^2 = dx^3$, and $\epsilon^3 = dx^1$, the situation is quite different. Indeed, the field equations can be shown to be

$$3\ddot{\alpha} + 9\dot{\alpha}^2 = 3\Lambda + \frac{1}{2}e^{-2\alpha+4\beta} + 18\dot{b}\dot{\beta}^2e^{-\alpha+2\beta}, \tag{3.2}$$

$$3\dot{\alpha}^2 = \Lambda + \frac{1}{4}e^{-2\alpha+4\beta} + \frac{3}{4}\dot{\beta}^2 + 6\dot{b}^2 + 18\dot{b}\dot{\beta}^2e^{-\alpha+2\beta}, \tag{3.3}$$

$$\dot{b} = Ce^{-3\alpha} + \frac{1}{6}e^{-3\alpha+6\beta} - \frac{3}{4}\dot{\beta}^2e^{-\alpha+2\beta}. \tag{3.4}$$

Note that C is a parameter that can be set as zero following the symmetry properties of the field parameters [16]. We will hence set $C = 0$ from now on. The Kaloper's solutions can be obtained from the ansatz $\exp[2\beta - \alpha] = \eta =$ constant. It can then be shown that the field equations imply the simple relation $\dot{\beta} = \rho =$ constant. This leads immediately to the result $\dot{\alpha} = 2\rho =$ constant. Consequently, the field equations can be written as the following constraint equations:

$$6\rho^2\eta^4 - 27\rho^4\eta^2 - 72\rho^2 + \eta^2 + 6\Lambda = 0, \tag{3.5}$$

$$36\rho^2\eta^4 - 243\rho^4\eta^2 + 4\eta^6 - 270\rho^2 + 6\eta^2 + 24\Lambda = 0. \tag{3.6}$$

We can eliminate the Λ-term from the above equations to obtain a constraint equation independent of the cosmological constant Λ:

$$4\eta^6 + 12\rho^2\eta^4 + (2 - 135\rho^4)\eta^2 + 18\rho^2 = 0. \tag{3.7}$$

Accordingly, the expanding solutions can be solved as a function of the parameter $p > 2/9$:

$$\rho^2 = p\eta^2 = p\left[\frac{18p + 2}{(15p + 2)(9p - 2)}\right]^{1/2}. \tag{3.8}$$

As a result, the cosmological constant can be shown to be

$$\Lambda = \frac{1}{6}\left[\frac{(1134p^2 + 135p - 2)^2(18p + 2)}{(15p + 2)^3(9p - 2)}\right]^{1/2}. \tag{3.9}$$

Note that typos in the above equations written in Ref. [16] are corrected.

A first look at the set of Eqs. (3.5) and (3.6) shows that ρ and η can be solved as functions of Λ. The results reveal a more profound symmetry hidden in the equations. The relation $\rho^2 = p\eta^2$ between ρ and η implies that there are two sets of solutions associated with each given Λ. The relation between these two sets of solutions classified by the different choice of a p parameter is very interesting. We will compare the energy dependence of these two sets of solutions in the details of Sec. IV.

B. Perturbation equation

Perturbing the field equations gives

$$3\frac{d^2}{dt^2}\delta\alpha + \alpha_1\frac{d}{dt}\delta\alpha + \alpha_2\delta\alpha = \beta_1\frac{d}{dt}\delta\beta + \beta_2\delta\beta, \tag{3.10}$$

$$\alpha_3\frac{d}{dt}\delta\alpha + \alpha_4\delta\alpha = \beta_3\frac{d}{dt}\delta\beta + \beta_4\delta\beta, \tag{3.11}$$

where α_i are all constants:

$$\alpha_1 = 36\rho, \tag{3.12}$$

$$\alpha_2 = \eta^2 + 12\rho^2\eta^4 - 27\rho^4\eta^2, \tag{3.13}$$

$$\alpha_3 = \frac{1}{3}\alpha_1, \tag{3.14}$$

$$\alpha_4 = \frac{1}{2}\eta^2 + \eta^6 + 6\rho^2\eta^4 - \frac{81}{4}\rho^4\eta^2, \tag{3.15}$$

$$\beta_1 = 6\rho\eta^4 - 54\rho^3\eta^2, \tag{3.16}$$

$$\beta_2 = 2\alpha_2, \tag{3.17}$$

$$\beta_3 = \frac{3}{2}\rho + 3\rho\eta^4 - \frac{81}{2}\rho^3\eta^2, \tag{3.18}$$

$$\beta_4 = 2\alpha_4. \tag{3.19}$$

As a result, we can write the perturbation equation as

$$\mathcal{D}\,\delta A \equiv \begin{pmatrix} 3\nu^2 + \alpha_1\nu + \alpha_2 & -\beta_1\nu - \beta_2 \\ \alpha_3\nu + \alpha_4 & -\beta_3\nu - \beta_4 \end{pmatrix}\begin{pmatrix} \delta\alpha \\ \delta\beta \end{pmatrix} = 0 \tag{3.20}$$

with

$$\delta\alpha = \alpha_0 \exp[\nu t], \qquad \delta\beta = \beta_0 \exp[\nu t]. \tag{3.21}$$

Here α_0, β_0 are constant parameters. Therefore, nontrivial solutions for $\delta\alpha$ and $\delta\beta$ exist only when $\det\mathcal{D} = 0$. The determinant equation turns out to be

$$3\beta_3\nu^2 + B\nu + D = 0 \tag{3.22}$$

with $B = \alpha_1\beta_3 + 6\alpha_4 - \alpha_3\beta_1 = 18\rho\beta_3$. In addition, $D = 2\alpha_1\alpha_4 - \alpha_2\beta_3 - \alpha_4\beta_1 - 2\alpha_2\alpha_3$, which can also be shown to be

$$D = -\frac{3}{2}\rho\eta^2(-9 + 4\eta^8 - 99\rho^2\eta^2 - 36\rho^2\eta^6 + 567\rho^4 - 48\eta^4 + 81\rho^4\eta^4). \tag{3.23}$$

As a result, the perturbation equation becomes

$$3\beta_3(\nu^2 + 6\rho\nu) + D = 0 \tag{3.24}$$

that can be solved directly to give the following solutions:

$$\nu = -3\rho \pm \sqrt{9\rho^2 - \frac{D}{3\beta_3}}. \tag{3.25}$$

The coefficient D can be written as a function of p:

$$D = -3\sqrt[4]{2}\sqrt{p}(9p + 1)^{3/4} \times (9p - 2)^{-7/4}(15p + 2)^{-11/4}F(p), \tag{3.26}$$

with

$$F(p) = 153\,090p^4 - 4617p^3 - 18\,036p^2 - 2916p - 128. \tag{3.27}$$

Note that $\beta_3(p)$ as a function of p can be shown to take the following form:

$$\beta_3 = -\frac{9p(117p + 10)}{2(15p + 2)(9p - 2)}. \tag{3.28}$$

Therefore, β_3 is always negative for all $p > 2/9$. In addition, the constraint $p > 2/9$ has to be observed according to the definition (3.8) of the expanding solutions. Hence, the unstable mode exists only when $D > 0$, or equivalently $F = 153\,090p^4 - 4617p^3 - 18\,036p^2 - 2916p - 128 < 0$.

The roots to the polynomial equation $F(p) = 0$ can be shown (see the appendix) to be close to -0.1797, -0.1307, -0.0839, $0.424\,422$. As a result, there is only one positive root $p_0 \cong 0.424\,422$ in the region $p > 2/9$. It is thus clear that $F < 0$ only when $2/9 < p < p_0$ in the region $p > 2/9$. In fact, we can also show explicitly that $F(p)$ is a monotonically increasing function of p for all $p > p_0$. The numerical plot of $F(p)$ can also be shown to confirm this result.

Therefore, for all expanding solutions with $p > 2/9$, we reach the conclusion that $\frac{2}{9} < p < p_0$ is the region for the existence of an unstable mode. The only positive root $p_0 \cong 0.424\,422$ to the equation $F(p) = 0$ represents the upper limit of the unstable mode. p_0 then serves as the critical point of stability associated with the expanding solutions.

IV. DYNAMICAL ANALYSIS

A. Scale factor and cosmological constant

By setting $\beta = \beta_0 + \rho t$, the expanding scale factors can be shown to be $a_1(t) \equiv \exp[\alpha + \beta] = \exp[3\beta_0] \times \exp[3\rho t]/\eta$, and $a_2(t) \equiv \exp[\alpha - \beta] = \exp[\beta_0]\exp[\rho t]/\eta$. Hence, the expanding rate of the expanding solutions depends on the field variable ρ. The numerical result of ρ is shown in Fig. 1. Note that the V-shape curve of the function ρ attains its minimum at $p = p_1 \cong 0.4378$.

We can also show analytically that when p is extremely close to $2/9$, such that $p = 2/9 + \epsilon$ for a small positive parameter ϵ, the scale factor ρ approaches the following limit:

$$\rho \rightarrow \frac{1}{3}(2\epsilon)^{-1/4}. \tag{4.1}$$

On the other hand, when $p \gg 1$, $\rho \rightarrow (2p/15)^{1/4}$. It is apparent that the analytic results shown above agree with the numerical results, shown in Fig. 1, in both $p \rightarrow 2/9$ and $p \gg 1$ limits.

In addition, we can show that

$$\rho'(p) = \frac{Q(p)}{2^{7/4}p^{1/2}}[(9p + 1)^3(15p + 2)^5(9p - 2)^5]^{-1/4} \tag{4.2}$$

with $Q(p) = 1215p^3 - 216p^2 - 120p - 8$ as a function of p. Hence, the minimum of ρ will occur when $Q(p) = 0$. We can show (see the appendix) that the roots of the polynomial equation $Q(p) = 0$ are $p \cong -0.1371$,

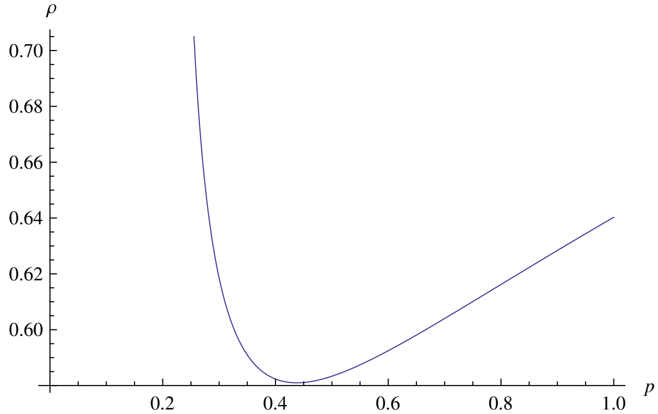

FIG. 1 (color online). ρ is plotted as a function of the parameter p. The local minimum of $\rho \cong 0.5810$ occurs when $p = p_1 \cong 0.4378$.

-0.0869, and a positive root $p = p_1 \cong 0.4378$. Note that, for $p > 2/9$, the minimum of ρ occurs when $p = p_1$. And p_1 is very close to the stability limit, $p = p_0$, of the anisotropically expanding solutions. The corresponding minimum of ρ is $\rho(p_1) \cong 0.5810$.

Similarly, the numerical plot of Λ is shown in Fig. 2. It can be shown that the curve of the function $\Lambda(p)$ is also a V-shape curve that attains its minimum exactly at the stability limit $p = p_0$. Indeed, we can show that Λ' approaches the following form:

$$\Lambda'(p) = \frac{3F(p)}{2^{3/2}}(9p+1)^{-1/2}(15p+2)^{-9/2} \times (9p-2)^{-3/2}(1134p^2+135p-2)^{-1}. \tag{4.3}$$

Hence, Λ attains its minimum when $F(p) = 0$, which coincides exactly with the stability limit $p = p_0$.

In particular, we can also show analytically that when p is extremely close to $2/9$, such that $p = 2/9 + \epsilon$, Λ takes the following limit:

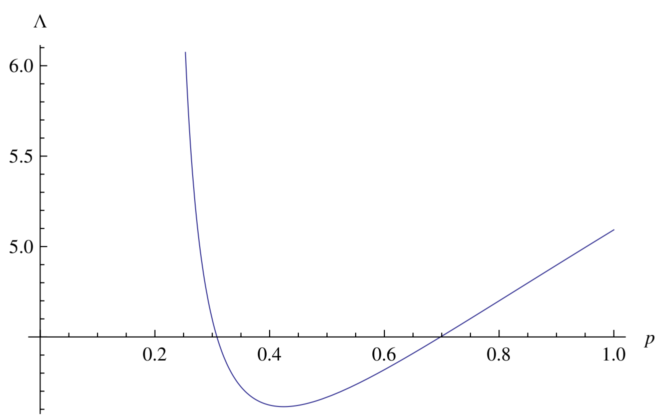

FIG. 2 (color online). There are two different p assigned for each Λ except the stable limit $p = p_0$. The local minimum of $\Lambda \cong 4.1145$ occurs when $p = p_0 \cong 0.424\,422$.

$$\Lambda \to \frac{21}{16}(2\epsilon)^{-1/2}, \tag{4.4}$$

and $\Lambda \to (63/5) \cdot (2p/15)^{1/2}$ when $p \gg 1$. Hence, the analytic results in both limits of p also agree with the numerical results shown in Fig. 2.

Note that for a given Λ, there are two independent expanding solutions classified by two different choices of p. One is in the unstable zone $p = p_< < p_0$; the other one is in the stable zone $p = p_> > p_0$. There is always a dual pair of solutions for each Λ except for the stable limit $\Lambda(p = p_0) \cong 4.1145$.

In other words, for the same cosmological constant with $\Lambda \gg 1$, there is an unstable inflationary solution near the region $p_< \cong 2/9$ and another inflationary solution with $p_> \gg 1$. The corresponding expanding factor is $\rho \cong 4\sqrt{\Lambda}/(3\sqrt{21})$ near the region $p = 2/9 + \epsilon$. Similarly, at the region $p \gg 1$, the corresponding expanding factor is $\rho \cong \sqrt{15\Lambda}/(3\sqrt{21})$. Apparently, the expanding scale $\rho(\Lambda)$ is bigger near the unstable region than the large p region for the same Λ. This is also indicated by the fact that the local minimum of the function $\rho(p)$ appears at $p = p_1 > p_0$. This result implies that stronger inflation (when $p \to 2/9$) will lead to unstable expanding solutions. As a result, the system will eventually become unstable, after an inflation of a time period of the order of $t \cong 1/(3\rho)$. Consequently, a natural exit of the inflationary phase can be achieved automatically.

B. Energy conditions for the LCS theory

In summary, we have shown that, other than the minimum value of $\Lambda \cong 4.1145$, there are two independent expanding solutions corresponding to any given Λ. One is in the unstable zone with $p = p_< < p_0$; the other is in the large $p = p_> > p_0$ region such that $\Lambda(p_>) = \Lambda(p_<)$.

We can also show that the $\mathcal{H}^<$, the T_{00} component of the energy momentum tensor associated with the unstable mode at the region $p < p_0$, is slightly smaller than the counterpart, $\mathcal{H}^>$, in the large $p > p_0$ region with the same given Λ. Indeed, we can show that

$$\mathcal{H}(p) = -\frac{1}{2}\Lambda\left[\frac{(27p+2)(9p+1)}{1134p^2+135p-2}\right]. \tag{4.5}$$

As a result,

$$\mathcal{H}^>(p) \to -\frac{3}{28}\Lambda \tag{4.6}$$

when $p_> \gg 1$. On the other hand, when $p_< = 2/9 + \epsilon$,

$$\mathcal{H}^<(p) \to -\frac{1}{7}\Lambda. \tag{4.7}$$

Here $\mathcal{H}^> = \mathcal{H}(p = p_>)$ and $\mathcal{H}^< = \mathcal{H}(p = p_<)$ denote, respectively, the T_{00} component of the energy momentum tensor evaluated at $p = p_>$ and $p = p_<$ associated with the same cosmological constant Λ such that

$\Lambda(p_>) = \Lambda(p_<)$ along with the definition that $p_> > p_0 > p_<$.

Apparently, $\mathcal{H}^> > \mathcal{H}^<$ for any given $\Lambda \gg 1$. In fact, the numerical result shown in Fig. 3 also confirms that $\mathcal{H}^>$ is always greater than $\mathcal{H}^<$ for any different allowed value of Λ.

In addition, we can show that $\mathcal{E}^> > \mathcal{E}^<$ for the set of anisotropically expanding solutions with the same Λ associated with two different choices of p. Here $\mathcal{E} = T_{00} + T/2$. Indeed, T can be written as

$$T = 12\dot{b}^2 = \frac{(9p-2)(9p+1)}{1134p^2 + 135p - 2}\Lambda. \tag{4.8}$$

Therefore, in the large-p region, $\mathcal{E} \to -\Lambda/14$. On the other hand, $\mathcal{E} \to -\Lambda/7$ in the small p limit. As a result, we can show that $\mathcal{E}^> > \mathcal{E}^<$ for all allowable values of Λ. The numerical result shown in Fig. 4 also confirms that $\mathcal{E}^> > \mathcal{E}^<$ for all $p > 2/9$.

This is an interesting result. Note that the DEC holds when $T_{00} \equiv \mathcal{H} > 0$, and the SEC holds when $T_{00} + T/2 \equiv \mathcal{E} > 0$. It is apparent that $\mathcal{H} < 0$ and $\mathcal{E} < 0$ for the expanding solutions in LCS theory. Therefore, the DEC and SEC are both violated as mentioned earlier. Consequently, the energy constraints turn on a set of inequalities, following Eq. (1.18),

$$\Lambda + \mathcal{H} = \Lambda_0 \le \frac{1}{3}K^2 \le \Lambda_1 = \Lambda - \mathcal{E}. \tag{4.9}$$

Therefore, the upper and lower bounds, in contrast to the case with a single Λ, turn on a stability window with a width $\Gamma = -(\mathcal{E} + \mathcal{H}) > 0$. Apparently, Eq. (4.9) implies that $\sigma^2 = 0$ if $\mathcal{H} + \mathcal{E} \ge 0$, or equivalently $\Gamma \le 0$. This is, however, not the case for the LCS expanding solutions discussed here.

For the case of the LCS expanding solutions, we can put the inequality in a more transparent form. Indeed, we can show that the window of the large-p solution is smaller than the window of the small-p solution. Indeed, the result is

$$\Lambda_0^< \le \Lambda_0^> \le \frac{1}{3}K^2 \le \Lambda_1^> \le \Lambda_1^<. \tag{4.10}$$

Equation (4.10) implies that $\Gamma_> < \Gamma_<$ with $\Gamma_> = \Lambda_1^> - \Lambda_0^>$ and $\Gamma_< = \Lambda_1^< - \Lambda_0^<$. If the window of stability is smaller, the solution should be more unlikely to be a stable solution. This inequality may be interpreted as: the smaller that $\mathcal{E}$ and $\mathcal{H}$ are, the more unlikely the solutions will remain stable. Hence, the above result indicates clearly that the large-p solution is energetically more unfavorable for stability, as compared to the small-p solution.

In addition, the energy components $\mathcal{H}_> > \mathcal{H}_<$ and $\mathcal{E}_> > \mathcal{E}_<$ imply that the small-p solutions are energetically more stable than the large-p solutions. Therefore, large-p solutions should be unstable and evolving toward the more stable small-p solutions in the long run. In fact, the argument regarding the energy conditions $\mathcal{H}_> > \mathcal{H}_<$ and $\mathcal{E}_> > \mathcal{E}_<$ is equivalent to the argument regarding the width $\Gamma_> < \Gamma_<$. Both arguments indicate large-p solutions are unlikely to be stable solutions. Although this is not a rigorous proof of the claim that large-p solutions are also unstable, the inequality shown above will hopefully shed light on the final answer.

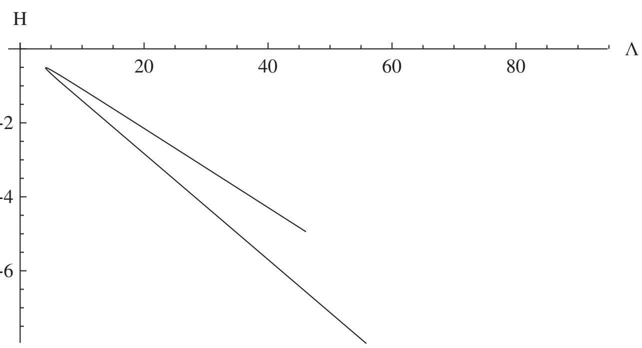

FIG. 3. $\mathcal{H}(p)$ versus $\Lambda(p)$ are plotted from $p = 2/9$ to $p = 100$. The upper curve represents $\mathcal{H}(p)$ associated with the large $p > p_0$ region. The plot indicates that $\mathcal{H}^> > \mathcal{H}^<$ for any allowable Λ.

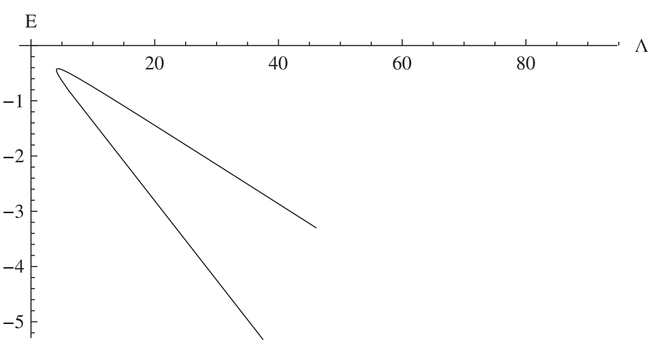

FIG. 4. $\mathcal{E}(p)$ versus $\Lambda(p)$ are plotted from $p = 2/9$ to $p = 100$. The upper curve represents $\mathcal{E}(p)$ associated with the large $p > p_0$ region. The plot indicates that $\mathcal{E}^> > \mathcal{E}^<$ for any allowable Λ.

C. Isotropic limit

Note also that the field equations become

$$3\dot{\alpha}^2 = \Lambda + \frac{1}{4}e^{-2\alpha} + \frac{1}{6}e^{-6\alpha} \tag{4.11}$$

in the isotropic limit when $\beta = 0$. Perturbing the fields $\delta\alpha$ leads to the following equations:

$$\left(6\dot{\alpha}\nu + \frac{1}{2}e^{-2\alpha} + e^{-6\alpha}\right)\delta\alpha = 0, \tag{4.12}$$

with $\delta\alpha = \alpha_0 \exp[\nu t]$ and α_0 a constant parameter. Therefore, we can obtain the following solution:

$$\nu = \nu_1 = -\frac{\exp[-2\alpha] + 2\exp[-6\alpha]}{12\dot{\alpha}}, \tag{4.13}$$

which is always negative for any expanding solutions with $\dot{\alpha} > 0$. Therefore, any isotropically expanding solutions are always stable. Note also that the expanding solutions will approach $\alpha \to \sqrt{\Lambda/3}$ at the future infinity. Hence the isotropically expanding solutions are indeed the stable future solutions for the LCS theory.

V. BIANCHI TYPE I EXPANDING SOLUTIONS

The solutions in the other class A Bianchi models are in general very difficult to solve. For heuristic purposes, we will present a set of anisotropically expanding solutions in the Bianchi type I space. The field equations turn out to be completely integrable after a few appropriate rearrangements of the field equations. In addition, it is known that $A_{\mu\nu} = 0$ and $\Omega_L = 0$ for the Bianchi type I spaces. Therefore, the solutions to the LCS term are given simply as $\dot{b} = C/V$. We will try to solve for the field equations with $C \neq 0$ in this section. A set of expanding solutions will be discussed here with a detailed analysis on the parameters characterizing the initial conditions of the various field variables.

First of all, the Bianchi type I metric can be read off from the following equation:

$$\begin{aligned} ds^2 &= -dt^2 + a_1^2 dx^2 + a_2^2 dy^2 + a_3^2 dz^2 \\ &= -dt^2 + \exp[2\alpha - 4\sigma_+]dx^2 \\ &\quad + \exp[2\alpha + 2\sigma_+ + 2\sqrt{3}\sigma_-]dy^2 \\ &\quad + \exp[2\alpha + 2\sigma_+ - 2\sqrt{3}\sigma_-]dz^2. \end{aligned} \tag{5.1}$$

The equation $G^\mu{}_\mu + G^0{}_0 = -2(3\dot{H} + 9H^2) = -6\Lambda$ can be simplified as

$$\ddot{\alpha} + 3\dot{\alpha}^2 = \Lambda. \tag{5.2}$$

This equation can be solved by defining the volume factor $V = a_1 a_2 a_3 = \exp[3\alpha]$. Indeed, we can write Eq. (5.2) as

$$\ddot{V} = 3\Lambda V, \tag{5.3}$$

which is a linear equation in V. This can be solved to give a linear combination of the exponential solutions,

$$V = a\exp[\sqrt{3\Lambda}t] + b\exp[-\sqrt{3\Lambda}t]. \tag{5.4}$$

We can therefore show that $\exp[3\alpha]$ becomes

$$\begin{aligned} V &= \exp[3\alpha] \\ &= \exp[3\alpha_0]\left[\cosh\sqrt{3\Lambda}t + \frac{\dot{\alpha}_0}{\sqrt{\Lambda/3}}\sinh\sqrt{3\Lambda}t\right] \end{aligned} \tag{5.5}$$

with $\alpha_0 = \alpha(t=0)$ and $\dot{\alpha}_0 = \dot{\alpha}(t=0)$ as the appropriate initial values. The axion field b can be solved to give $\dot{b} = C/V$. Therefore, $\dot{b}$ can be expressed as

$$\dot{b} = C\exp[-3\alpha] \tag{5.6}$$

with the help of Eq. (5.5). In addition, the field equation $G^1{}_1 + G^0{}_1 = -2\Lambda$ can be shown to give

$$(\partial_t + 3\dot{\alpha})\dot{\sigma}_+ = 0 \tag{5.7}$$

that has the following solution:

$$\dot{\sigma}_+ = k_+\exp[-3\alpha] \tag{5.8}$$

with k_+ an integration constant. Moreover, we can also show that $\dot{\alpha}^2$ takes the following simple form when it is combined with $-\Lambda/3$:

$$\dot{\alpha}^2 - \frac{\Lambda}{3} = \left(\dot{\alpha}_0^2 - \frac{\Lambda}{3}\right)\exp[6\alpha_0]\exp[-6\alpha]. \tag{5.9}$$

As a result, from the Friedmann equation

$$G^0{}_0 = -3(\dot{\alpha}^2 - \dot{\sigma}_+^2 - \dot{\sigma}_-^2) = -\Lambda - 6\dot{b}^2, \tag{5.10}$$

we can show that the following solution to $\dot{\sigma}_-$,

$$\begin{aligned} \dot{\sigma}_- = \pm&\left[\dot{\alpha}_0^2 - \frac{\Lambda}{3} - (k_+^2 + 2C^2)\exp[-6\alpha_0]\right]^{1/2} \\ &\times\left[\cosh\sqrt{3\Lambda}t + \frac{3\dot{\alpha}_0}{\sqrt{3\Lambda}}\sinh\sqrt{3\Lambda}t\right]^{-1}, \end{aligned} \tag{5.11}$$

exists only when

$$\dot{\alpha}_0^2 \geq \frac{\Lambda}{3} + (k_+^2 + 2C^2)\exp[-6\alpha_0]. \tag{5.12}$$

In summary, we have found a set of solutions of the following form:

$$\dot{\sigma}_+ = k_+\exp[-3\alpha] \tag{5.13}$$

$$\dot{\sigma}_- = k_-\exp[-3\alpha] \tag{5.14}$$

$$\dot{b} = C\exp[-3\alpha] \tag{5.15}$$

$$\dot{\alpha}^2 - \frac{\Lambda}{3} = k_\alpha^2\exp[-6\alpha] \tag{5.16}$$

with

$$k_- = \pm\left[\dot{\alpha}_0^2 - \frac{\Lambda}{3} - (k_+^2 + 2C^2)\exp[-6\alpha_0]\right]^{1/2}\exp[3\alpha_0], \tag{5.17}$$

$$k_\alpha^2 = \left(\dot{\alpha}_0^2 - \frac{\Lambda}{3}\right)\exp[6\alpha_0]. \tag{5.18}$$

This implies that the constraint (5.12) becomes

$$\dot{\alpha}_0^2 = \frac{\Lambda}{3} + (k_+^2 + k_-^2 + 2C^2)\exp[-6\alpha_0], \tag{5.19}$$

or equivalently,

$$k_\alpha^2 = k_+^2 + k_-^2 + 2C^2. \tag{5.20}$$

An equation of the form (5.13) can be integrated directly to give

$$\sigma_+ = \sigma_1 + \left[\frac{k_+ \exp[-3\alpha_0]}{3\sqrt{\dot\alpha_0^2 - \Lambda/3}}\right]\left\{\left[\ln\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right] + \ln\left[\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right]\right\} \tag{5.21}$$

with $\sigma_1 = \sigma_+(0)$. Similarly, the solutions of σ_- are

$$\sigma_- = \sigma_2 + \left[\frac{k_- \exp[-3\alpha_0]}{3\sqrt{\dot\alpha_0^2 - \Lambda/3}}\right]\left\{\left[\ln\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right] + \ln\left[\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right]\right\} \tag{5.22}$$

with $\sigma_2 = \sigma_-(0)$. Likewise, $\dot b$ can also be integrated to give

$$b = b_0 + \left[\frac{C \exp[-3\alpha_0]}{3\sqrt{\dot\alpha_0^2 - \Lambda/3}}\right]\left\{\left[\ln\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} - \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right] + \ln\left[\frac{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}} + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}{\sqrt{\dot\alpha_0 + (\Lambda/3)^{1/2}}\exp[\sqrt{3\Lambda}t] + \sqrt{\dot\alpha_0 - (\Lambda/3)^{1/2}}}\right]\right\} \tag{5.23}$$

with $b_0 = b(0)$. Note that both $\sigma_\pm \to$ constant at time infinity. Therefore, anisotropy will decrease as the Universe expands. In particular, we can show explicitly that the expanding solutions we found are stable under perturbations of the following form: $(\delta\alpha, \delta\sigma_+, \delta\sigma_-) = (k_1, k_2, k_3)\exp[\nu t]$. Indeed, we can cast the perturbation equations derived from perturbing the field Eqs. (5.2), (5.7), and (5.10), as a matrix equation:

$$\mathcal{D}\begin{pmatrix}\delta\alpha\\ \delta\sigma_+\\ \delta\sigma_-\end{pmatrix} \equiv \begin{bmatrix} \nu + 6\dot\alpha, & 0, & 0\\ 3\dot\sigma_+, & \nu + 3\dot\alpha, & 0\\ \dot\alpha\nu + 6C^2\exp[-6\alpha], & -\dot\sigma_+\nu, & -\dot\sigma_-\nu \end{bmatrix} \times \begin{pmatrix}\delta\alpha\\ \delta\sigma_+\\ \delta\sigma_-\end{pmatrix} = 0. \tag{5.24}$$

A nontrivial solution exists only when $\det\mathcal{D} = 0$. Therefore, we can show that $\nu = -3\dot\alpha$ or $\nu = -6\dot\alpha$ for the perturbation. Consequently, these sets of solutions are stable solutions.

In the limit $\sigma_- = \sigma_1$, $k_- = 0$. In this case, $a_2 = a_3$. This will imply that the constraint (5.19) becomes

$$\dot\alpha_0^2 = \frac{\Lambda}{3} + (k_+^2 + 2C^2)\exp[-6\alpha_0]. \tag{5.25}$$

Note that the field equations in the Bianchi type VII_0 and type II under the constraint $a_2 = a_3$ are identical to the field equations in the Bianchi type I space discussed here. Therefore, they all have the same anisotropically expanding solutions shown here in this section.

Moreover, for the Friedmann-Robertson-Walker metric, namely, $a_1 = a_2 = a_3$ or equivalently $\sigma_\pm = 0$, the constraint (5.19) becomes

$$\dot{\alpha}_0^2 = \frac{\Lambda}{3} + 2C^2 \exp[-6\alpha_0]. \tag{5.26}$$

The constraint (5.19) is quite interesting because it puts all the initial conditions on the field variables α_0, $\dot{\alpha}_0$, $k_\pm$, and C binded together with the cosmological constant Λ. It implies that the initial inflation prescribed by $\dot{\sigma}_0$ has to be large enough to cover all the contribution from the left-hand side of the constraint (5.19). Or we can look at it equivalently from a different viewpoint. It implies that the contributions from the cosmological constant Λ, the anisotropy $k_\pm$, and the LCS term C tend to drive the expansion wilder than the system without these contributions.

This result can be interpreted as: more energy is required to drive the anisotropy of this inflationary solution. It requires more energy to keep the LCS contribution related to the parameter C. It also requires more energy to sustain the anisotropic growth signified by the parameters $k_\pm$. This is in a sense consistent with the idea of Coleman's conjecture [37] stating that the most stable vacuum comes with the maximal symmetry of the system. It is apparent that the isotropic Friedmann-Robertson-Walker metric has a maximal symmetry group of rotation and translation as compared to the smaller symmetry subgroup of translation associated with the Bianchi type I spaces. Roughly speaking, Coleman's conjecture is another expression of Hawking's no-hair conjectures. All results shown in this paper appear to support the tendency of the evolution toward the lowest available stable vacuum of the system.

VI. CONCLUSION

We have shown that, other than the minimum value of $\Lambda \cong 4.1145$, there are two independent expanding solutions for any given value of Λ. One is in the unstable zone with $p = p_< < p_0$; the other is in the large-p region ($p = p_> > p_0$) such that $\Lambda(p_>) = \Lambda(p_<)$. We also show that $\mathcal{H} = T_{00}$ and $\mathcal{E} = T_{00} + T/2$ associated with the unstable mode in the small-p region ($p < p_0$) are larger than the counterpart in the large-p region ($p > p_0$) for the same Λ. Indeed, the energy components $\mathcal{H}_> > \mathcal{H}_<$ and $\mathcal{E}_> > \mathcal{E}_<$ imply that the small-p solutions are energetically more stable than the large-p solutions. Therefore, large-p solutions should be unstable and evolving toward the more stable small-p solutions in the long run.

We have also shown that the energy constraints turn on a set of inequalities, following a revised version of the Wald's inequalities shown in Eq. (1.18). Indeed, it can be shown that

$$\Lambda + \mathcal{H} = \Lambda_0 \leq \frac{1}{3}K^2 \leq \Lambda_1 = \Lambda - \mathcal{E}.$$

Therefore, the upper and lower bounds, in contrast to the case with a single Λ, turn on a stability window with a width $\Gamma = -(\mathcal{E} + \mathcal{H}) > 0$. Apparently, Eq. (4.9) implies that $\sigma^2 = 0$ if $\mathcal{H} + \mathcal{E} \geq 0$, or equivalently $\Gamma \leq 0$. This is, however, not the case for the LCS expanding solutions discussed here.

Therefore, these energy inequalities also squeeze the window of instability of the large-p solutions as compared to the small-p solutions. Indeed, we have shown that $\Gamma_< > \Gamma_>$ from Eq. (4.10). This result appears to indicate that the large-p solutions are more unlikely to remain stable as anisotropically expanding solutions. Although this is not a rigorous proof, this result does reveal some important messages carried by the modified Wald inequality (4.10) associated with the energy conditions (1.2) and (1.3). Hopefully, further understanding of the properties associated with Hawking's conjecture may become more transparent with the help of the results shown in this paper.

The system is inevitably unstable once an unstable mode, that will carry the system off the anisotropically expanding phase, is found. It is not, however, enough to claim that an expanding solution is stable simply from the result that no unstable mode can be found in the perturbation directions being considered. It is still possible that an unstable mode could exist in different directions such as the inhomogeneous perturbations, or the perturbations of additional physical fields that have not been included here. Additional directions of perturbations may not be easy to perform in a complete and thorough manner. For example, we have also tried to perturb the metric filed along the direction g_{33}. The resulting stability modes are the same as the ones found in this paper plus an additional more stable mode associated with the g_{33} perturbation.

For heuristic reasons, we also present a complete analysis of a set of anisotropically expanding solutions of the same model in the Bianchi type I space. This set of exact and anisotropically expanding solutions is known to be unstable following the proof by Robert Wald. [4] We show explicitly that the anisotropy field variables $\sigma_\pm$ will both go to a constant at time infinity. The result indicates that the lowest energy vacuum solution, namely, the isotropic Friedmann-Robertson-Walker space appears to be the most stable vacuum state.

All the results shown in this paper indicate that the LCS theory does carry interesting information concerning the evolution of our physical universe. Further understanding of the implications of this theory could be important to the understanding of the evolution of our early universe.

ACKNOWLEDGMENTS

This research is supported in part by the NSC of Taiwan under Contract No. NSC 98-2112-M-009-02-MY3.

APPENDIX: POLYNOMIAL EQUATIONS

The roots of a quartic polynomial equation

$$x^4 + ax^3 + by^2 + cy + d = 0 \tag{A1}$$

with arbitrary coefficients a, b, c, d can be shown to be

$$x = -\frac{f}{2} \pm \frac{1}{2}\left[\frac{3a^2 - 4f^2 - 8b}{4} + \left(\frac{8c - 4ab + a^3}{4f}\right)\right]^{1/2} - \frac{a}{4}, \tag{A2}$$

$$x = \frac{f}{2} \pm \frac{1}{2}\left[\frac{3a^2 - 4f^2 - 8b}{4} - \left(\frac{8c - 4ab + a^3}{4f}\right)\right]^{1/2} - \frac{a}{4} \tag{A3}$$

with

$$f = \left\{\Upsilon \cos\left[\frac{1}{3}\cos^{-1}\frac{4S}{\Upsilon^3}\right] - \frac{2b}{3} + \frac{a^2}{4}\right\}^{1/2}, \tag{A4}$$

and

$$\Upsilon^2 = \frac{4}{9}b^2 + \frac{16}{3}d - \frac{4}{3}ac, \qquad S = \frac{2}{27}b^3 + c^2 - \frac{abc}{3} - \frac{8bd}{3} + a^2 d. \tag{A5}$$

In addition, the roots of a cubic polynomial equation

$$x^3 + ax^2 + by + c = 0 \tag{A6}$$

with arbitrary coefficients a, b, c can be shown to be

$$x = -\frac{a}{3} + \frac{2\sqrt{a^2 - 3b}}{3}\cos\left\{\frac{1}{3}\cos^{-1}\left[4\left(\frac{9ab - 2a^3 - 27c}{27}\right) \times \left(\frac{4a^2 - 12b}{9}\right)^{-3/2}\right] + \frac{2n}{3}\pi\right\}, \tag{A7}$$

with $n = 0, 1, 2$ representing three different roots.

[1] A. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[3] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).
[4] R. Wald, Phys. Rev. D **28**, 2118 (1983).
[5] J. D. Barrow, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983), p. 267.
[6] W. Boucher and G. W. Gibbons, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983), p. 273.
[7] A. A. Starobinskii, Phys. Lett. **91B**, 99 (1980).
[8] L. G. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).
[9] J. D. Barrow, Phys. Lett. B **187**, 12 (1987).
[10] J. D. Barrow, Phys. Lett. B **235**, 40 (1990); J. D. Barrow and P. Saich, Phys. Lett. B **249**, 406 (1990); J. D. Barrow and A. R. Liddle, Phys. Rev. D **47**, R5219 (1993); A. D. Rendall, Classical Quantum Gravity **22**, 1655 (2005).
[11] J. D. Barrow, Nucl. Phys. **B296**, 697 (1988).
[12] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).
[13] K. I. Maeda, Phys. Rev. D **39**, 3159 (1989).
[14] S. Kanno, M. Watanabe, and J. Soda, Phys. Rev. Lett. **102**, 191302 (2009).
[15] S. Kanno, J. Soda, and M. Watanabe, arXiv:1010.5307v2.J. Cosmol. Astropart. Phys. 12 (2010) 024.
[16] E. Weber, J. Math. Phys. (N.Y.) **25**, 3279 (1984); H. H. Soleng, Classical Quantum Gravity **6**, 1387 (1989); N. Kaloper, Phys. Rev. D **44**, 2380 (1991).
[17] J. D. Barrow, Phys. Lett. B **180**, 335 (1986).
[18] J. D. Barrow, Nucl. Phys. **B310**, 743 (1988).
[19] J. D. Barrow, Phys. Lett. B **183**, 285 (1987).
[20] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[21] A. D. Linde, Phys. Lett. B **175**, 395 (1986); A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).
[22] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University, New York, 1986).
[23] A. Berkin, Phys. Rev. D **44**, 1020 (1991).
[24] E. Bruning, D. Coule, and C. Xu, Gen. Relativ. Gravit. **26**, 1197 (1994).
[25] E. E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).
[26] M. Tanimoto, V. Moncrief, and K. Yasuno, Classical Quantum Gravity **20**, 1879 (2003).
[27] S. Hervik, in CMS Summer Meeting, Halifax, NS, Canada, 2004 (unpublished).
[28] H.-J. Schmidt, arXiv:gr-qc/0407095.
[29] J. D. Barrow and Sigbjorn Hervik, Phys. Rev. D **73**, 023007 (2006).
[30] J. D. Barrow and Sigbjorn Hervik, Phys. Rev. D **74**, 124017 (2006).
[31] W. F. Kao, Eur. Phys. J. C **53**, 87 (2007).
[32] W. F. Kao and I. C. Lin, J. Cosmol. Astropart. Phys. 01 (2009) 022.
[33] W. F. Kao, Phys. Rev. D **79**, 043001 (2009).
[34] W. F. Kao, Eur. Phys. J. C **65**, 555 (2009).
[35] W. F. Kao and I. C. Lin, Phys. Rev. D **83**, 063004 (2011).
[36] R. Kantowski and R. K. Sachs, J. Math. Phys. (N.Y.) **7**, 443 (1966).
[37] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures* (Cambridge University Press, Cambridge, England, 1988), ISBN 0-521-31827-0; , Phys. Rev. D **15**, 2929 (1977); C. G. Callan, Jr. and S. Coleman, Phys. Rev. D **16**, 1762 (1977).