



Drawn k -in-a-row games

Sheng-Hao Chiang^a, I-Chen Wu^{b,*}, Ping-Hung Lin^b

^a National Experimental High School at Hsinchu Science Park, Hsinchu, Taiwan

^b Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan

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ABSTRACT

Wu and Huang (2005) [12] and Wu et al. (2006) [13] presented a generalized family of k -in-a-row games, called $Connect(m, n, k, p, q)$. Two players, *Black* and *White*, alternately place p stones on an $m \times n$ board in each turn. Black plays first, and places q stones initially. The player who first gets k consecutive stones of his/her own horizontally, vertically, or diagonally wins. Both tie the game when the board is filled up with neither player winning. A $Connect(m, n, k, p, q)$ game is drawn if neither has any winning strategy. Given p , this paper derives the value $k_{draw}(p)$, such that $Connect(m, n, k, p, q)$ games are drawn for all $k \geq k_{draw}(p)$, $m \geq 1$, $n \geq 1$, $0 \leq q \leq p$, as follows. (1) $k_{draw}(p) = 11$. (2) For all $p \geq 3$, $k_{draw}(p) = 3p + 3d - 1$, where d is a logarithmic function of p . So, the ratio $k_{draw}(p)/p$ is approximately 3 for sufficiently large p . The first result was derived with the help of a program. To our knowledge, our $k_{draw}(p)$ values are currently the smallest for all $2 \leq p < 1000$.

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1. Introduction

A generalized family of k -in-a-row games, called $Connect(m, n, k, p, q)$, [12,13], was introduced and presented by Wu et al. Two players, *Black* and *White*, alternately place p stones on empty squares¹ of an $m \times n$ board in each turn. Black plays first, and places q stones initially. The player who first gets k consecutive stones of his/her own horizontally, vertically, or diagonally wins. Both players tie the game when the board is filled up with neither player winning. For example, *Tic-tac-toe* is $Connect(3, 3, 3, 1, 1)$, *Go-Moku* in the free style (a traditional five-in-a-row game) is $Connect(15, 15, 5, 1, 1)$, and *Connect6* [13], played on the traditional Go board, is $Connect(19, 19, 6, 2, 1)$.

In the past, many researchers have been engaged in solving $Connect(m, n, k, p, q)$ games. One player, either *Black* or *White*, is said to *win* a game, if he/she has a winning strategy such that he/she wins for all the subsequent moves. Allis et al. [1,2] solved *Go-Moku* with *Black* winning. Herik et al. [9] and Wu et al. [12,13] also mentioned several k -in-a-row games with *Black* winning.

A game is said to be *drawn* if neither player has any winning strategy. For simplicity of discussion in this paper, $Connect(k, p)$ refers to the collection of $Connect(m, n, k, p, q)$ games for all $m \geq 1$, $n \geq 1$, $0 \leq q \leq p$. $Connect(k, p)$ is said to be *drawn* if all $Connect(m, n, k, p, q)$ games in $Connect(k, p)$ are drawn. Given p , this paper derives the value $k_{draw}(p)$, such that $Connect(k_{draw}(p), p)$ games are drawn. Since drawn $Connect(k, p)$ games also imply drawn $Connect(k+1, p)$, the value $k_{draw}(p)$ should be as small as possible.

In the past, Zetters [15] derived that $Connect(8, 1)$ is drawn. Pluhar [11] derived tight bounds $k_{draw}(p) = p + \Omega(\log_2 p)$ for all $p \geq 1000$ (see Theorem 1 in [11]). However, the requirement that $p \geq 1000$ is unrealistic in real games. Thus, it is

* Corresponding author. Tel.: +886 3 5731855; fax: +886 3 5733777.

E-mail addresses: jiang555@ms37.hinet.net (S.-H. Chiang), icwu@csie.nctu.edu.tw, icwu@cs.nctu.edu.tw (I.-C. Wu), bhlin@csie.nctu.edu.tw (P.-H. Lin).

¹ Practically, stones are placed on empty intersections of Renju or Go boards. In this paper, when we say squares, we mean intersections.

important to obtain tight bounds when $p < 1000$. Hsieh and Tsai [10] have recently derived that $k_{draw}(p) = 4p + 7$ for all positive p . The ratio $R = k_{draw}(p)/p$ is approximately 4 for sufficiently large p .

In this paper, **Theorem 1** (below) shows that $k_{draw}(2) = 11$, while the result in [10] is 15. **Theorem 2** derives a general bound $k_{draw}(p) = 3p + 3d - 1$ for all $p \geq 1$, where d is a logarithmic function of p , namely $P(d - 1) < p \leq P(d)$ and $P(d) = 2^d - d - 2$. When compared with [10], our $k_{draw}(p)$ values are smaller for all positive p , but they are the same for $k_{draw}(4)$. The ratio $R = k_{draw}(p)/p = 3 + (3d - 1)/p$ is approximately 3 for sufficiently large p . Section 2 modifies the games slightly into those in a different version, named *Maker–Breaker*. Both Sections 3 and 4 will use this version to prove **Theorems 1** and **2**, respectively. When compared with a preliminary version [6], this paper derives a tighter bound for $k_{draw}(3)$ and a more general result, specifically as follows. For all the drawn games, $Connect(\infty, \infty, k, p, p)$, derived in [6], this paper also shows that all games in $Connect(k, p)$ are also drawn, based on the Maker–Breaker argument.

Theorem 1. $Connect(11, 2)$ is drawn. \square

Theorem 2. Consider all $p \geq 1$. Let d be an integer and $P(d - 1) < p \leq P(d)$, where $P(d) = 2^d - d - 2$. Then, $Connect(3p + 3d - 1, p)$ games are drawn. \square

2. Maker–Breaker version

According to the strategy-stealing argument raised by Nash (see [5]), White has no winning strategy in $Connect(m, n, k, p, p)$, that is, when $q = p$. Therefore, for $Connect(m, n, k, p, p)$, either Black wins or White ties. For simplicity of combinatorial analysis, many researchers [3,7,11] followed an asymmetric version of rules, called *Maker–Breaker*, where White does not win in all cases (e.g., even if White connects up to k consecutive stones). So, all White can do is to break, that is, to prevent Black from winning (connecting up to k consecutive stones). In contrast to Maker–Breaker, the version with the original rules is called *Maker–Maker*. Obviously, if White has a strategy to tie a $Connect$ game in the Maker–Breaker version, White can tie the game in the original version (Maker–Maker) by simply following the same strategy. For simplicity of discussion in this paper, let $MBConnect(k, p)$ denote the game $Connect(\infty, \infty, k, p, p)$ in the Maker–Breaker version. **Corollary 1** shows an important property for $MBConnect(k, p)$.

Corollary 1. Assume that $MBConnect(k, p)$ is drawn. Then, $Connect(k, p)$ is drawn. That is, for all $m \geq 1, n \geq 1, 0 \leq q \leq p$, $Connect(m, n, k, p, q)$ games are drawn. \square

The reasons why **Corollary 1** is satisfied are as follows.

1. According to the strategy-stealing argument (also mentioned in [13]), if Black has a winning strategy in $Connect(m, n, k, p, q)$, then Black simply follows the strategy to win in $Connect(m, n, k, p, q + 1)$. On the other hand, if Black has no winning strategy in $Connect(m, n, k, p, q + 1)$, then Black has no winning strategy in $Connect(m, n, k, p, q)$ either. Similarly, if White has no winning strategy in $Connect(m, n, k, p, q)$, White has no winning strategy in $Connect(m, n, k, p, q + 1)$. Assume that $Connect(m, n, k, p, p)$ is drawn. Then, Black has no winning strategy in $Connect(m, n, k, p, p)$. From the previous paragraph, we derive that, for all $0 \leq q \leq p$, Black has no winning strategy in $Connect(m, n, k, p, q)$. On the other hand, since White in $Connect(m, n, k, p, 0)$ is equivalent to Black in $Connect(m, n, k, p, p)$, White does not win in $Connect(m, n, k, p, 0)$ either. From the previous paragraph, we derive that, for all $0 \leq q \leq p$, White has no winning strategy in $Connect(m, n, k, p, q)$. Thus, since neither has any winning strategy, $Connect(m, n, k, p, q)$ games are drawn for all $0 \leq q \leq p$.
2. If Black has a winning strategy in $Connect(m, n, k, p, q)$ in the Maker–Breaker version, then Black simply follows the strategy to win in $Connect(m + 1, n, k, p, q)$, $Connect(m, n + 1, k, p, q)$, or even $Connect(\infty, \infty, k, p, q)$ in the Maker–Breaker version. On the other hand, if Black has no winning strategy in $Connect(\infty, \infty, k, p, q)$ in the Maker–Breaker version, then Black does not win in $Connect(m, n, k, p, q)$ in the Maker–Breaker version for all $m \geq 1, n \geq 1$, either.

Assume that $MBConnect(k, p)$ is drawn. For the second reason, for all $m \geq 1, n \geq 1$, $Connect(m, n, k, p, p)$ games are drawn in the Maker–Breaker version, as well as in the original version. For the first reason, $Connect(m, n, k, p, q)$ games are drawn for all $m \geq 1, n \geq 1, 0 \leq q \leq p$. Thus, $Connect(k, p)$ is drawn and **Corollary 1** is satisfied.

On the basis of **Corollary 1**, Sections 3 and 4 both simply derive drawn $MBConnect(k, p)$ from **Theorems 1** and **2**, respectively, instead of deriving drawn $Connect(k, p)$ directly. Moreover, to prove both theorems, we also need to define new Maker–Breaker games for smaller boards B , named $MBBoard(B, p)$, in **Definition 1**.

Definition 1. $MBBoard(B, p)$ is a Maker–Breaker game defined as follows.

1. The *game board* B is composed of a set of squares and a set of *lines*, each of which covers a subset of squares. For simplicity of discussion, all lines are (vertically, horizontally, or diagonally) straight and solid in all figures in the rest of this paper, as illustrated in **Fig. 1**.
2. In Move $2i - 1$, where $i \geq 1$, Black is allowed to place p' stones on the game board B , where $p' \leq p$. In Move $2i$, White places p' or fewer stones.
3. Black wins when *occupying some line*. Note that Black is said to occupy a line if all the squares covered by the line are occupied by black stones. \square

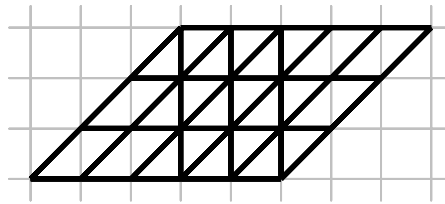


Fig. 1. The game board B_2 .

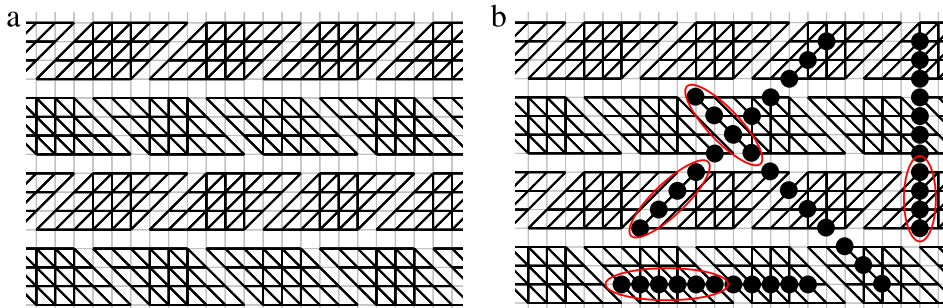


Fig. 2. (a) Partitioning the infinite board into disjoint B_2 . (b) Covering one complete solid line for each segment of 11 consecutive squares.

The game $MBBoard(B, p)$ is said to be a *drawn* game if Black has no winning strategy, that is, White has some strategy to prevent Black from winning in all cases.

In the above game, the game board B can be viewed as a kind of *hypergraph* G [4,8]. All squares in B are vertices in G , while all (solid) lines in B are *edges*, or so-called *hyperedges* in G , covering a set of vertices. For example, the board in Fig. 1 includes 6×4 squares with 4 horizontal, 3 vertical, and 6 diagonal lines (from the lower left to the upper right). The corresponding hypergraph includes 24 vertices and 13 (i.e., $4 + 3 + 6$) edges, accordingly. In the rest of this paper, we still use the terms game boards, lines, and squares, instead of graphs, edges, and vertices.

3. Proof of Theorem 1

The infinite board is partitioned into an infinite number of disjoint B_2 (without overlap and vacancy) as shown in Fig. 2(a), where B_2 is the game board shown in Fig. 1. From Lemma 1 (below), since $MBBoard(B_2, 2)$ is drawn, White has some strategy S such that none of the solid lines are occupied by Black. Let White follow S to play inside each B_2 . Observed from Fig. 2(b), all segments of 11 consecutive squares vertically, horizontally, and diagonally must cover entirely one solid line among these B_2 . Since none of these solid lines are occupied by Black from Lemma 1, none of the segments contain all 11 black stones. Thus, $MBCConnect(11, 2)$ is drawn. From Corollary 1, $Connect(11, 2)$ is drawn. \square

Lemma 1. $MBBoard(B_2, 2)$ is drawn.

Proof. A program was written to verify that none of the solid lines in B_2 are occupied by Black. The program is briefly described in Section 3.2. An intuition is given in Section 3.1. \square

3.1. Intuition for Lemma 1

This subsection gives an intuition for the correctness of Lemma 1. Move 1 (by Black) is classified into the following cases.

1. Black only places one stone in the board, as illustrated in Fig. 3(a).
2. Black places two stones.
 - 2.1 Both are placed on the two squares marked “1” in Fig. 3(b), called *middle squares* for this game board.
 - 2.2 One of the two stones is placed on either of the two middle squares.
 - 2.3 Neither of the two stones is placed on the two middle squares.

In Case 2.1, White replies by placing two stones, as shown in Fig. 3(b); and in all the other cases, White replies by placing one stone on one of the two middle squares. Here, only Case 1 in Fig. 3(a) and Case 2.1 in Fig. 3(b) are illustrated. Intuitively, it is hard for Black to occupy a horizontal line, since the horizontal lines contain two more squares than the vertical and diagonal lines. Therefore, let us ignore and remove the horizontal lines for simplicity of analysis.

After Move 2 (by White), Fig. 4 shows the boards with active vertical and diagonal lines only. Let an *active line* be a line that does not yet contain a white stone. Since Black is never able to cover all the squares of some inactive line (not active),

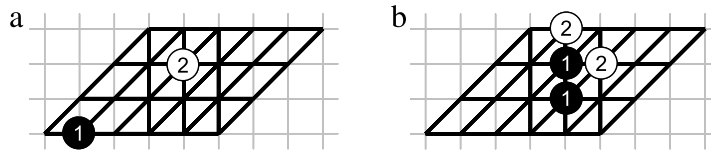


Fig. 3. The first two moves: (a) in Case 1, and (b) in Case 2.1.

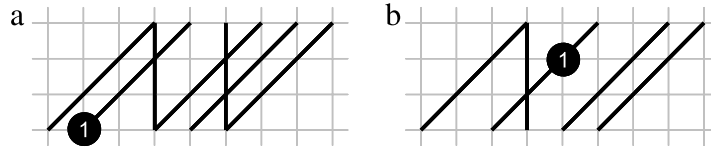


Fig. 4. The active vertical and diagonal lines after Move 2 (by White) in (a) Case 1, and (b) Case 2.1.

inactive lines are irrelevant to the results of games. Hence, the inactive lines can be removed from a board. In Fig. 4(a), the middle vertical line and the third diagonal line (from the left) become inactive and get removed after Move 2. In Fig. 4(b), the rightmost two vertical lines and the second and fourth diagonal lines (from the left) also become inactive and get removed, similarly.

A game board is called a *tree* if all the lines form no cycles in the board, as illustrated in both cases in Fig. 4. Lemma 2 (below) shows that a game is drawn if its game board is a tree which contains at most one black stone and in which each line covers at least four squares. Thus, from Lemma 2, the two games in Fig. 4 are drawn.

Lemma 2. *In a tree B_T , assume that there exists at most one black stone on B_T and that each line in B_T covers at least four squares. Then, $MBBoard(B_T, 2)$ is drawn.*

Proof. Assume that there exists one black stone on some square s . Black cannot win in his/her next move for the following reason. Since Black can place at most two stones in a move, one line contains at most three stones (together with the one on s). Since each line covers at least four squares, Black cannot win in the next move.

Let Black place one stone on another square s' in the next move. Since the game board is a tree, we find at most one path (a sequence of lines) from s to s' , and then let White place one stone on one of these lines in the path, if any. (Note that, if both s and s' are on the same line, White simply places a stone on that line.) Thus, B_T is broken into some trees, each of which contains at most one black stone. If Black places two stones in the next move, simply use two stones to break the game board as above. Thus, this lemma holds by induction. \square

To prove Lemma 1 rigidly, we also need to consider the case that some horizontal line may be occupied by Black. Thus, the proof for this unfortunately becomes tedious. In practice, we wrote a program to prove it by searching all cases exhaustively, as briefly described in the next subsection.

3.2. Program description for Lemma 1

The program to prove Lemma 1 uses a recursive search routine to search the game space and to find a strategy for White to tie the game. When it is Black's turn, the search routine searches all possible Black moves exhaustively, and verifies that Black does not win in any of the moves and any of their subsequent moves recursively. For each of these Black moves, the search routine chooses a White move to play such that Black does not win subsequently. The search routine does not search deeper moves when Black occupies some line, or when it is provable that Black has no winning way subsequently, e.g., there are no more active lines.

After running the above program, it was proved that White is able to tie the game. The program searched 1291,140,480 game positions in 17,104 s on a PC with AMD Athlon™ 64 × 2 Dual Processor with 5200 + 2.70 GHz. However, for the purpose of publishing the search tree, a method described in [14] was employed to optimize the size of the search tree. Then, under the optimization, the program ran in 37 s and searched 844,618 game positions. The search tree was published in [14].

4. Proof of Theorem 2

In this proof, similar to that of Theorem 1, the infinite board is partitioned into an infinite number of disjoint game boards $B_N(L)$ and $B_{-Z}(L)$ vertically interleaved without overlap and vacancy, as shown in Fig. 6. The game board² $B_Z(L)$ is shown

² The game board $B_N(L)$ is so named in this paper since the board shape consists of many N s, while the game board $B_Z(L)$ is so named since the parts different from $B_N(L)$ look like Z s.

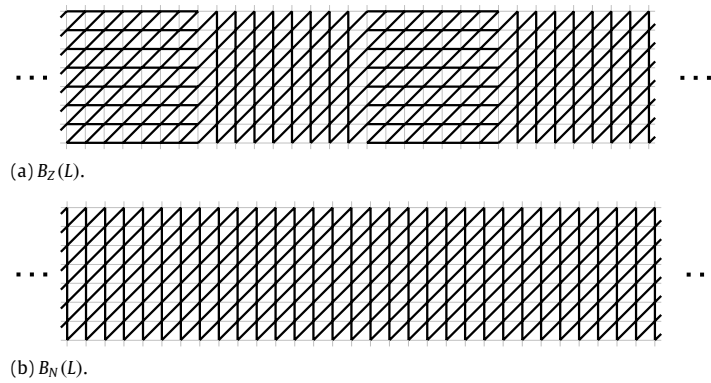


Fig. 5. Two game boards: (a) $B_Z(L)$ and (b) $B_N(L)$.

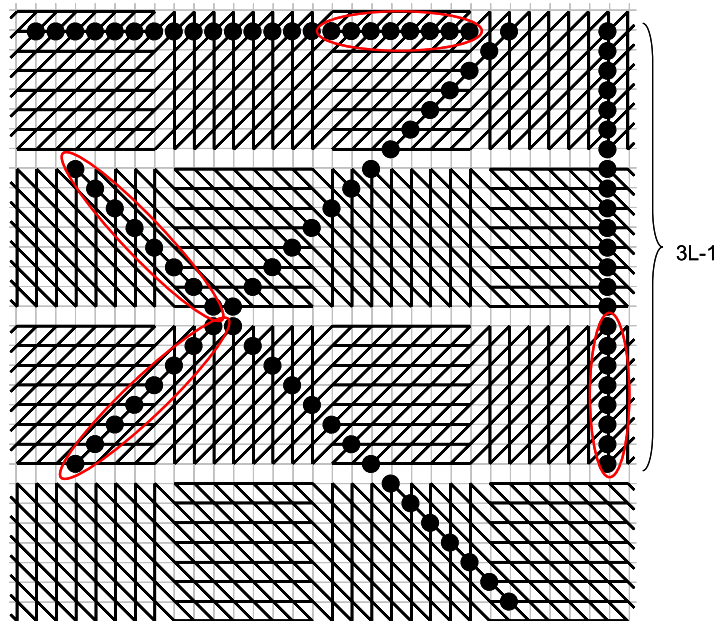


Fig. 6. Partitioning the infinite board into disjoint $B_Z(L)$.

in Fig. 5(a), where each (solid) line covers L squares and the game board extends infinitely to both sides. The game $B_{-Z}(L)$ is a horizontal mirror of $B_Z(L)$. Fig. 5(b) also shows another similar game board $B_N(L)$, which will be used in this section. Let $MBBoardZ(L, p)$ denote the game $MBBoard(B_Z(L), p)$, and $MBBoardN(L, p)$ denote $MBBoard(B_N(L), p)$, for simplicity of discussion. This proof will show that the following three properties are satisfied.

Property 1. If $MBBoardZ(L, p)$ is drawn, then $MBConnect(3L - 1, p)$ is drawn.

Property 2. If $MBBoardN(L, p)$ is drawn, then $MBConnect(3L - 1, p)$ is drawn.

Property 3. Consider all $p \geq 1$. Let $P(d - 1) < p \leq P(d)$, where $P(d) = 2^d - d - 2$. Then, $MBBoardN(p + d, p)$ games are drawn.

First, Property 1 is satisfied for the following reason. As observed in Fig. 6, all segments of $3L - 1$ consecutive squares vertically, horizontally, and diagonally must contain one whole solid line among these $B_Z(L)$ and $B_{-Z}(L)$. Assume that the game $MBBoardZ(L, p)$ is drawn. Then, White has some strategy S such that Black cannot occupy any solid lines inside each $B_Z(L)$ and $B_{-Z}(L)$. Thus, by following the strategy S inside each $B_Z(L)$ and $B_{-Z}(L)$, White prevents Black from occupying any segment of $3L - 1$ consecutive squares completely. Thus, $MBConnect(3L - 1, p)$ is drawn.

Then, both Properties 2 and 3 are shown in Sections 4.1 and 4.2, respectively. Section 4.1 shows that the game board $B_Z(L)$ is isomorphic to $B_N(L)$, in the sense of hypergraphs [4,8], and that Property 2 is satisfied from the isomorphism and Property 1. Section 4.2 proves that Property 3 is satisfied for all $MBBoardN$ games listed in Property 3. Thus, Theorem 2 is satisfied from Corollary 1, Property 2 and Property 3. \square

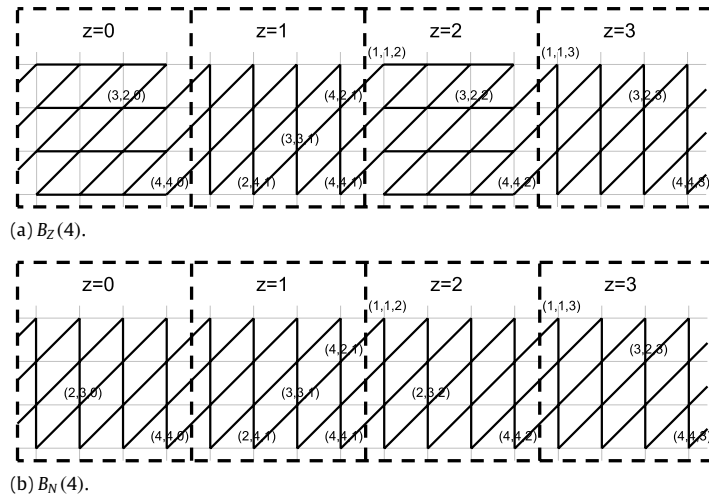


Fig. 7. Coordinate mapping between $B_Z(4)$ and $B_N(4)$.

4.1. Isomorphism

Both game boards $B_Z(L)$ and $B_N(L)$ are hypergraph isomorphic [4,8] according to the following mapping. Let every L neighboring vertical or horizontal solid lines be grouped into one zone in both $B_Z(L)$ and $B_N(L)$, as shown respectively in Fig. 7(a) and (b). In both game boards, each square has a coordinate (x, y, z) , where the square is in the x th column (from the left) and in the y th row (from the top) in zone z . Let each square at (x, y, z) on $B_Z(L)$ be mapped into the one at (x, y, z) on $B_N(L)$ when z is even, and at (y, x, z) on $B_N(L)$ when z is odd. All solid lines (or hyperedges) on $B_Z(L)$ are mapped into those on $B_N(L)$ accordingly, except that the i th horizontal line (from the top) of $B_Z(L)$ is mapped to the i th vertical line (from the left) of $B_N(L)$ in zone z , where z is odd.

Lemma 3. Consider both $MBBoardZ(L, p)$ and $MBBoardN(L, p)$ games over all L and p . Then, $MBBoardZ(L, p)$ is drawn if and only if $MBBoardN(L, p)$ is drawn.

Proof. According to the above mapping from $B_Z(L)$ to $B_N(L)$, placing one stone at (x, y, z) in $B_Z(L)$ is equivalent to placing one stone at (x, y, z) in $B_N(L)$ when z is even, and at (y, x, z) when z is odd, and vice versa. Since both $B_Z(L)$ and $B_N(L)$ are hypergraph isomorphic for the mapping, one solid line of $B_Z(L)$ is occupied by Black if and only if the mapped solid line of $B_N(L)$ is. Therefore, $MBBoardZ(L, p)$ is drawn if and only if $MBBoardN(L, p)$ is drawn. \square

From Lemma 3 and Property 1, Property 2 is satisfied.

4.2. Drawn $MBBoardN$ games

This section will prove that Property 3 is satisfied. First, we introduce the concept of exclusive squares in Section 4.2.1, which is used in the remaining subsections. In order to prove that all $MBBoardN$ games are drawn in Property 3, we derive some initial drawn $MBBoardN$ games in Section 4.2.2, and derive induction rules for $MBBoardN$ games in Section 4.2.3. Finally, Section 4.2.4 concludes that Property 3 is satisfied.

4.2.1. Game boards with exclusive squares

In this subsection, we introduce the concept of *exclusive squares*, on which Black is not allowed to place stones. The game boards with exclusive squares are defined in Definition 2 (below).

Definition 2. $MBBoardX(B, b)$ is a Maker–Breaker game defined as follows.

1. The game board B is the same as that in Definition 1, except for the following. For each line, one extra square is added as an *exclusive square*, as illustrated with solid bullets in Fig. 8(a)–(c).
2. In Move $2i - 1$, where $i \geq 1$, Black is allowed to place any (positive) number of black stones, say p' (≥ 1) black stones, on the game board B . However, Black is *not* allowed to place stones on these exclusive squares. In Move $2i$, White is allowed to place p' or fewer white stones on any squares (including exclusive squares).
3. Black wins if the following condition holds. An active line contains *more than* b black stones at time t_{2i} (when Black is to play), where $i \geq 0$. Time t_j indicates the moment after Move j and before Move $j + 1$, and t_0 indicates the initial moment. \square

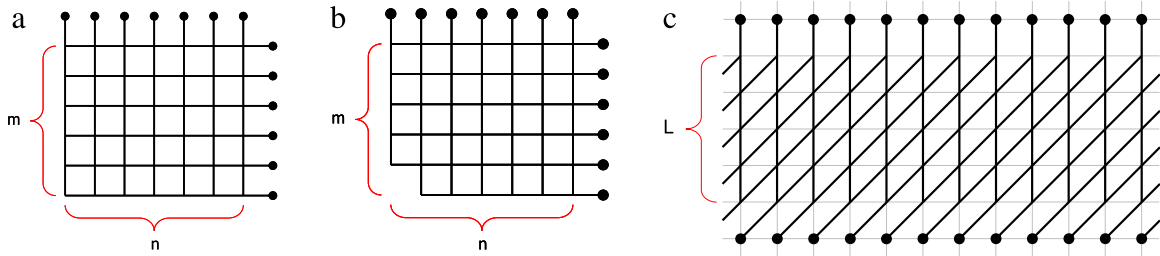


Fig. 8. Three game boards with exclusive squares (solid bullets). (a) $B_{recX}(m, n)$. (b) $B_{recX-}(m, n)$. (c) $B_{NX}(L)$.

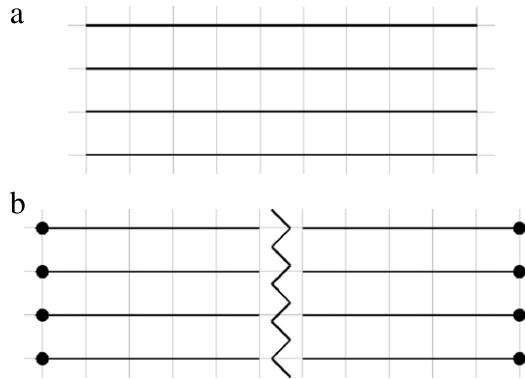


Fig. 9. An illustration. (a) The original game board. (b) Partitioned game boards with exclusive squares.

The game $MBBoardX(B, b)$ is said to be a drawn game if White has some strategy to prevent Black winning in all cases.

The motivation of using exclusive squares is to partition a game board into two or more game boards with exclusive squares and then to use Lemma 4 (below) to derive some properties from the partitioned game boards. Let us illustrate it by a simple game $MBBoard(B, 9)$ as follows. Let the board B contain disjoint lines each with 10 squares (which are not covered by any other lines), as shown in Fig. 9(a). Then, partition the board B into two, one named B_{left} containing 5 squares of each line and the other B_{right} containing the other 5, and add exclusive squares to all lines as shown in Fig. 9(b). Clearly, both games $MBBoardX(B_{left}, 0)$ and $MBBoardX(B_{right}, 0)$ are drawn, for the following reason. Whenever Black places one or more stones on some line, White places one stone on the exclusive square of the line to defend. From Lemma 4, we obtain that $MBBoard(B, 10 - (0 + 0) - 1)$ is drawn; that is, $MBBoard(B, 9)$ is drawn. Obviously, it is true that $MBBoard(B, 9)$ is drawn, from the following observation. Whenever Black places one or more stones on some active line, White places one stone on that line in the next move to make it inactive. Note that Black must leave one square unoccupied in an active line, so White is allowed to place a stone on that line.

Lemma 4. Consider a game board B , where each line covers at least L squares. Partition³ the game board B into two disjoint game boards, B_1 and B_2 . Assume that both games $MBBoardX(B_1, b_1)$ and $MBBoardX(B_2, b_2)$ are drawn and that $L - (b_1 + b_2) > 1$. Then, White has some strategy in $MBBoard(B, L - (b_1 + b_2) - 1)$ such that each active line in B contains at most $b_1 + b_2$ black stones at all times t_{2i} (when Black is to play), where $i \geq 0$. Implicitly, $MBBoard(B, L - (b_1 + b_2) - 1)$ is drawn.

Proof. It suffices to prove by induction that White has some strategy such that each active line in B contains at most $b_1 + b_2$ black stones at all times t_{2i} , where $i \geq 0$. This implies that $MBBoard(B, L - (b_1 + b_2) - 1)$ is drawn, since Black cannot occupy any active line (at most $b_1 + b_2$ black stones) in the next move (at most $L - (b_1 + b_2) - 1$ black stones), and each line covers at least $L \geq (b_1 + b_2) + L - (b_1 + b_2) - 1 = L - 1$ squares.

It is trivial that the induction hypothesis is true initially.

Assume that the induction hypothesis is true at t_{2i} , when Black is to move. Consider Black's next move. Since Black can place at most $L - b_1 - b_2 - 1$ stones in a move, each active line must leave one square unoccupied. Now, investigate the black stones of this move in B_1 . Since $MBBoardX(B_1, b_1)$ is drawn according to the assumption, White must have some strategy for the game such that each active line contains at most b_1 black stones in B_1 at t_{2i+2} . Thus, White simply follows the strategy to place stones at the edge of B_1 . In the case that White needs to place a stone on the exclusive square in one active line in B_1 , White uses the following strategy. If the corresponding line in B is inactive (e.g., the line contains a white stone at the edge of B_2), simply ignore this line. Otherwise, if it is active, White simply places one stone on the unoccupied square of the line

³ In the partitioning, we assume that each square belongs to either B_1 or B_2 and that each pair of squares in either B_1 or B_2 is covered by one line if they are also covered by the same line in B .

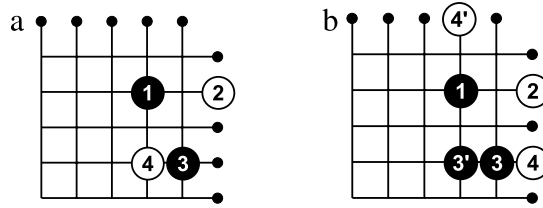


Fig. 10. Two cases for $B_{recX}(m, n)$.

as described above. Note that it does not matter even if the unoccupied square is at the edge of B_2 . Thus, White ensures that each active line contains at most b_1 black stones at the edge of B_1 at t_{2i+2} . Similarly, White also ensures that each active line contains at most b_2 black stones in B_2 at t_{2i+2} . Thus, the induction hypothesis is true at t_{2i+2} . \square

In this paper, we consider three game boards with exclusive squares, as shown in Fig. 8. The first game board, denoted by $B_{recX}(m, n)$ and shown in Fig. 8(a), consists of m horizontal lines and n vertical lines, each of which contains one extra exclusive square. The second, denoted by $B_{recX-}(m, n)$ and shown in Fig. 8(b), is the same as $B_{recX}(m, n)$ except that the square at the lower-left corner is removed. The third, shown in Fig. 8(c), is the original $B_N(L)$ extended with one exclusive square for each line. For simplicity, let $MBBoardNX(L, b)$ denote the game $MBBoardX(B_N(L), b)$ and $B_{NX}(L)$ denote the game board $B_N(L)$ with extra exclusive squares. Three properties related to the above three boards are shown respectively in Lemma 5, Lemma 6, and Lemma 7 (below).

Lemma 5. $MBBoardX(B_{recX}(m, n), 1)$ is drawn over all m and n .

Proof. Let variables $\sigma_R(r)$ and $\sigma_C(c)$ respectively be the number of black stones in the r th horizontal line and that in the c th vertical line, if still active, and be 0, otherwise. Let variable $\sigma = \sum_R \sigma_R(r) + \sum_C \sigma_C(c)$. For this proof, it suffices to prove that White has a strategy such that $\sigma \leq 1$ at all times t_{2i} (when Black is to play), where $i \geq 0$.

Assume by induction that $\sigma \leq 1$ at some t_{2i} . Assume that, in Move $2i + 1$, Black places only one stone on square s at row r and column c . Obviously, Move $2i + 1$ increases σ by at most two (one for the vertical line and the other for the horizontal line). That is, $\sigma \leq 3$. White uses the following strategy to make Move $2i + 2$ such that $\sigma \leq 1$ at t_{2i+2} .

1. When $\sigma \leq 1$, simply place a stone randomly on one empty square, if any.
2. When $\sigma \leq 2$, simply choose one active line containing a black stone and block it by placing one white stone on the exclusive square in that line. Then, σ is at most 1.
3. When $\sigma = 3$ and an active line contains two black stones, simply block the active line by placing one white stone on the exclusive square in that line. Then, σ is at most 1.
4. In the remaining case that $\sigma = 3$ and none of the active lines contains two black stones, assume some $\sigma_R(r') = 1$, where $r' \neq r$, without loss of generality. Thus, the square s' at row r' and column c (both lines are active) must be empty (otherwise, we are in Case 3, since two black stones are in the same column). Therefore, simply place one white stone on s' . Since the stone blocks the two active lines in row r' and column c , σ is back to 1. This is illustrated by Moves 3 and 4 in Fig. 10(a).

However, if Black places several black stones, say p' black stones, in Move $2i + 1$, we separate the move into p' submoves, each with one stone only. Then, White pretends that Black makes submoves one by one, and therefore follows the above strategy to place stones, except for the following case. If White is to place one stone on an empty square s' in some submove M as in Case 4, but one of the subsequent submoves M' places one black stone on s' too, the strategy needs to be changed as follows.

5. Place two white stones respectively on the exclusive squares of the two active lines in row r' and column c containing s' . Thus, σ is back to 1 too. Thus, for M' , White replies by placing no more stones. In this case, the two white stones together are viewed as a reply to the two black stones at submoves M and M' . This case is illustrated by the example in Fig. 10(b). For Move 3, Black places two stones at 3 and 3'. Assume Black to make submoves in the sequence 3 and then 3'. For 3, White cannot reply by placing a stone on 3', since it will be occupied by Black. Therefore, White places stones on 4 and 4' to make σ back to 1, instead.

From the above strategy, $\sigma \leq 1$ is maintained at all times t_{2i} . Thus, this lemma holds. \square

Lemma 6. $MBBoardX(B_{recX-}(m, n), 1)$ is drawn over all m and n .

Proof. This proof is the same as that in Lemma 5, except for the first black stone and White's reply. The first black stone is placed on the board in the following three positions: (1) in the leftmost vertical line, (2) in the bottom horizontal line, and (3) in the rest of the rectangle. In Case 1, let White reply by placing one white stone on the leftmost vertical line as shown in Fig. 11(a), thus making this vertical line inactive. Now, the variable σ is only 1. Then, we simply follow the strategy described in Lemma 5 to maintain $\sigma \leq 1$. Similarly, in Case 2, let White reply by placing one on the bottom horizontal line. In Case 3, let White place one on the leftmost vertical line without loss of generality, while blocking the first black stone in

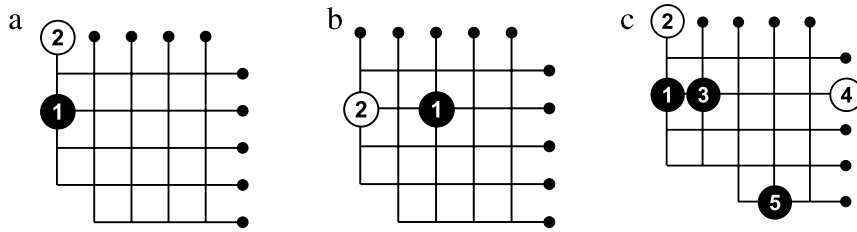


Fig. 11. (a) and (b): Two cases for $B_{recX-}(m, n)$ in and (c) another case for the board missing two corner squares.

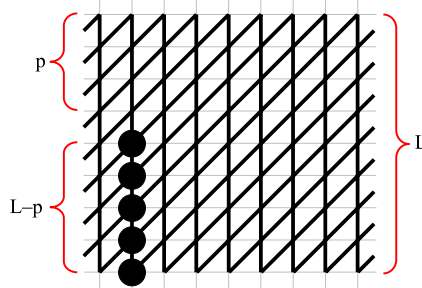


Fig. 12. The case that Black already occupies $L - p$ stones on an active line.

the same horizontal line as shown in Fig. 11(b). Similarly, since the variable σ is only 1, simply follow the strategy described in Lemma 5 to maintain $\sigma \leq 1$. Thus, White is able to maintain $\sigma \leq 1$ in all cases. That is, $MBBoardX(B_{recX-}(m, n), 1)$ is drawn. (Note that we may not maintain $\sigma \leq 1$ when two corner squares are missing, as illustrated in Fig. 11(c).) \square

Lemma 7. As described above, assume that the game $MBBoardN(L, p)$ is drawn. Then, $MBBoardNX(L, L - p - 1)$ is drawn.

Proof. Since $MBBoardN(L, p)$ is drawn, White has a strategy S such that all active lines have at most $L - p - 1$ black stones at all times t_{2i} (when Black is to play). Otherwise, if an active line contains at least $L - p$ black stones, Black wins by simply placing p stones on this line, as illustrated in Fig. 12.

In the game $MBBoardNX(L, L - p - 1)$, assume that Black still places at most p black stones in Move $2i + 1$, where $i \geq 0$. Then, White simply follows strategy S (without placing stones on exclusive squares) such that all active lines in $B_{NX}(L)$ contain at most $L - p - 1$ black stones at all times t_{2i+2} (when Black is to play).

Assume that Black makes a move with more than p black stones. We separate the move into several submoves, each with at most p black stones. Then, White pretends that Black makes submoves one by one, and for each submove simply follows S to play, but with the following exceptional case. By following S , assume that White needs to make a submove on some empty squares, but some subsequent Black submoves will place stones on these empty squares. Without loss of generality, assume that White makes a submove M on an empty square s , but some subsequent Black submove M' will place a stone on s . Then, the strategy is changed as follows.

1. Place two white stones respectively on the exclusive squares of the two lines containing s , instead. The reason is similar to that in Case 5 in Lemma 5. Both lines containing s are no longer active. Let the black stone at s be added into M and removed from M' . Thus, the reply to M still prevents Black from having active lines with more than $L - p - 1$ black stones. Although the reply to M uses one more stone, M has one more stone on s too.

Thus, all active lines in the game $MBBoardNX(L, L - p - 1)$ have at most $L - p - 1$ black stones at all t_{2i} (when Black is to play). That is, $MBBoardNX(L, L - p - 1)$ is drawn. \square

4.2.2. Initial drawn games

In this subsection, initial $MBBoardN(4, 1)$, $MBBoardNX(2, 1)$ and $MBBoardNX(3, 2)$ games are shown to be drawn in Lemma 8, Lemma 9, and Lemma 10 respectively.

Lemma 8. $MBBoardN(4, 1)$ is drawn.

Proof. Let us transform $B_N(4)$ into $B_{N-}(4)$ by shortening the solid lines, as shown in Fig. 13. Since $B_{N-}(4)$ is a tree and there are no black stones initially, $B_{N-}(4)$ is drawn, from Lemma 2. Obviously, this implies that $B_N(4)$ with extra longer lines is drawn too. \square

Lemma 9. $MBBoardNX(2, 1)$ is drawn.

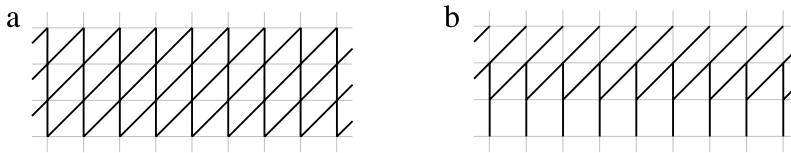


Fig. 13. (a) $B_N(4)$. (b) $B_{N-}(4)$, the same as $B_N(4)$ except that all the solid lines are shortened.

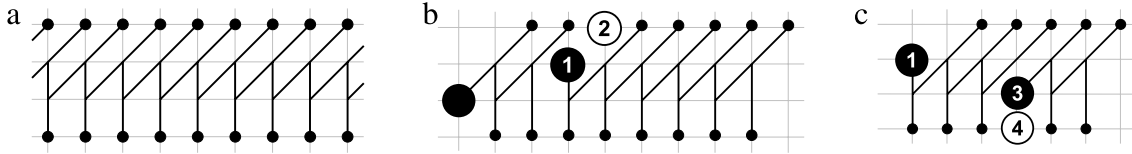


Fig. 14. (a) $B_{NX}(2)$. (b) The tree broken by White, Move 2. (c) The tree broken by White, Move 4.

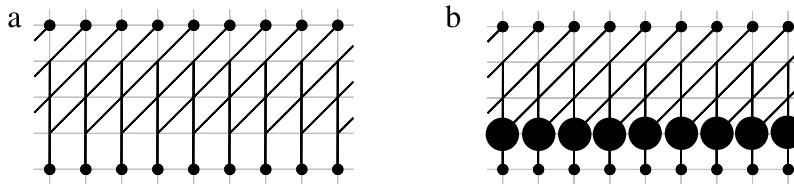


Fig. 15. (a) $B_{NX}(3)$. (b) $B_{NX}(3)$ occupied by some black stones initially.

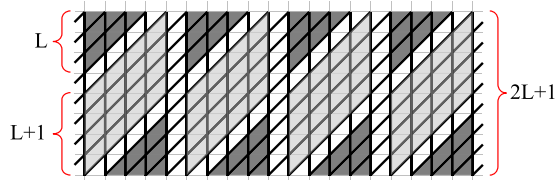


Fig. 16. Partitioning $B_N(2L + 1)$ into dark gray and light gray zones.

Proof. The game board $B_{NX}(2)$ is a tree, as shown in Fig. 14(a). First, we assume that Black places one stone for each move. It suffices to prove that White has a strategy such that at all times t_{2i} (when Black is to play) each of the trees (formed by all the active lines) satisfies that only the leftmost (active) line, if it exists, contains one black stone. For example, in Fig. 14(b), for Move 1 (by Black), Move 2 (by White) blocks the diagonal line on Move 1; and in Fig. 14(c), for Move 3, Move 4 blocks the vertical line containing the stone of Move 3. Thus, it is easy to see that no active lines contain two black stones at all times t_{2i} . If Black places several stones in one move, we simply pretend that Black places stones one at a time. White simply follows the above strategy without being disturbed by Black’s multi-stone moves, since White replies by placing stones on exclusive squares where Black cannot place stones. Thus, $MBBoardNX(2, 1)$ is drawn. \square

Lemma 10. $MBBoardNX(3, 2)$ is drawn.

Proof. For game board $B_{NX}(3)$ as shown in Fig. 15(a), assume that all squares above the bottom exclusive squares are initially occupied by black stones, as shown in Fig. 15(b). By ignoring these squares with black stones, the game board becomes $B_{NX}(2)$. From Lemma 9, at all times t_{2i} (when Black is to play), Black occupies at most one of the remaining two squares plus the one already shown in Fig. 15(b), that is, at most two. Thus, $MBBoardNX(3, 2)$ is drawn. \square

4.2.3. Induction rules

In this subsection, four induction rules are shown in Lemma 11, Lemma 12, Lemma 13, and Lemma 14 respectively.

Lemma 11. Assume that $MBBoardNX(L, b)$ is drawn, where $0 < b < L$. Then, $MBBoardN(2L + 1, 2L - b - 1)$ is drawn too.

Proof. Partition the game board $B_N(2L + 1)$ into dark gray and light gray game boards, as shown in Fig. 16. Half of the dark gray board can be squeezed into $B_N(L)$, as shown in Fig. 17. The light gray game board is the union of disjoint $B_{recX}(L + 1, L + 1)$. Since $MBBoardNX(L, b)$ is drawn from the assumption and $MBBoardX(B_{recX}(L + 1, L + 1), 1)$ is drawn from Lemma 5, $MBBoardN(2L + 1, (2L + 1) - (b + 1) - 1) = MBBoardN(2L + 1, 2L - b - 1)$ is drawn from Lemma 4. \square

Lemma 12. Assume that $MBBoardNX(L, b)$ is drawn, where $0 < b < L$. Then, $MBBoardN(2L + 2, 2L - b)$ is drawn too.

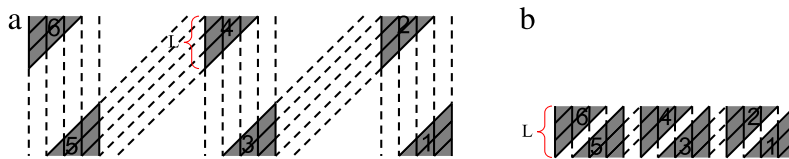


Fig. 17. (a) Half of the dark gray game board. (b) Squeezing the game board in (a) into a $B_N(L)$.

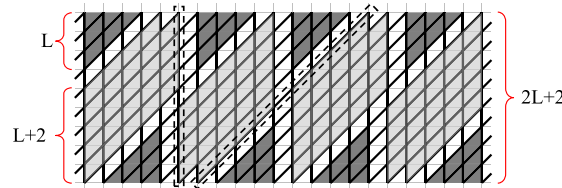


Fig. 18. Partitioning $B_N(2L + 2)$ into light gray and dark gray zones.

Table 1

List of drawn $MBBoardN$ games derived from Property 4, where $2 \leq p \leq 4$.

Drawn games	Drawn games derived from Lemma 11 or Lemma 12.
$MBBoardNX(2, 1)$	$\rightarrow MBBoardN(5, 2)$ and $MBBoardN(6, 3)$
$MBBoardNX(3, 2)$	$\rightarrow MBBoardN(7, 3)$ and $MBBoardN(8, 4)$

Table 2

List of drawn $MBBoardN$ games derived from Property 5, where $5 \leq p \leq 13$.

Drawn games	Drawn games derived from Lemmas 13 and 14.
$MBBoardN(4, 1)$	$\rightarrow MBBoardN(9, 5)$ and $MBBoardN(10, 6)$
$MBBoardN(5, 2)$	$\rightarrow MBBoardN(11, 7)$ and $MBBoardN(12, 8)$
$MBBoardN(6, 3)$	$\rightarrow MBBoardN(13, 9)$ and $MBBoardN(14, 10)$
$MBBoardN(7, 3)$	$\rightarrow MBBoardN(15, 10)$ and $MBBoardN(16, 11)$
$MBBoardN(8, 4)$	$\rightarrow MBBoardN(17, 12)$ and $MBBoardN(18, 13)$

Proof. This proof is similar to that in Lemma 11, except that $B_{recX}(L + 2, L + 2)$ is used (instead of B_{recX}) and some lines marked in dashed boxes in Fig. 18 are covered by two $B_{recX}(L + 2, L + 2)$. For the lines covered by two $B_{recX}(L + 2, L + 2)$, since each active line in $B_{recX}(L + 2, L + 2)$ contains at most one black stone, each of these lines, if active, contains at most two black stones when Black is to play. For the other lines, we can still use Lemma 4 to derive that each line, if active, contains at most $b + 1$ black stones when Black is to play. Since $b + 1 \geq 2$, all lines contain at most $b + 1$ black stones when Black is to play. Thus, the game $MBBoardN(2L + 2, (2L + 2) - (b + 1) - 1) = MBBoardN(2L + 2, 2L - b)$ is drawn. \square

Lemma 13. Assume that $MBBoardN(L, p)$ is drawn. Then, $MBBoardN(2L + 1, L + p)$ is drawn too.

Proof. Since $MBBoardN(L, p)$ is drawn, $MBBoardNX(L, L - p - 1)$ is drawn from Lemma 7. From Lemma 11, $MBBoardN(2L + 1, 2L - (L - p - 1) - 1) = MBBoardN(2L + 1, L + p)$ is drawn. Thus, this lemma holds. \square

Lemma 14. Assume that $MBBoardN(L, p)$ is drawn. Then, $MBBoardN(2L + 2, L + p + 1)$ is drawn too.

Proof. Since $MBBoardN(L, p)$ is drawn, $MBBoardNX(L, L - p - 1)$ is drawn from Lemma 7. From Lemma 12, $MBBoardN(2L + 2, 2L - (L - p - 1)) = MBBoardN(2L + 2, L + p + 1)$ is drawn. Thus, this lemma holds. \square

4.2.4. The proof for Property 3

This subsection concludes in Lemma 15 that Property 3 is satisfied.

Lemma 15. Property 3 is satisfied.

Proof. Initially, the three games, $MBBoardN(4, 1)$, $MBBoardNX(2, 1)$ and $MBBoardNX(3, 2)$, are shown to be drawn in Lemma 8, Lemma 9, and Lemma 10, respectively. From Lemma 11 or Lemma 12, we obtain the drawn $MBBoardN$ games, for all $2 \leq p \leq 4$, as shown in Table 1. Then, from Lemmas 13 and 14, we obtain the drawn $MBBoardN$ games, for all $5 \leq p \leq 13$, as shown in Table 2. By induction, all the remaining drawn $MBBoardN$ games in Property 3 can be derived from Lemmas 13 and 14. \square

5. Conclusion

The contributions of this paper are listed as follows.

- With the help of a program, this paper shows that $Connect(11, 2)$ is drawn. Note that drawn $Connect(k, p)$ implies drawn $Connect(m, n, k', p, q)$ for all $k' \geq k, m \geq 1, n \geq 1, 0 \leq q \leq p$. In contrast, the best known result [10] in the past was drawn $Connect(15, 2)$.
- This paper shows that $Connect(k_{draw}(p), p)$ games are drawn for all $p \geq 3$, where $k_{draw}(p) = 3p + 3d - 1$ and d is a logarithmic function of p . Specifically, d is an integer such that $P(d - 1) < p \leq P(d)$ and $P(d) = 2^d - d - 2$. The values $k_{draw}(p)$ derived in this paper are currently the smallest for all $2 \leq p < 1000$ (the value is the same as that in [10] when $p = 4$).

Although this paper presents tighter bound for k , many interesting problems are still open. The following are two examples.

- Derive lower $k_{draw}(p)$ for $p < 1000$, especially for small p , e.g., $1 \leq p \leq 10$. These problems are more realistic in real games. For example, $Connect(5, 1)$ favors Black [1,2], while $Connect(8, 1)$ is drawn [17]. There is still a gap between 5 and 8. When $p = 2$, the gap is even wider. Currently, the conjecture by most $Connect6$ players are that $Connect6$, $Connect(19, 19, 6, 2, 1)$, is drawn, and that Black wins in $Connect(19, 19, 6, 2, 2)$. Both are still open problems. A search approach similar to those in [15,16] is perhaps helpful to solve the latter. However, from our experiences, it is very difficult to use the search approach to solve the former. It is also an important open problem to solve all $Connect(n, 2)$, where $7 \leq n \leq 10$.
- Derive general tighter bounds than those in this paper and those in [11] simultaneously.

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