Analysis of Diversity-Multiplexing Tradeoff in a Cooperative Network Coding System

Li-Chun Wang, Wei-Cheng Liu, and Sau-Hsuan Wu

***Abstract*—This paper addresses the analysis of diversity-multiplexing tradeoff (DMT) for a decode-and-forward (DF)-based cooperative network coding (CNC) system. The exact outage probability is also provided. Our results show that network coding can assist the relay node to improve multiplexing and diversity gain.**

***Index Terms*—Cooperative communications, decode-and-forward, diversity-multiplexing tradeoff, network coding, outage probability.**

I. Introduction

Cooperative communication has been attracting a great deal of attention recently. Relay nodes in a cooperative communication system can assist the transmitter in sending information to the receiver. This process is similar to a virtual multiple-input multiple-output (MIMO) system because the nodes in a cooperative network form a virtual antenna array. Clearly, cooperative communication systems can provide diversity gain, referred to as *cooperative diversity* similar to the space diversity provided by MIMO antenna techniques. Many cooperative communication protocols have been proposed to improve diversity gain, and these fall into three main categories. The first category is based on amplify-and-forward (AF) [1], such as incremental AF (IAF) [1] and nonorthogonal AF (NAF) [2]. The second category is based on space-time coding (STC), such as in [3]–[5]. The third category is based on decode-and-forward (DF) systems, such as selection DF (SDF) [1], dynamic DF (DDF) [2], enhanced static DF (ESDF), and enhanced dynamic DF (EDDF) [6]. Cooperative communications can also enable ad hoc networks to extend their coverage and improve capacity [7]. In a multi-user environment, users can share resources via the cooperative communication system.

As previously mentioned, a cooperative communication system can be viewed as a virtual MIMO system. Hence, cooperative communication systems offer the diversity-multiplexing tradeoff (DMT) feature of MIMO systems [8]. The DMTs of a number of well-known cooperative protocols have been identified in the literature, including AF, SDF, DF, IAF [1], DDF, NAF [2], STC1–3 [3]–[5], ESDF, and EDDF [6]. The IAF protocol provides the optimal DMT performance among the aforementioned protocols. In addition to improvements in diversity gain, the means by which multiplexing gain can be provided by taking advantage of relays, has also received a great deal of attention.

Combining network coding with cooperative communication, referred to as *cooperative network coding (CNC)* or time division broadcast (TDBC) protocol [7], has the potential to exploit multiplexing gain from relay nodes. Moreover, CNC has the advantage of not requiring any changes to the physical layer, as well as high bandwidth efficiency [7]. Previous studies related to CNC are listed in [7], [9]–[19].

Outage probability and DMT are two common means by which the performance of cooperative protocols is measured. The DMT analysis of CNC has not previously appeared in the literature. Through the analysis of DMT, it can be observed that exploiting the "XOR in the air" approach to cooperative protocols could improve multiplexing gain, diversity gain, or both compared with traditional cooperative protocols. Moreover, DMT is able to provide greater insight than simply using capacity results. Capacity is only a quantity representing the upper bound of the error-free transmission rate in the lossy channel, while DMT is a tradeoff curve that can provide a tradeoff between various transmission rates and error rates in the erroneous channel. This was our motivation in deriving outage probability and DMT of the CNC.

In this paper, we derived the exact outage probability and DMT of the CNC protocol. DMT is the limit of the slope of outage probability versus the signal-to-noise ratio (SNR) curve in the dB domain as SNR approaches infinity. The difficulty in analyzing DMT is the need to express the limit as a function between diversity gain and multiplexing gain, rather than as a single value. The uniqueness of our method in determining DMT is the fact that we divide the problem of expressing the limit into several simpler sub-problems that can be solved individually. As seen in Theorem 2, the proof is succinct and easy to follow.

The remainder of this paper is organized as follows. In Section II, we describe the system model and introduce the CNC protocol. We provided definitions of important terminology and analyze the outage probability and DMT of the CNC protocol in Section III. Numerical results are shown in Section IV, and conclusions are presented in Section V.

II. System Model and CNC Protocol

A. System Model

Figure 1 shows the system model for the CNC with a single relay node. Nodes A and B transmit and receive user

Paper approved by C. Fragouli, the Editor for Network Coding and Network Information Theory of the IEEE Communications Society. Manuscript received January 26, 2009; revised January 11, August 24, October 27, 2010, and January 1, 2011.

This work is supported by the National Science Council, Taiwan, under the contract NSC95-2221-E-009-147.

L.-C. Wang is with the Department of Electrical Engineering, National Chiao Tung University, Hsinchu, Taiwan (e-mail: lichun@cc.nctu.edu.tw).

W.-C. Liu is with the Department of Communications Engineering, National Chung Cheng University, Chia-Yi, Taiwan (e-mail: comwcliu@ccu.edu.tw).

S.-H. Wu is with the Department of Electrical Engineering, National Chiao Tung University, Hsinchu, Taiwan (e-mail: sauhsuan@cm.nctu.edu.tw).

Digital Object Identifier 10.1109/TCOMM.2011.062311.090011

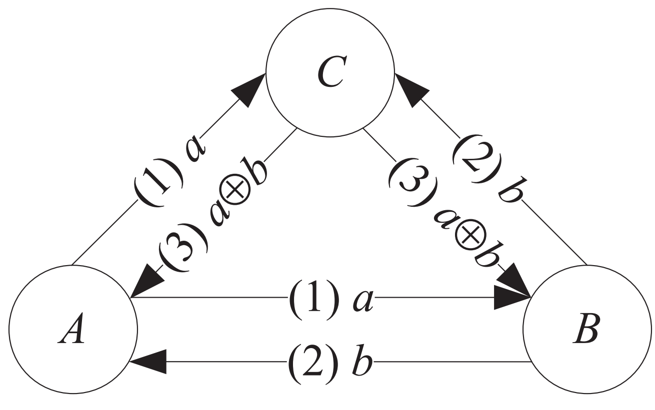

Fig. 1. System model and the CNC protocol, where phase (1): A sends a to B and C; phase (2): B sends b to A and C; phase (3): C broadcasts $a \oplus b$ to A and B.

data, and relay C forwards data. We denote the channel gains between nodes X and Y as h_{XY}, where $X, Y \in \{A, B, C\}$. In addition to additive white Gaussian noise (AWGN), the radio channel effect h_{XY} experienced at each node is assumed to be independent and identically distributed (i.i.d.) complex normal random variables with zero mean and unity variance. We consider the half-duplex nodes which are unable transmit or receive data simultaneously. In the considered scenario, nodes A and B can directly communicate with each other.

B. CNC Protocol

In Fig. 1 the CNC protocol is illustrated for the case using a single relay node. In phases (1) and (2), A and B transmit information a and b, respectively. Then C decodes a and b in the binary form and computes $c = a \oplus b$, where $\oplus$ is the bitwise exclusive or (XOR) operator. In phase (3), node C broadcasts the mixed information c to A and B. In this manner, A is able to obtain information b via the operation $c \oplus a = b$ and node B is able to obtain information a via the operation $c \oplus b = a$. In this case, the relay node plays the role of DF [1].

III. Outage Probability and Analysis of DMT

A. Definition

In this subsection, we define SNR, multiplexing gain r, outage probability, and diversity gain d for the considered CNC system. The SNR is defined as

$$\text{SNR} := \frac{\mathrm{E}\{|x_k[n]|^2\}}{N_0}, \tag{1}$$

where $x_k[n]$ is the transmitted signal containing information k and $k \in \{a, b, c\}$. $\mathrm{E}\{Z\}$ is the expectation of a random variable Z.

Denote R as the data rate for each channel, where R can be a function of SNR [8]. The multiplexing gain r is defined as

$$r := \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\log \text{SNR}}. \tag{2}$$

Note that the base of the $\log$ function is 2 in this paper.

The diversity gain d is defined as

$$d := -\lim_{\text{SNR}\to\infty} \frac{\log[P_{\text{out}}(\text{SNR})]}{\log \text{SNR}}, \tag{3}$$

where $P_{\text{out}}(\text{SNR})$ is the outage probability of the overall system.

B. Analysis of Outage Probability

The outage probability of the CNC protocol with a single relay node is analyzed by the following theorem:

Theorem 1: The outage probability of the CNC protocol with a single relay node is characterized by

$$P_{\text{out}}^{\text{CNC}} = \left[1 - \exp\left(-\frac{2^{3R/2}-1}{\text{SNR}}\right)\right]^2 \cdot \left[1 + \exp\left(-\frac{2^{3R/2}-1}{\text{SNR}}\right)\right]. \tag{4}$$

Proof: Let E_{XY} denote the outage event on the link between nodes X and Y, where $X, Y \in \{A, B, C\}$ but $X \neq Y$. Clearly, $E_{XY} = E_{YX}$. Events E_{AB}, E_{BC}, and E_{CA} are independent. Furthermore,

$$\begin{aligned}\mathrm{P}[E_{XY}] &= \mathrm{P}\left[\frac{2}{3}\log(1 + |h_{XY}|^2\text{SNR}) < R\right] \\ &= \mathrm{P}\left[|h_{XY}|^2 < \frac{2^{3R/2}-1}{\text{SNR}}\right] \\ &\overset{(a)}{=} 1 - \exp\left(-\frac{2^{3R/2}-1}{\text{SNR}}\right) := p.\end{aligned} \tag{5}$$

The factor $\frac{2}{3}$ in the first line is due to the transmission of two independent packets a and b in three time slots (phases). The equality (a) holds because $|h_{XY}|^2$ is an exponential random variable with unity mean.

The outage event for node A is

$$E_A = E_{BA} \cap E_{CA}. \tag{6}$$

The equation holds because the outage event for node A occurs if both events E_{BA} and E_{CA} occur. Similarly, the outage event for node B can be written as

$$E_B = E_{AB} \cap E_{CB}. \tag{7}$$

The outage event for the overall system is expressed as

$$E_{\text{out}}^{\text{CNC}} = E_A \cup E_B = E_{AB} \cap (E_{AC} \cup E_{BC}). \tag{8}$$

The outage probability of the CNC system is

$$\begin{aligned}P_{\text{out}}^{\text{CNC}} &= \mathrm{P}[E_{AB} \cap (E_{AC} \cup E_{BC})] \\ &= \mathrm{P}[E_{AB}](\mathrm{P}[E_{AC}] + \mathrm{P}[E_{BC}] - \mathrm{P}[E_{AC} \cap E_{BC}]) \\ &= p \cdot (2p - p^2) = p^2(2-p).\end{aligned} \tag{9}$$

Substituting (5) into (9), we can get (4). ■

C. Analysis of DMT

The following theorem gives the DMT for the CNC protocol with a single relay node.

Theorem 2: The DMT achieved by the CNC protocol with a single relay node is characterized by $d_{\text{CNC}}(r) = 2 - 3r$ for $0 < r < 2/3$.

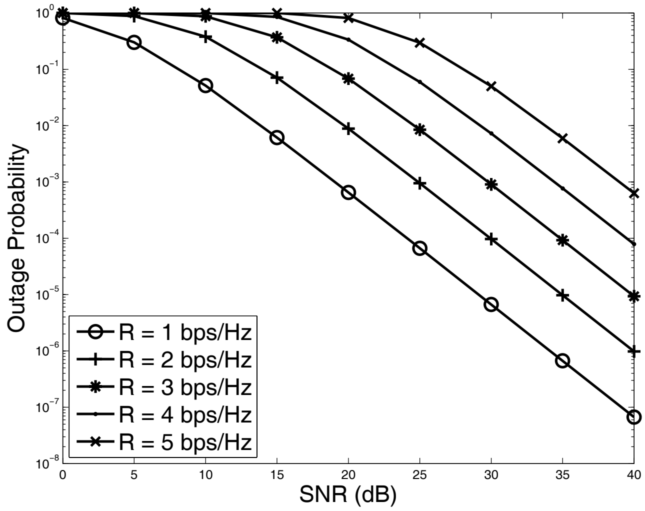

Fig. 2. Outage probability for the cooperative network coding (CNC) protocol when the data rate R is equal to 1, 2, 3, 4, 5 bps/Hz.

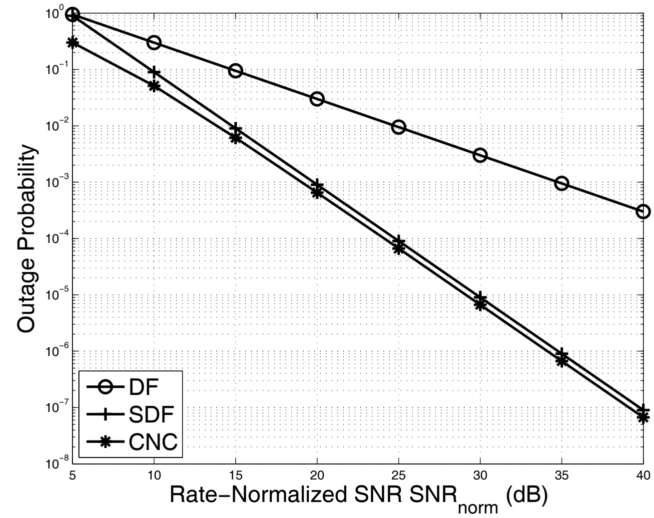

Fig. 3. Comparison of outage probability for decode-and-forward (DF), selection decode-and-forward (SDF), and cooperative network coding (CNC) protocol when the data rate R is equal to 1 bps/Hz.

Proof:

$$\begin{aligned} d_{\text{CNC}}(r) &= -\lim_{\text{SNR}\to\infty} \frac{\log P_{\text{out}}^{\text{CNC}}}{\log \text{SNR}} \\ &= -\lim_{\text{SNR}\to\infty} \frac{\log[p^2(2-p)]}{\log \text{SNR}} \\ &= 2\lim_{s\to 0} \frac{\log p}{\log s} + \lim_{s\to 0} \frac{\log(2-p)}{\log s}, \end{aligned} \tag{10}$$

where p is defined in (5) and $s = 1/\text{SNR}$. Substituting $R = r \log \text{SNR}$ [8] into p, we have

$$p = 1 - \exp(s - s^{1-3r/2}) \ . \tag{11}$$

Then we can obtain $\lim_{s\to 0} p = 0$ for $1 - 3r/2 > 0$ or $r < 2/3$. Hence, the second term of (10) is equal to zero. Apply the l'Hôpital's rule two times on the first term of (10), we have

$$\begin{aligned} d_{\text{CNC}}(r) &= \lim_{s\to 0} \Big[2s + (2-3r)(1 - s^{1-3r/2}) \\ &\quad + \frac{3r}{1 - (1-3r/2)s^{-3r/2}} \Big] \\ &= 2 - 3r \end{aligned} \tag{12}$$

for $0 < r < 2/3$. ■

IV. Numerical Results

Figure 2 shows the outage probability for the CNC protocol based on Theorem 1 when the data rate R is equal to 1, 2, 3, 4, 5 bps/Hz. As expected, when the data rate increases, the system is more likely to undergo outage and the outage probability also increases.

Figure 3 is a comparison of outage probability for DF, SDF [1], and the CNC protocols when the data rate R is equal to 1 bps/Hz. For the SDF protocol, if the SNR of the received signal at the relay exceeds the threshold, the relay performs the DF operation; otherwise, the relay remains idle [20]. The x-axis is the rate-normalized SNR defined by [1]

$$\text{SNR}_{\text{norm}} := \frac{\text{SNR}}{2^R - 1}. \tag{13}$$

Estimation of the outage probability for DF and SDF in the high SNR region is performed according to [1, Table I]. From this figure, it is clear that the DF protocol has the greatest outage probability, and SDF and CNC protocols have nearly the same outage probability. Furthermore, we observe that SDF and CNC protocols have larger diversity gain than the DF protocol does.

Figure 4 illustrates a comparison of DMT regarding a 2×1 multiple-input single-output (MISO) system, with the CNC protocol, SDF, and DF for a single relay node. Note that the DMT of the 2×1 MISO system is the upper bound of the DMT in single-relay cooperative communication systems [6]. From Fig. 4, we can see that the CNC protocol improved both the diversity gain and multiplexing gain compared with the DF protocol. The maximum diversity and multiplexing gain that the CNC protocol was able to achieve were 2 and 2/3, respectively, while the maximum diversity and multiplexing gain of the DF protocol were 1 and 1/2, respectively. Furthermore, the CNC protocol outperformed the SDF protocol (an enhanced version of DF), and its maximal diversity gain and multiplexing gain were 2 and 1/2, respectively. Hence, we can conclude that using network coding at the relay node improves not only diversity gain but also multiplexing gain.

V. Conclusions

In this paper, we investigated the DMT of a CNC protocol integrating the concept of DF relay transmission in a cooperative communication system with the combination of information involved in network coding. The proposed CNC protocol is suitable for enabling efficient information exchange between two users. We provided two theorems to illustrate outage probability and DMT analytical results. We also provided a comparison of DMT for our CNC protocol with 2×1 MISO system, SDF, and DF. We have clearly demonstrated that our CNC protocol improved both the diversity and multiplexing gain compared with DF protocols.

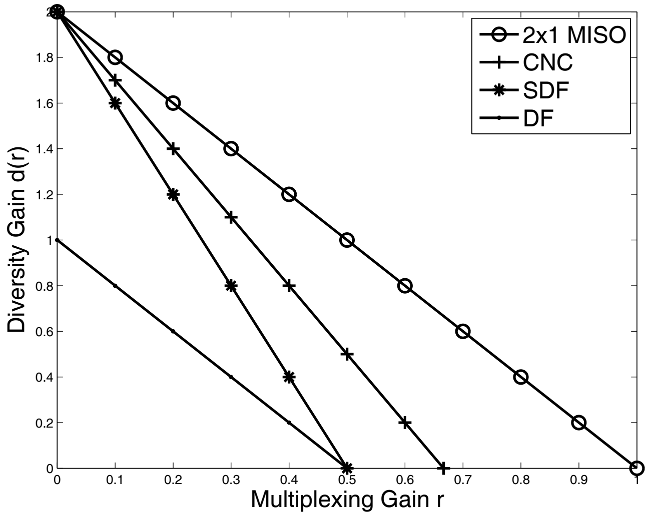

Fig. 4. Comparison of diversity-multiplexing tradeoff the 2×1 multiple-input single-output (MISO) system, cooperative network coding (CNC), selection decode-and-forward (SDF), and decode-and-forward (DF).

REFERENCES

[1] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062-3080, Dec. 2004.

[2] K. Azarian, H. E. Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4152-4172, Dec. 2005.

[3] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415-2425, Oct. 2003.

[4] R. U. Nabar, H. Bolcskei, and F. W. Kneubuhler, "Fading relay channels: performance limits and space-time signal design," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 6, pp. 1099-1109, Aug. 2004.

[5] N. Prasad and M. K. Varanasi, "Diversity and multiplexing tradeoff bounds for cooperative diversity protocols," in *Proc. IEEE Int. Symp. Inf. Theory*, p. 271, June 2004.

[6] ——, "High performance static and dynamic cooperative communication protocols for the half duplex fading relay channel," in *Proc. IEEE Global Commun. Conf.*, pp. 1-5, Nov. 2006.

[7] S. J. Kim, P. Mitran, and V. Tarokh, "Performance bounds for bi-directional coded cooperation protocols," in *Proc. 27th Int. Conf. Distrib. Comput. Syst. Workshops*, p. 83, June 2007.

[8] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073-1096, May 2003.

[9] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1204-1216, July 2000.

[10] B. Rankov and A. Wittneben, "Spectral efficient protocols for nonregenerative half-duplex relaying," in *Allerton Conf. Commun., Control, Comput.*, Sep. 2005. Available: http://www.nari.ee.ethz.ch/wireless/pubs/p/allerton2005

[11] X. Bao and J. Li, "On the outage properties of adaptive network coded cooperation (ANCC) in large wireless networks," in *Proc. IEEE Int. Conf. Acoustics, Speech Signal Process.*, vol. 4, pp. 57-60, May 2006.

[12] Y. Chen, S. Kishore, and J. Li, "Wireless diversity through network coding," in *Proc. IEEE Wireless Commun. Netw. Conf.*, vol. 3, pp. 1681-1686, Apr. 2006.

[13] C. Hausl and J. Hagenauer, "Iterative network and channel decoding for the two-way relay channel," in *Proc. Int. Conf. Commun.*, vol. 4, pp. 1568-1573, June 2006.

[14] C. Hausl and P. Dupraz, "Joint network-channel coding for the multiple-access relay channel," in *Proc. Third Annual IEEE Commun. Society Conf. Sensor, Mesh Ad Hoc Commun. Netw.*, vol. 3, pp. 817-822, Sep. 2006.

[15] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Médard, and J. Crowcroft, "XORs in the air: practical wireless network coding," in *Proc. ACM SIGCOMM*, pp. 243-254, Sep. 2006.

[16] P. Popovski and H. Yomo, "Bi-directional amplification of throughput in a wireless multi-hop network," in *Proc. IEEE 63rd Veh. Technol. Conf. Spring*, pp. 588-593, May 2006.

[17] W. Chen, K. B. Letaief, and Z. Cao, "Opportunistic network coding for wireless networks," in *Proc. Int. Conf. Commun.*, pp. 4634-4639, June 2007.

[18] S. Katti, S. Gollakota, and D. Katabi, "Embracing wireless interference: analog network coding," in *Proc. ACM SIGCOMM*, pp. 397-408, Aug. 2007.

[19] L. Lv, H. Yu, and J. Yang, "Opportunistic cooperative network-coding based on space-time coding for bi-directional traffic flows," in *Proc. IEEE Fourth Workshop Netw. Coding, Theory, Applicat.*, pp. 43-48, Jan. 2008.

[20] K. J. R. Liu, A. K. Sadek, W. Su, and A. Kwasinski, *Cooperative Communications and Networking*. Cambridge University Press, 2009.