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Anisotropically expanding universe in massive gravity

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We study the cosmological implications of a ghost-free nonlinear massive gravity theory with a cosmological constant. We find that the massive terms serve as an effective cosmological constant for a large class of metric spaces with a compatible fiducial metric associated with the massive terms. A specific solution is solved as an example for this model under the Bianchi type I space and a compatible Bianchi type I fiducial metric. A stability analysis indicates that this set of solutions tends to be stable. Nonetheless, the anisotropically expanding solution will, in general, turn unstable when an additional scalar field with negative kinetic energy is introduced.

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I. INTRODUCTION

A unique ghost-free linearized massive gravity theory was formulated by Fierz and Pauli (FP) in 1939 [1]. This theory is known to propagate 5 degrees of freedom for a massive spin-2 graviton in the neighborhood of the flat Minkowski metric space. With an analysis of constraints, Boulware and Deser (BD) showed that FP theory has an extra negative energy degree of freedom in the nonlinear level [2]. This additional negative energy degree of freedom was later referred to as the BD ghost.

From 2009 to 2010, de Rham, Gabadadze, and Tolley proposed a nonlinear massive gravity theory that was later shown to be free of the BD ghost in 2011 [3,4]. The de Rham, Gabadadze, and Tolley ghost-free proof was first shown to be true with a flat reference (or fiducial) metric [4]. It has also been shown to be free of the BD ghost in the fully nonlinear level and in the presence of an arbitrary fiducial metric in 2012 [5,6]. A nice review on this subject can also be found in Ref. [7].

A consistent theory of massive gravity is also needed from observational considerations. Incorporating the recent discovery of dark energy and the associated cosmological constant problem has inspired investigations of the long-range corrections of general relativity. Massive gravity is certainly a plausible approach to the quest of a revised theory of gravity. The existence of a consistent ghost-free massive gravity theory has resulted in much activity in the study of all possible implications of this theory to the evolution of the early universe [5,6,8–23].

In particular, it was shown in Ref. [9] that the nonlinear massive gravity theory does not admit spatially flat homogeneous and isotropic cosmological solutions. It does admit some spatially open homogeneous and isotropic cosmological solutions [10,11]. In addition, some anisotropic solutions [9,12,13] and some inhomogeneous solutions [9,14,15] have been found recently. There are also some studies on the black holes physics [16,17]. Additionally, a ghost-free bimetric theory [19] has been proposed by Hassan and Rosen in Refs. [20,21], and ghost-free multimetric theories have been studied in Ref. [23].

Note that many studies done earlier in Refs. [4–6,8–18,20,21,23] only focus on the isotropic fiducial metric, while the physical metric is generalized to the isotropic or anisotropic metric spaces. On the other hand, the results shown in Refs. [9,12–15] indicate that the massive gravity theory will still remain ghost-free for all general fiducial metrics. Therefore, we propose to study a more general scenario of the nonlinear massive gravity theory with a more general fiducial metric. In addition, a cosmological constant will also be included for heuristic reasons that will be studied in this paper.

As a result, the reference metric will be treated as additional auxiliary fields similar to the Stuckelberg fields. A new model-independent method will be introduced to derive the field equations for the universal properties associated with the complicated structure hidden in the massive terms. As a result, we will also show that the massive terms will serve as an effective cosmological constant for a large class of metric spaces with a compatible fiducial metric associated with the massive terms.

A specific set of solutions will be solved as an example for this model under the Bianchi type I space and a compatible Bianchi type I fiducial metric. A stability analysis will also be performed to show that this set of solutions tends to be stable [21,24–26]. Moreover, we will show that the anisotropically expanding solution will, in general, become unstable when an additional scalar field with negative kinetic energy is introduced.

This paper will be organized as follows: (i) A brief review of the motivation of this research is given in the present section. (ii) An introduction and review of the nonlinear massive gravity theory will be presented in Sec. II. A heuristic derivation of the field equations will also be presented in this section. (iii) The analysis of the

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universal properties associated with a general reference metric will be presented in Sec. III. In particular, we will show that the massive terms will serve as an effective cosmological constant in this section. (iv) The field equations will be presented in Sec. IV for this model in the presence of a Bianchi type I physical space and similar reference metric space. (v) We will solve the reference metric equations and derive the exact value of the effective cosmological constant in Sec. V. (vi) A set of anisotropic solutions and its stability analysis will be presented in Sec. VI. We will also show that the presence of a scalar phantom field will affect the stability of the cosmological solution in this section. (vii) Finally, concluding remarks and discussions will be given in Sec. VII.

II. THE STÜCKELBERG FORMULATION

It is known that the physical metric $g_{\mu\nu}$ can be expanded around a reference (or fiducial) metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

along with the introduction of a spin-2 field $h_{\mu\nu}$. The field $h_{\mu\nu}$ is not, however, covariant under diffeomorphism. This is simply because the background reference metric breaks the diffeomorphism symmetry. In order to preserve diffeomorphism, we need to introduce a covariant reference metric that observes diffeomorphism. This can be done with the help of the Stückelberg fields ϕ^a ($a = 0$, 1, 2, 3). Indeed, we can introduce a background fiducial (or reference) metric f_{ab} in order to define $Z_{\mu\nu}$ as [4,8,9]

$$Z_{\mu\nu} \equiv f_{ab}\partial_\mu \phi^a \partial_\nu \phi^b. \tag{2.2}$$

As a result, we can expand the metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = Z_{\mu\nu} + H_{\mu\nu}. \tag{2.3}$$

Here the Roman letters a, b, c will denote flat-space indices to be raised or lowered by the flat Minkowski metric η_{ab}, while the Greek letters μ, ν, α will denote curved space indices to be raised or lowered by the physical metric $g_{\mu\nu}$. Note that $\phi^a = x^a + \pi^a$ is known to be the linear expansion of the Stückelberg field around the unitary gauge $\phi^a = x^a$. It is apparent that $Z_{\mu\nu} = \eta_{\mu\nu}$ when (i) the fiducial metric is taken as the Minkowski flat metric, namely, $f_{ab} = \eta_{ab}$, and (ii) the unitary gauge $\phi^a = x^a$ is also adopted.

Note that $Z_{\mu\nu}$ and $H_{\mu\nu}$ are both covariant under diffeomorphism since ϕ^a are introduced as scalar fields under diffeomorphism. In fact, the introduction of the Stückelberg field ϕ^a is known to be useful in the process of extracting physically relevant massive graviton interaction terms in a covariant way. Indeed, the massive Lagrangian can be formulated with the help of the background metric $Z_{\mu\nu}$ by defining the tensor field $\mathcal{K}^\mu{}_\nu$ as [4,8,9]:

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - M^\mu{}_\nu, \tag{2.4}$$

with [27]

$$Z^\mu{}_\nu \equiv g^{\mu\alpha} Z_{\alpha\nu} \equiv M^\mu{}_\rho M^\rho{}_\nu. \tag{2.5}$$

For convenience, we will occasionally write the above equations in matrix notation as

$$\mathcal{K} = \delta - M, \tag{2.6}$$

$$M^2 = g^{-1} Z. \tag{2.7}$$

Here δ denotes the unit matrix. Note that we have defined a 4×4 metric A via the relation $(A)_{\mu\nu} \equiv A^\mu{}_\nu$. We also defined the multiplication of two matrices as $(AB)_{\mu\nu} = (A)_{\mu\alpha}(B)_{\alpha\nu} = A^\mu{}_\alpha B^\alpha{}_\nu$.

It is known that the action of the ghost-free nonlinear massive spin-2 field theory is given by [4–6,8–18,20,21,23]

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{g}\{R - 2\Lambda + m_g^2(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4)\}, \tag{2.8}$$

with M_p the Planck mass, Λ the cosmological constant, m_g the graviton mass, $\alpha_{3,4}$ the free parameters, and $g \equiv -\det g_{\mu\nu}$ the determinant of the physical metric $g_{\mu\nu}$. In addition, the massive terms $\mathcal{L}_i$ ($i = 2 - 4$) are defined as

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \tag{2.9}$$

$$\mathcal{L}_3 = \frac{1}{3}[\mathcal{K}]^3 - [\mathcal{K}][\mathcal{K}^2] + \frac{2}{3}[\mathcal{K}^3], \tag{2.10}$$

$$\mathcal{L}_4 = \frac{1}{12}[\mathcal{K}]^4 - \frac{1}{2}[\mathcal{K}]^2[\mathcal{K}^2] + \frac{1}{4}[\mathcal{K}^2]^2 + \frac{2}{3}[\mathcal{K}][\mathcal{K}^3] - \frac{1}{2}[\mathcal{K}^4]. \tag{2.11}$$

Here we will use the bracket notation $[A] \equiv \mathrm{tr}A = \sum_i A^i{}_i$ to denote the trace of any arbitrary matrix A. For example, we have [4,8,9] $[\mathcal{K}] = \mathrm{tr}\mathcal{K}$, $[\mathcal{K}]^2 = (\mathrm{tr}\mathcal{K})^2$, and $[\mathcal{K}^2] = \mathrm{tr}\mathcal{K}^2$.

The variation of the action (2.8) with respect to the physical metric $g_{\mu\nu}$ leads to the modified Einstein field equation

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \Lambda g_{\mu\nu} + m_g^2(X_{\mu\nu} + \alpha_4 Y_{\mu\nu}) = 8\pi G T_{\mu\nu}, \tag{2.12}$$

with $X_{\mu\nu}$ and $Y_{\mu\nu}$ defined as

$$X_{\mu\nu} = \mathcal{K}_{\mu\nu} - [\mathcal{K}]g_{\mu\nu} - (\alpha_3+1)\Big\{\mathcal{K}^2_{\mu\nu} - [\mathcal{K}]\mathcal{K}_{\mu\nu} + \frac{[\mathcal{K}]^2 - [\mathcal{K}^2]}{2}g_{\mu\nu}\Big\} + (\alpha_3+\alpha_4) \times \Big\{\mathcal{K}^3_{\mu\nu} - [\mathcal{K}]\mathcal{K}^2_{\mu\nu} + \frac{1}{2}\mathcal{K}_{\mu\nu}\{[\mathcal{K}]^2 - [\mathcal{K}^2]\}\Big\} - \frac{\alpha_3+\alpha_4}{6}\{[\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\}g_{\mu\nu}, \tag{2.13}$$

$$Y_{\mu\nu} = -\frac{\mathcal{L}_4}{2}g_{\mu\nu} + \tilde{Y}_{\mu\nu}, \tag{2.14}$$

$$\tilde{Y}_{\mu\nu} = \frac{1}{6}[\mathcal{K}]^3\mathcal{K}_{\mu\nu} - \frac{1}{2}[\mathcal{K}][\mathcal{K}^2]\mathcal{K}_{\mu\nu} + \frac{1}{3}[\mathcal{K}^3]\mathcal{K}_{\mu\nu} - \frac{1}{2}[\mathcal{K}]^2\mathcal{K}^2_{\mu\nu} + \frac{1}{2}[\mathcal{K}^2]\mathcal{K}^2_{\mu\nu} + [\mathcal{K}]\mathcal{K}^3_{\mu\nu} - \mathcal{K}^4_{\mu\nu}. \tag{2.15}$$

In order to derive the variational equations associated with the massive terms, we will introduce a new and simple method to derive the variational equations. We will also show a few general properties of this model with the new method that will be presented shortly.

Note first that the massive terms will remain ghost-free with a more general fiducial metric. Therefore, we will consider the effect of a more general fiducial metric. By introducing this metric, an additional set of field variables and hence more degrees of freedoms will be introduced to the system. The Lagrangian of the system will then be a functional of the physical metric $g_{\mu\nu}$, the Stückelberg fields ϕ^a, and the fiducial metric f_{ab}. The Stückelberg fields ϕ^a and the fiducial metric f_{ab} will be treated similarly as extra auxiliary fields here. A complete set of field equations can thus be obtained by varying the physical metric $g_{\mu\nu}$, the Stückelberg fields ϕ^a, and also the fiducial metric f_{ab}.

We will now show that the reference metric should be chosen in a consistent way to adopt a consistent set of solutions to a set of Euler-Lagrange equations derived from all fields involved. Indeed, we can vary the full Lagrangian in order to derive the field equations. To be more specific, we can show that, for the massive terms $\mathcal{L}_M$,

$$\delta(\sqrt{g}\mathcal{L}_M) = \frac{1}{2}\sqrt{g}\mathcal{L}_M g^{\mu\nu}\delta g_{\mu\nu} + \sqrt{g}\frac{\delta\mathcal{L}_M}{\delta\mathcal{K}^\mu{}_\nu}\delta\mathcal{K}^\mu{}_\nu = \frac{1}{2}\sqrt{g}\mathcal{L}_M[g^{-1}\delta g] - \sqrt{g}[A\delta M], \tag{2.16}$$

with $A^\nu{}_\mu \equiv \delta\mathcal{L}_M/\delta\mathcal{K}^\mu{}_\nu$. Note that the massive Lagrangian $\mathcal{L}_M$ is defined as

$$\mathcal{L}_M = \mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4. \tag{2.17}$$

From the definition $M^2 = g^{-1}Z$ in Eq. (2.5), we can show that

$$(\delta M)M + M\delta M = (\delta g^{-1})Z + g^{-1}\delta Z. \tag{2.18}$$

Therefore, we will have

$$A\delta M + AM(\delta M)M^{-1} = A(\delta g^{-1})ZM^{-1} + Ag^{-1}(\delta Z)M^{-1}. \tag{2.19}$$

Taking the trace, we can derive the results

$$2[A\delta M] = -[MAg^{-1}\delta g] + [MAZ^{-1}\delta Z] = -2[t\delta g] + 2[s\delta Z]. \tag{2.20}$$

Here we have used the fact that $[A, M] = 0$; namely, A and M commute with each other. This follows from the fact that A can be shown to be a polynomial functional of M. Note that we have also defined some new matrices B, t, and s as

$$B^\mu{}_\nu = M^\mu{}_\alpha A^\alpha{}_\nu;\ t = \frac{(Bg^{-1} + g^{-1}B^T)}{4};\quad s = \frac{(BZ^{-1} + Z^{-1}B^T)}{4} \tag{2.21}$$

for convenience. We have also written the energy momentum tensors s and t as apparently symmetric tensors. This is partly why the definitions of t and s are a little bit more complicated than what we expected. Consequently, we have

$$\frac{2}{\sqrt{g}}\delta(\sqrt{g}\mathcal{L}_M) = \mathcal{L}_M[g^{-1}\delta g] + [t\delta g] - [s\delta Z]. \tag{2.22}$$

The last term in Eq. (2.22) can be expanded as

$$[s\delta Z] = s^{\mu\nu}\delta(f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b) = s^{\mu\nu}\{(\delta f_{ab})\partial_\mu\phi^a\partial_\nu\phi^b + 2f_{ab}\partial_\mu\phi^a\partial_\nu\delta\phi^b\}. \tag{2.23}$$

Taking the integration-by-parts of $s^{\mu\nu}f_{ab}\partial_\mu\phi^a\partial_\nu\delta\phi^b$, Eq. (2.23) reduces to

$$[s\delta Z] = s^{\mu\nu}(\delta f_{ab})\partial_\mu\phi^a\partial_\nu\phi^b - 2D_\nu(s^{\mu\nu}f_{ab}\partial_\mu\phi^a)\delta\phi^b. \tag{2.24}$$

Here the covariant derivative shows up as an effect of the $\sqrt{g}$ volume factor in the full set of actions. As a result, we have

$$\frac{2}{\sqrt{g}}\delta(\sqrt{g}\mathcal{L}_M) = (\mathcal{L}_M g^{\mu\nu} + t^{\mu\nu})\delta g_{\mu\nu} - s^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b\delta f_{ab} + 2D_\nu(s^{\mu\nu}f_{ab}\partial_\mu\phi^a)\delta\phi^b. \tag{2.25}$$

Note that the variational equations of the Stückelberg field read

$$D_\nu(s^{\mu\nu}f_{ab}\partial_\mu\phi^a) = \frac{1}{2}s^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^c(\partial_{\phi^0}f_{ac})\delta_{b0}. \tag{2.26}$$

In addition, the variational equation of the fiducial metric f_{ab} is

$$s^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b = 0. \tag{2.27}$$

If the matrix $\partial_\nu\phi^b$ is nonsingular, the inverse of this matrix exists. Note that, for example, under the unitary gauge, $\partial_\nu\phi^b = \delta_\nu{}^b$ is nonsingular. We can then show that the fiducial metric equation is simply

$$s^{\mu\nu} = 0. \tag{2.28}$$

It is apparent at this point that, if the matrix $\partial_\nu\phi^b$ is nonsingular, the solution to the fiducial metric equations is simply $s = 0$. In addition, the solution to the $s = 0$ equation is a subset of the solutions to the Stückelberg field equations $D_\nu(s^{\mu\nu}f_{ab}\partial_\mu\phi^a) = (1/2)s^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^c(\partial_{\phi^0}f_{ac})\delta_{b0}$.

In other words, the solutions to the fiducial metric equations can be thought of as the most probable or more stable subset of solutions to the full set of constraint equations if the matrix $\partial_\nu\phi^b$ is nonsingular.

In addition, the Stückelberg field ϕ^a is introduced to simulate the effect of a gauge parameter. And a change of the Stückelberg field can be represented by a change of the fiducial metric once we allow the fiducial metric fields to be arbitrary fields. Indeed, it can be shown that we can transfer the change of ϕ^a ($\rightarrow \phi'^a$) to f'_{ab} by the relation

$$Z_{\mu\nu} = f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = f'_{ab}\partial_\mu\phi'^a\partial_\nu\phi'^b. \tag{2.29}$$

In particular, for $\phi^a = x^a$, we have

$$f_{\mu\nu} = f'_{ab}\partial_\mu\phi'^a\partial_\nu\phi'^b. \tag{2.30}$$

As a result, we can faithfully embed the dynamics of the Stückelberg field to the dynamics of a more general fiducial metric field. As a result, the fiducial metric equations $s = 0$ can be treated as a set of complete equations governing the central effect of the massive Lagrangian. We will show later that the choice of fiducial metric cannot be random. The choice of fiducial metric has to obey the variational equations. This will impose a constraint on the choice of the fiducial metric such that it will work coherently with the physical metric.

A. Effective cosmological constant

We can also show that $s_{\mu\nu} = 0$ will imply $t_{\mu\nu} = 0$ too. The proof is quite straightforward. Indeed, we can show that

$$B = -Z^{-1}B^TZ = -M^{-2}(g^{-1}B^Tg)M^2 \tag{2.31}$$

if $s = 0$ (or $BZ^{-1} = -Z^{-1}B^T$). Here we have also used the definition $M^2 = g^{-1}Z$ (hence $Z = gM^2$) to derive the above equation. Hence we come to the conclusion that

$$g^{-1}B^Tg = -M^2BM^{-2} = -B \tag{2.32}$$

which is equivalent to the statement $t = 0$ (or $Bg^{-1} = -g^{-1}B^T$). Note that we have used the commuting property $[B, M^2] = 0$. This also follows directly from the fact that $B(K) = MA(K)$ is also a polynomial functional of M.

Hence we prove that the fiducial metric equation $s^{\mu\nu} = 0$ implies that part of the massive-related energy momentum tensor $t^{\mu\nu}$ also vanishes automatically. As a result, the Bianchi identity ensures that the energy momentum tensor associated with the massive terms is conserved:

$$D_\mu T^{\mu\nu}_M \equiv -\frac{1}{2}D_\mu(\mathcal{L}_M g^{\mu\nu} + t^{\mu\nu}) = 0 \tag{2.33}$$

with $T^{\mu\nu}_M \equiv -\frac{1}{2}(\mathcal{L}_M g^{\mu\nu} + t^{\mu\nu})$ the energy momentum tensor associated with the massive Lagrangian $\mathcal{L}_M$. Therefore, we are led to the result

$$\partial^\nu\mathcal{L}_M = -D_\mu t^{\mu\nu} = 0 \tag{2.34}$$

provided that $t^{\mu\nu} = 0$. Therefore, we have shown that for a large class of fiducial metrics f_{ab}, the effect of the massive action is nothing more than a contribution as an effective cosmological constant.

In summary, we have shown that the fiducial equation $s^{\mu\nu} = 0$ will imply $t^{\mu\nu} = 0$. On the other hand, it is also true that $t^{\mu\nu} = 0$ implies $s^{\mu\nu} = 0$ and hence the fiducial metric equation will be observed too. Therefore, the vanishing of the energy momentum tensor $t^{\mu\nu} = 0$ will also be consistent with the complete set of equations.

Note also that, with the matrix notation and new approach introduced in this section, we can derive the variational equations (2.12) and (2.14) straightforwardly. In conclusion, we are left with the modified Einstein equation (2.12) of the following form:

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \Lambda g_{\mu\nu} - \frac{m_g^2}{2}\mathcal{L}_M g_{\mu\nu} = 0 \tag{2.35}$$

once an explicit fiducial metric is assumed. We will then need to compute the explicit value of the effective cosmological constant $\Lambda_M = -m_g^2/(2\mathcal{L}_M)$ from the constraint equation $s^{\mu\nu} = 0$.

B. Alternative proof

For heuristic reasons, we will introduce another way of parametrizing the matrix algebra involved in this section. This will provide another view on the proof that the massive terms will serve as an effective cosmological constant.

Indeed, we can define the hatted tensor $\hat{K}_{\mu\nu} = g_{\mu\alpha}K^\alpha_\nu$ via the matrix formulation $\hat{K} = gK$. In short, we are trying to bring the matrix from a mapping of the type T(1,1) tensor to the type T(0,2) tensor. This will make the matrix multiplication and transport operation more transparent. The multiplication of two hatted matrices will then be defined as $\hat{A}g^{-1}\hat{B}$. Therefore, we have

$$gM^2 = (gM)g^{-1}(gM) = \hat{M}g^{-1}\hat{M}, \tag{2.36}$$

$$\hat{K} = g - \hat{M}, \tag{2.37}$$

$$(\hat{K})^2 = \hat{K} g^{-1} \hat{K}. \tag{2.38}$$

If $\hat{M}$ is chosen to be symmetric, $\hat{K}$ will also be symmetric. Hence there exists a similar transformation that diagonalizes the matrix $K' = S\hat{K}S^{-1}$.

Upon doing this, we can write

$$\hat{t} = \frac{1}{2}\hat{B}, \tag{2.39}$$

$$\hat{s} = \frac{1}{2}\hat{B} \cdot \hat{Z}^{-1} = \frac{1}{2}\hat{B} g^{-1} \hat{Z}^{-1} \tag{2.40}$$

such that $\hat{t}$ and $\hat{s}$ are both symmetric. This follows directly from the fact that $f(\hat{M}) g^{-1} f(\hat{M})$ is symmetric if $\hat{M}$ is symmetric. As a result, in this hatted coordinate, the field equations read

$$D_\mu [\mathcal{L}_M g^{\mu\nu} + \hat{t}^{\mu\nu}] = 0, \tag{2.41}$$

$$\hat{s} = 0. \tag{2.42}$$

Therefore, the vanishing of $\hat{s} = 0$ implies immediately that $\hat{B} = 0$, and hence $\hat{t} = 0$. As a result, the massive Lagrangian $\mathcal{L}_M$ acts as an effective cosmological constant.

III. SOME ADDITIONAL PROPERTIES OF THE FIDUCIAL FIELD EQUATIONS

A. The recurrence relation of the massive Lagrangian

It is known that the quartic massive terms do not contribute to the field equations. We will briefly review the algebra hidden in the determinant of $M = I - K$. The determinant of an $n \times n$ matrix M_{ab} can be shown as

$$|M| \equiv \det M = \frac{1}{n!} e^{a_1, a_2, \ldots, a_n} e^{b_1, b_2, \ldots, b_n} M_{a_1 b_1} M_{a_2 b_2} \cdots M_{a_n b_n} \tag{3.1}$$

with the help of the flat Levi-Civita tensor $e^{a_1, a_2, \ldots, a_n}$ that is totally symmetric in n dimensions. The inverse of the matrix can thus be shown as

$$M^{ab} = \frac{\tilde{M}_{ba}}{|M|} \tag{3.2}$$

with $\tilde{M}_{ba}$ the cofactor of M_{ab} defined as

$$\tilde{M}_{ab} = \frac{s(ab)}{(n-1)!} e^{a, a_2, \ldots, a_n} e^{b, b_2, \ldots, b_n} \langle M_{ab} \rangle M_{a_2 b_2} M_{a_3 b_3} \cdots M_{a_n b_n}. \tag{3.3}$$

$s(ab)$ in the above definition represents the sign derived from the permutation of a and b with respect to the indices a_i and b_i from their original position in order to bring the ordered series from $(a_2, a_3, \ldots, a, \ldots, a_n; b_2, b_3, \ldots, b, \ldots, b_n)$ to $(a, a_2, a_3, \ldots, a_n; b, b_2, b_3, \ldots, b_n)$. In addition, the notation $\langle M_{ab} \rangle$ denotes the omission of the matrix element M_{ab} from the definition of the cofactor $\tilde{M}_{ab}$ in Eq. (3.3). It can therefore be shown that $M^{ab} M_{bc} = \delta^a{}_c$ by direct calculation.

Now let us assume that $n = 4$ in four-dimensional space. We can show that

$$|M| = \frac{1}{2} \sum_{i=0}^{4} (-1)^i \mathcal{L}_i \tag{3.4}$$

with $\mathcal{L}_0 = 2$, $\mathcal{L}_1 = 2[K]$ and

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2],$$
$$\mathcal{L}_3 = \frac{1}{3}[\mathcal{K}]^3 - [\mathcal{K}][\mathcal{K}^2] + \frac{2}{3}[\mathcal{K}^3],$$
$$\mathcal{L}_4 = \frac{1}{12}[\mathcal{K}]^4 - \frac{1}{2}[\mathcal{K}]^2[\mathcal{K}^2] + \frac{1}{4}[\mathcal{K}^2]^2 + \frac{2}{3}[\mathcal{K}][\mathcal{K}^3] - \frac{1}{2}[\mathcal{K}^4],$$

defined earlier in Eqs. (2.9), (2.10), and (2.11). Therefore, the massive Lagrangian is nothing more than the polynomial components of $2|M|$. In addition, the variation of $|M|$, or the variation equation of $\mathcal{L}_n$, with respect to $K^a{}_b$, can be shown to obey the following equations:

$$\frac{\delta |M|}{\delta K^a{}_b} K^a{}_c = |M| \delta^b{}_c - \tilde{M}_c{}^b = |M| \delta^b{}_c - \frac{\delta |M|}{\delta K^c{}_b}. \tag{3.5}$$

Note that we have written the upper and lower indices correctly in order to respect the original tensor properties. These equations can be written as a set of recurrence relations order by order according to their power in $O(K)$. The result is

$$\frac{\delta \mathcal{L}_n}{\delta K^a{}_b} K^a{}_c = \mathcal{L}_{n-1} \delta^b{}_c - \frac{\delta \mathcal{L}_{n-1}}{\delta K^a{}_b} K^a{}_c, \tag{3.6}$$

by collecting all $O(K^n)$ components in the equation. In writing the above equation, we have resummed the matrix component $K^a{}_b$ as a type $T(1, 1)$ tensor. It is easy to show that the equations derived above also hold for $K^a{}_b$. Note that this recurrence relation can also be checked directly from the fiducial metric equations. In particular, the $n = 5$ recurrence relation implies immediately that

$$\mathcal{L}_4 \delta^b{}_c - \frac{\delta \mathcal{L}_4}{\delta K^a{}_b} K^a{}_c = 0 \tag{3.7}$$

since the expansion of $|M|$ in Eq. (3.4) terminates when $n \geq 5$, namely, $\mathcal{L}_5 = 0$. Equation (3.7) shows exactly that the $O(K^4)$ contribution to the energy momentum tensor vanishes identically. It also shows that the field equations given earlier obey a unique recurrence relation that also keeps the physics free of the FP ghost.

B. The traceless quartic terms

Note that we can also show that the trace of $Y_{\mu\nu}$ vanishes in an alternative approach. This is in fact a general feature of the massive terms. For heuristic reasons, we will present the proof as follows. The tensor $Z_{\mu\nu}$ is defined as

$$Z_{\mu\nu} \equiv M_{ab}\partial_\mu \phi^a \partial_\nu \phi^b \tag{3.8}$$

and $M^2 = g^{-1}Z$. If we perform a global scale transformation with a scale factor Ω,

$$\phi^a \to \phi'^a = \Omega^{-1}\phi^a, \tag{3.9}$$

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{3.10}$$

we will have the result that

$$\sqrt{g}\mathcal{L}_M(M^4) \tag{3.11}$$

is invariant under the scale transformation. Here the quartic term $L_M(M^4)$ denotes all components of the massive term L_M that are quartic in M. As a result, we can show that the energy momentum tensor associated with the M-quartic part $L_M(M^4)$ is traceless. To be more specific, we have

$$\begin{aligned}\delta(\sqrt{g}\mathcal{L}_M(M^4)) &= \frac{\delta(\sqrt{g}\mathcal{L}_M(M^4))}{\delta g_{\mu\nu}}\delta g_{\mu\nu} \\ &= 2\frac{\delta(\sqrt{g}\mathcal{L}_M(M^4))}{\delta g_{\mu\nu}} g_{\mu\nu}\Omega\delta\Omega = 0.\end{aligned} \tag{3.12}$$

Therefore, the trace of the energy momentum tensor associated with the M-quartic part,

$$\frac{\delta(\sqrt{g}\mathcal{L}_M(M^4))}{\delta g_{\mu\nu}} g_{\mu\nu} = 0, \tag{3.13}$$

does vanish. And the reason that the whole quartic term $Y_{\mu\nu}$ is traceless is because $Y_{\mu\nu}$ always appears in a combination as a functional of $\mathcal{K}$. The traceless $Y_{\mu\nu}$ means that $\tilde{Y}_{\mu\nu}$ can be written as

$$\tilde{Y}_{\mu\nu} = \frac{1}{4}\tilde{Y}g^{\mu\nu} \tag{3.14}$$

with $\tilde{Y}$ the trace of $\tilde{Y}_{\mu\nu}$, i.e., $\tilde{Y} \equiv \tilde{Y}_{\mu\nu}g^{\mu\nu}$.

IV. BIANCHI TYPE I PHYSICAL METRIC AND FIDUCIAL METRIC

In order to study the nontrivial contribution from a more general reference metric f_{ab}, we will assume that the reference metric belongs to the Bianchi type metric similar to the physical metric. We will also try to find analytic solutions with the Stückelberg scalar fields chosen in the unitary gauge $\phi^a = x^a$. As shown earlier, we will treat the reference metric and the Stückelberg scalar fields as auxiliary fields. A variation method will then be performed to obtain the field equations along with the physical metric.

It was shown earlier that the fiducial equation $s = 0$ will naturally lead to the result $t = 0$. This set of consistent solutions will hence lead to the result $\mathcal{L}_M =$ constant. Thus, the contribution from the massive terms will simply act as an effective cosmological constant to the physical metric equation. In conclusion, we are left with the modified Einstein equation (2.12) in the following form:

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + (\Lambda + \Lambda_M)g_{\mu\nu} = 0. \tag{4.1}$$

Therefore, we need to compute the exact value of the effective cosmological constant $\Lambda_M = -m_g^2\mathcal{L}_M/2$.

We will try to study the effect of the massive gravity theory with the Bianchi type I physical metric $g_{\mu\nu}$ and similarly the Bianchi type I fiducial metric f_{ab} represented by $Z_{\mu\nu}$:

$$\begin{aligned}g_{\mu\nu}dx^\mu dx^\nu = &-N_1^2(t)dt^2 + \exp[2\alpha_1(t) - 4\sigma_1(t)]dx^2 \\ &+ \exp[2\alpha_1(t) + 2\sigma_1(t)](dy^2 + dz^2),\end{aligned} \tag{4.2}$$

$$\begin{aligned}Z_{\mu\nu} = &-N_2^2(\phi^0)\partial_\mu\phi^0\partial_\nu\phi^0 \\ &+ \exp[2\alpha_2(\phi^0) - 4\sigma_2(\phi^0)]\partial_\mu\phi^1\partial_\nu\phi^1 \\ &+ \exp[2\alpha_2(\phi^0) + 2\sigma_2(\phi^0)] \\ &\times(\partial_\mu\phi^2\partial_\nu\phi^2 + \partial_\mu\phi^3\partial_\nu\phi^3),\end{aligned} \tag{4.3}$$

with N_1 and N_2 the lapse functions. Note that N_1 is introduced here to obtain the Friedmann equation from its variational equation (or the Euler-Lagrange equation) [4,9,28]. It can be set as $N_1 = 1$ by reparametrizing the time coordinate. Note that we cannot do the same thing on N_2 once the time coordinate has been chosen as $N_1 = 1$. Therefore, N_2 will be left as a free parameter to be solved from the field equation.

The variational equation of the physical metric will lead to the following equations:

$$3(\dot{\alpha}_1^2 - \dot{\sigma}_1^2) - \Lambda = \Lambda_M, \tag{4.4}$$

$$2\ddot{\alpha}_1 + 3\dot{\alpha}_1^2 + 3\dot{\sigma}_1^2 - \Lambda = \Lambda_M, \tag{4.5}$$

$$\ddot{\sigma}_1 + 3\dot{\alpha}_1\dot{\sigma}_1 = 0. \tag{4.6}$$

Here the result $\mathcal{L}_M$ will serve as an effective cosmological constant $\mathcal{L}_M = -2\Lambda_M/m_g^2$ under the constraint fiducial metric equation derived from $s_{\mu\nu} = 0$.

In order to derive the constraint equations, we will define the following parameters as in Ref. [21]:

$$[\mathcal{K}]^n = (4 - \gamma - A - 2B)^n, \tag{4.7}$$

$$[\mathcal{K}^n] = (1-\gamma)^n + (1-A)^n + 2(1-B)^n, \tag{4.8}$$

$$\gamma = \frac{N_2}{N_1}; \qquad A = \epsilon\eta^{-2}; \qquad B = \epsilon\eta, \tag{4.9}$$

$$\epsilon = \exp[\alpha_2 - \alpha_1]; \qquad \eta = \exp[\sigma_2 - \sigma_1], \tag{4.10}$$

$$R = \frac{6}{N_1^3}[-\dot{N}_1\dot{\alpha}_1 + N_1(\ddot{\alpha}_1 + 2\dot{\alpha}_1^2 + \dot{\sigma}_1^2)]. \tag{4.11}$$

Note that due to the structure of the massive terms (i.e., they are functionals of $[K^n]$) the fiducial metric is coherent with the physical metric in combinations as ϵ and η. We will come back to this point later to show that the choice of fiducial metric f_{ab} has to work coherently with the chosen physical metric $g_{\mu\nu}$ in order to accommodate the effect $\mathcal{L}_M =$ constant.

With these notations, the massive terms in Eqs. (2.9), (2.10), and (2.11) can be shown as

$$\mathcal{L}_2 = 2[B(2A + B) + (\gamma - 3)(A + 2B) + 3(2 - \gamma)], \tag{4.12}$$

$$\mathcal{L}_3 = -2[AB^2 + 3\gamma - 4 + (3 - 2\gamma)(A + 2B) + (\gamma - 2)B(2A + B)], \tag{4.13}$$

$$\mathcal{L}_4 = 2(\gamma - 1)(A - 1)(B - 1)^2. \tag{4.14}$$

Hence we can show that

$$\begin{aligned}\mathcal{L}_M &\equiv \mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4 \\ &= 2[(\gamma\gamma_2 - \gamma_1)(A + 2B) + (\gamma_2 - \gamma\gamma_3)B(2A + B) \\ &\quad + (\gamma\alpha_4 - \gamma_3)AB^2 - \gamma\gamma_1 + (3\gamma_1 - 3\gamma_2 + \gamma_3)],\end{aligned} \tag{4.15}$$

with the parameters γ_i defined as [12,13]

$$\gamma_1 = 3 + 3\alpha_3 + \alpha_4; \qquad \gamma_2 = 1 + 2\alpha_3 + \alpha_4; \qquad \gamma_3 = \alpha_3 + \alpha_4. \tag{4.16}$$

Note that the unitary gauge condition has been used in deriving the above equations.

V. THE CONSTRAINT EQUATIONS

We will try to evaluate the effective cosmological constant Λ_M in this section. Note that the fiducial metric only shows up in the massive Lagrangian $\mathcal{L}_M$. Hence we only need to obtain the variational equation by varying $\mathcal{L}_M$ with respect to the fiducial metric f_{ab}. This can be done effectively by varying with respect to N_2, α_2 and σ_2. The constancy of the massive Lagrangian is also apparent as a unique property that there is no time derivative in the fiducial metric. As a result, the variational equations

$$\frac{\partial\mathcal{L}_M}{\partial N_2} = 0, \qquad \frac{\partial\mathcal{L}_M}{\partial\alpha_2} = 0, \qquad \frac{\partial\mathcal{L}_M}{\partial\sigma_2} = 0 \tag{5.1}$$

simply become a statement that $\mathcal{L}_M$ is independent of N_2, α_2 and σ_2.

In addition, the variational equations can also be obtained effectively by the variation with respect to an equivalent set of new variables A, B and γ. To be more specific, we can show that the variational equations are related by

$$\frac{\partial\mathcal{L}}{\partial N_2} = \frac{\partial\mathcal{L}}{\partial\gamma} = 0, \qquad \frac{\partial\mathcal{L}}{\partial\alpha_2} = A\frac{\partial\mathcal{L}}{\partial A} + B\frac{\partial\mathcal{L}}{\partial B} = 0, \qquad \frac{\partial\mathcal{L}}{\partial\sigma_2} = -2A\frac{\partial\mathcal{L}}{\partial A} + B\frac{\partial\mathcal{L}}{\partial B} = 0. \tag{5.2}$$

Because $A = \epsilon\eta^{-2} > 0$, $B = \epsilon\eta > 0$ as shown in Eq. (4.9), it is straightforward to show that the variational equations with respect to N_2, α_2 and σ_2 [(5.1)] are equivalent to the variational equations with respect to A, B and γ:

$$\frac{\partial\mathcal{L}}{\partial\gamma} = 0, \qquad \frac{\partial\mathcal{L}}{\partial A} = 0, \qquad \frac{\partial\mathcal{L}}{\partial B} = 0. \tag{5.3}$$

As a result, the fiducial equations can be shown as

$$\frac{\partial\mathcal{L}}{\partial\gamma} = -2\hat{m}[\gamma_1 - \gamma_2(A + 2B) + \gamma_3 B(2A + B) - \alpha_4 AB^2] = 0, \tag{5.4}$$

$$\frac{\partial\mathcal{L}}{\partial A} = 2\hat{m}[(\gamma\alpha_4 - \gamma_3)B^2 + 2(\gamma_2 - \gamma\gamma_3)B - \gamma_1 + \gamma\gamma_2] = 0, \tag{5.5}$$

$$\frac{\partial\mathcal{L}}{\partial B} = 4\hat{m}[(\gamma\alpha_4 - \gamma_3)AB + (\gamma_2 - \gamma\gamma_3)(A + B) - \gamma_1 + \gamma\gamma_2] = 0 \tag{5.6}$$

with $\hat{m} = M_p^2 e^{3\alpha_1} N_1 m_g^2/2$. We can thus combine Eqs. (5.5) and (5.6) to derive

$$(B - A)[\gamma_2 - \gamma\gamma_3 - (\gamma_3 - \gamma\alpha_4)B] = 0. \tag{5.7}$$

A. $A = B$ solutions

Note that $A = B$ is a trivial solution to the above equation. Indeed, $A = B$ implies that $\eta = 1$ and hence $\sigma_1 = \sigma_2$. This means that the anisotropy factors of the physical metric and fiducial metric agree with each other. In addition, the constraint equations (5.4) and (5.5) reduce to

$$(1 - A)[3 + 3(1 - A)\alpha_3 + (1 - A)^2\alpha_4] = 0, \tag{5.8}$$

$$2A + N_2 - 3 + (1 - A)(A - 3 + 2N_2)\alpha_3 + (N_2 - 1)(1 - A)^2\alpha_4 = 0. \tag{5.9}$$

It is easy to show that the solution $N_2 = A$ will turn Eq. (5.9) into Eq. (5.8). Therefore, $N_2 = A$ is a solution to the above equations. As a result, Eq. (5.8) implies that $A = 1$ or

$$3 + 3(1 - A)\alpha_3 + (1 - A)^2\alpha_4 = 0. \tag{5.10}$$

The set of solutions with $A = N_2 = 1$ is a trivial solution with a vanishing effective cosmological constant $\Lambda_M = 0$. This corresponds to the case in which the fiducial metric is identical to the physical metric.

In addition to the trivial solutions $A = N_2 = 1$, Eq. (5.10) also admits two more solutions:

$$A = N_2 = 1 + \frac{2\alpha_3 \pm \sqrt{9\alpha_3^2 - 12\alpha_4}}{2\alpha_4}, \tag{5.11}$$

requiring the constraint on the field parameters $3\alpha_3^2 > 4\alpha_4$. In addition, the effective cosmological constant $\Lambda_M = -m_g^2 \mathcal{L}_{\mathcal{M}}/2$ with

$$\mathcal{L}_M = 2(A-1)^2[\alpha_4 A^2 - 2(2\alpha_3 + \alpha_4)A + 4\alpha_3 + \alpha_4 + 6]. \tag{5.12}$$

With the solution $A = B = N_2$ given by Eq. (5.10) we can show that the effective cosmological constant is

$$\Lambda_M = \frac{3m_g^2}{2\alpha_4^3}\Big[9\alpha_3^4 + 6\alpha_4^2 - 18\alpha_3^2\alpha_4 \pm \alpha_3(3\alpha_3^2 - 4\alpha_4)\sqrt{3(3\alpha_3^2 - 4\alpha_4)}\Big]. \tag{5.13}$$

B. $A \neq B$ solutions

For the case where $A \neq B$ (or $\eta \neq 1$), we can show that Eq. (5.7) implies

$$B = \frac{\gamma_2 - N_2\gamma_3}{\gamma_3 - N_2\alpha_4}. \tag{5.14}$$

At this point, we note that the solution $s_{\mu\nu} = 0$ will imply that the massive Lagrangian is a constant independent of time. This will imply, for the Bianchi type I fiducial metric space, that A, B and N_2 are all constants in time. This then implies that $\alpha_2 \propto \alpha_1$ and $\sigma_2 \propto \sigma_1$.

In fact, we can show that $Z_{\mu\nu} = f_{\mu\nu} = f_{ab}\delta^a_\mu \delta^b_\nu$ under the unitary gauge $\phi^a = x^a$. Therefore, the definition $M^2 = g^{-1}Z$ implies that $(M^2)^\mu{}_\nu = g^{\mu\alpha} f_{\alpha\nu}$. Therefore, $\mathcal{L}_M(M)$, a functional of $[M^n]$, for some integer n, is then a functional of $[(g^{-1}Z)^{n/2}]$. Hence, the result $\mathcal{L}_M$ is a constant implies that $[(g^{-1}Z)^{n/2}]$ or its appropriate combinations must be constants too.

In the model studied in this section, the physical metric space is chosen as the Bianchi type I space; the constancy of the massive terms implies that the consistent metric choice will have to be of a compatible type metric. As a result, A, B and N_2 can remain constants once $\alpha_2 \propto \alpha_1$ and $\sigma_2 \propto \sigma_1$ are set as the fixed point solutions.

Therefore, the choice of unitary gauge and the fiducial metric equation $s_{\mu\nu} = 0$ will only be a consistent choice of solutions if $g^{\mu\alpha} f_{\alpha\nu}$ turns out to be compatible with the requirement that $[(g^{-1}Z)^{n/2}]$ or its appropriate combinations are all constants too. This indicates that the fiducial metric and physical metric must be chosen in a consistent manner. For example, when we have set the physical metric space as the Bianchi type I space, the consistent choice of fiducial metric will have to be a similar Bianchi type I metric in order to accommodate a consistent solution under the unitary gauge.

Note also that $s_{\mu\nu} = 0$ is a solution to the fiducial metric equation; even the Stückelberg field is chosen in the unitary gauge. The only requirement is the existence of nonsingularity of the matrix $\partial_\mu \phi^a$. Therefore, the $s_{\mu\nu} = 0$ solution remains an appropriate solution to the whole system. As a result, this set of solutions will cause the massive terms to act as an effective cosmological constant. We will show in the next section that the $s_{\mu\nu} = 0$ solution tends to be a stable solution against the perturbations to the whole system.

In summary, a simple set of solutions under the unitary gauge chosen for the Stückelberg field will set the massive terms as an effective constant. This choice of gauge cannot however be made independent of the choice of a coherent set of physical and fiducial metrics. In short, the fiducial metric has to be in the same class as the physical metric in order to turn the massive terms into an effective cosmological constant, as in the example demonstrated in this section.

Substituting this solution into Eq. (5.5) or Eq. (5.6), we can derive an equation for N_2:

$$(\gamma_3^2 - \alpha_4\gamma_2)N_2^2 + (\alpha_4\gamma_1 - \gamma_2\gamma_3)N_2 + \gamma_2^2 - \gamma_1\gamma_3 = 0. \tag{5.15}$$

With the definitions of the γ_i's in Eq. (4.16), we can write Eq. (5.15) as

$$(\alpha_3^2 - \alpha_4)N_2^2 - (2\alpha_3^2 + \alpha_3 - 2\alpha_4)N_2 + \alpha_3^2 + \alpha_3 - \alpha_4 + 1 = 0. \tag{5.16}$$

Therefore, the solutions to this equation are

$$N_2^{\pm} = 1 + \frac{\alpha_3 \pm \sqrt{-3\alpha_3^2 + 4\alpha_4}}{2(\alpha_3^2 - \alpha_4)}. \tag{5.17}$$

Therefore, the constraint for the coupling constants α_3 and α_4 is $\alpha_4 > 3\alpha_3^2/4$. Note that N_2 will be taken as a positive constant for convenience. This is the constraint for the existence of the Bianchi type I solutions we study in this paper. Therefore, Eq. (5.4) implies that

$$A = N_2, \tag{5.18}$$

and hence

$$\epsilon = \exp[\alpha_2 - \alpha_1] = (AB^2)^{\frac{1}{3}}, \qquad \eta = \exp[\sigma_2 - \sigma_1] = \left(\frac{B}{A}\right)^{\frac{1}{3}}. \tag{5.19}$$

As a result, we can evaluate the effective cosmological constant as

$$\Lambda_M = -\frac{m_g^2}{2}(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) = \frac{m_g^2}{\alpha_4 - \alpha_3^2}. \tag{5.20}$$

It is apparent that $\alpha_4 - \alpha_3^2 > 0$ is the requirement that the effective cosmological constant is positive. On the other hand, it will contribute as a negative cosmological constant. In particular, it will cancel the effect of the genuine cosmological constant if $\Lambda = -\Lambda_M$. Or equivalently,

$$\alpha_3^2 - \alpha_4 = \frac{m_g^2}{\Lambda}. \tag{5.21}$$

In summary, we have two different sets of solutions corresponding to the cases (i) $A = B = N_2$ and (ii) $A = N_2 \neq B$, with the associated effective cosmological constant given, respectively, by

$$\Lambda_M = \frac{3m_g^2}{2\alpha_4^3}\Big[9\alpha_3^4 + 6\alpha_4^2 - 18\alpha_3^2\alpha_4 \pm \alpha_3(3\alpha_3^2 - 4\alpha_4)\sqrt{3(3\alpha_3^2 - 4\alpha_4)}\Big] \tag{5.22}$$

and

$$\Lambda_M = -\frac{m_g^2}{2}(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) = \frac{m_g^2}{\alpha_4 - \alpha_3^2}. \tag{5.23}$$

Note that the first set of solutions $A = B = N_2$ exists only when $3\alpha_3^2 - 4\alpha_4 > 0$, while the second set of solutions $A = N_2 \neq B$ exists only when $3\alpha_3^2 - 4\alpha_4 < 0$. Hence solutions exist for all physical parameters α_3 and α_4. Different combinations will just correspond to different choices of the fiducial metric that can couple to the massive gravity theories.

VI. ANISOTROPIC COSMOLOGICAL SOLUTION AND ITS STABILITY

Note that we are left with the field equations (4.6), (4.7), and (4.8), which could be written as

$$3(\dot\alpha_1^2 - \dot\sigma_1^2) = \Lambda_1 = \Lambda + \Lambda_M, \tag{6.1}$$

$$2\ddot\alpha_1 + 3\dot\alpha_1^2 + 3\dot\sigma_1^2 = \Lambda_1, \tag{6.2}$$

$$\ddot\sigma_1 + 3\dot\alpha_1\dot\sigma_1 = 0. \tag{6.3}$$

Eliminating the $\dot\sigma_1^2$ terms from Eqs. (6.1) and (6.2), we will have the following ordinary differential equation for α_1:

$$\ddot\alpha_1 + 3\dot\alpha_1^2 = \Lambda_1. \tag{6.4}$$

This equation can be solved by defining the volume factor $V = a_1a_2a_3 = \exp[3\alpha_1]$. Indeed, we can write Eq. (6.4) as

$$\ddot V = 3\Lambda_1 V \equiv 9H_1^2 V, \tag{6.5}$$

which is a linear equation in V. Here we have written $H_1^2 = \Lambda_1/3$ as the Hubble parameter. This can be solved to give a linear combination of the exponential solutions:

$$V = a\exp[3H_1t] + b\exp[-3H_1t]. \tag{6.6}$$

We can therefore show that $\exp[3\alpha_1]$ becomes

$$V = \exp[3\alpha_1] = \exp[3\alpha_0]\Big[\cosh 3H_1t + \frac{\dot\alpha_0}{H_1}\sinh 3H_1t\Big] \tag{6.7}$$

with $\alpha_0 = \alpha_1(t=0)$ and $\dot\alpha_0 = \dot\alpha_1(t=0)$ the appropriate initial values. In addition, the field equation (6.3) can be shown to give the following solution:

$$\dot\sigma_1 = k\exp[-3\alpha_1] \tag{6.8}$$

with k an integration constant. Moreover, the Friedmann equation implies that the following boundary condition has to be observed:

$$\dot\alpha_0^2 - H_1^2 = k^2\exp[-6\alpha_0]. \tag{6.9}$$

Equation (6.8) can be integrated directly to give

$$\sigma_1(t) = \sigma_0 + \Big[\frac{1}{3}\Big]\Big\{\ln\Big[\frac{\sqrt{\dot\alpha_0 + H_1}\exp[3H_1t] - \sqrt{\dot\alpha_0 - H_1}}{\sqrt{\dot\alpha_0 + H_1} - \sqrt{\dot\alpha_0 - H_1}}\Big] + \ln\Big[\frac{\sqrt{\dot\alpha_0 + H_1} + \sqrt{\dot\alpha_0 - H_1}}{\sqrt{\dot\alpha_0 + H_1}\exp[3H_1t] + \sqrt{\dot\alpha_0 - H_1}}\Big]\Big\} \tag{6.10}$$

with $\sigma_0 = \sigma_1(0)$. With the solution of σ_1 shown above, we can readily show that the Friedmann equation (6.1) gives nothing more than the boundary condition (6.9). All time-dependent terms just cancel each other. As a result, we show that the exact solutions (6.8) and (6.10) are a set of complete solutions to the field equations.

A. Stability analysis

Now we are ready to perturb the field equations in order to verify whether the solutions we found are stable or not. The Einstein equations take the following form: $G_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu}$. The perturbation of the Einstein tensor $\delta G_{\mu\nu}$ will be related to the perturbation of $T_{\mu\nu} = \mathcal{L}_m g_{\mu\nu} + t_{\mu\nu}$. But we have already shown that the fiducial equation $s_{\mu\nu} = 0$ implies the vanishing of $t_{\mu\nu}$. Once the perturbation $\delta s_{\mu\nu} = 0$ is included in the full set of the perturbation equations, we only need to include the effect of the perturbation derived from $\mathcal{L}_M$.

In addition, we have also shown that the fiducial metric equation $s_{\mu\nu} = 0$ reduces to a set of equations: $\delta\mathcal{L}_M/\delta N_2 = \delta\mathcal{L}_M/\delta A = \delta\mathcal{L}_M/\delta B = 0$. This set of equations also agrees with the condition that $\mathcal{L}_M =$ constant. Therefore, the perturbation of the massive term

$\mathcal{L}_M$ will be the same as the perturbation of the fiducial metric. Therefore, the complete set of perturbations will hence be equivalent to the set of perturbation equations derived from the Einstein equations by assuming Λ_1 is a constant irrelevant to the massive contents of the fiducial metric.

We introduce the exponential perturbations of the following form:

$$\begin{aligned}\delta\alpha_1 &= C_\alpha \exp[\kappa t]; \qquad \delta\sigma_1 = C_\sigma \exp[\kappa t],\\ \delta A &= C_A \exp[\kappa t]; \qquad \delta B = C_B \exp[\kappa t],\\ \delta N_2 &= C_{N_2} \exp[\kappa t].\end{aligned} \tag{6.11}$$

The perturbation of Eqs. (6.1), (6.8), (5.4), (5.5), and (5.6) around the cosmological solution defined by Eqs. (6.7), (6.10), (5.14), (5.17), and (5.18) will lead to the following perturbation equations written in matrix form:

$$\mathcal{D}\begin{pmatrix} C_\alpha \\ C_\sigma \\ C_A \\ C_B \\ C_{N_2}\end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0\\ A_{21} & A_{22} & 0 & 0 & 0\\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35}\\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45}\\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55}\end{bmatrix}\begin{pmatrix} C_\alpha \\ C_\sigma \\ C_A \\ C_B \\ C_{N_2}\end{pmatrix} = 0, \tag{6.12}$$

with $A_{ij}(\kappa)$ the perturbation function of κ. Note that we have eliminated the perturbation of the energy momentum tensor derived from the massive terms from the perturbation of α_1 and σ_1. These effects have already been included in the perturbations of A, B and N_2.

In order to admit a nontrivial solution to the perturbation equation, $\mathcal{D}$ has to be singular, i.e., $\det\mathcal{D} = 0$. It is therefore easy to see that the perturbation effect of the massive term decouples from the perturbation of the physical metric.

In addition, the perturbation derived from the $s_{\mu\nu} = 0$ part does not have any time derivative in it. Therefore, the perturbation of this part can only contribute to the $O(\kappa^0)$ coefficient. Therefore, the perturbation of the massive terms will not have any impact on the stability of the fixed point solutions, Eqs. (6.7) and (6.10). In short, the massive terms have nothing to do with the stability of the solution we found. This is true for all solutions of the form $s_{\mu\nu} = 0$ as long as it also implies the existence of the constraint $\mathcal{L}_M =$ constant.

Ignoring the effect of the massive term, the stability analysis reduces to the perturbation of α_1 and σ_1:

$$\mathcal{D}_1\begin{pmatrix} C_\alpha \\ C_\sigma\end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix} C_\alpha \\ C_\sigma\end{pmatrix} = 0. \tag{6.13}$$

As a result, we can show that

$$\mathcal{D}_1 = \begin{bmatrix} \kappa\dot\alpha_1, & -\kappa\dot\sigma_1\\ 3\dot\sigma_1, & \kappa\end{bmatrix}. \tag{6.14}$$

In order to admit a nontrivial solution to the perturbation equation, $\mathcal{D}_1$ has to be singular, i.e., $\det\mathcal{D}_1 = 0$. Therefore, this requirement leads to the equation

$$\kappa(\dot\alpha_1\kappa + 3\dot\sigma_1^2) = 0. \tag{6.15}$$

Thus, the perturbation equations admit two solutions, $\kappa = 0$ and

$$\kappa = \kappa_1 = -\frac{3k^2}{\dot\alpha_1\exp[6\alpha_1]}. \tag{6.16}$$

Since all solutions are nonpositive, the perturbation indicates that the solutions we found are in fact stable solutions. One also notes that $\dot\alpha_1 \to H_1$ and $\exp[\alpha_1] \to \infty$ at future infinity for expanding solutions. Therefore, the solution κ_1 tends to vanish at time infinity. In addition, we have also shown that, for this class of solutions in which the massive terms act as an effective cosmological constant, the massive terms will not affect the stability of the system.

Note that we can also perform the perturbation on the Stükelberg field along with the perturbations of the other fields. We will show that the result will not affect the stability analysis shown above. In fact, the perturbation of the Stükelberg field is equivalent to a different gauge choice. The gauge choice will not affect the stability analysis.

For heuristic reasons, we can also show this result from an observation that the perturbation on the Stükelberg field is equivalent to the perturbation on the s_{00} component of the fiducial metric equation. Indeed, it can be shown that the consistent time-dependent perturbation of the Stükelberg field against the unitary gauge, $\phi^a \to x^a + \varphi^a(t)$ is then $\delta\phi^a = \varphi^a(t)$. In addition, the perturbation on $Z_{\mu\nu} = f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$ gives $\delta Z_{\mu\nu} = f_{a\nu}\partial_\mu\varphi^a + f_{a\mu}\partial_\nu\varphi^a + f_{ab}\partial_\mu\varphi^a\partial_\nu\varphi^b$. Hence we can show that the perturbation $\delta Z_{\mu\nu}$ contributes only to the following components:

$$\delta Z_{00} = f_{00}[2\dot\varphi^0 + (\dot\varphi^0)^2] + \sum_{i=1}^{3} f_{ii}(\dot\varphi^i)^2, \tag{6.17}$$

$$\delta Z_{0i} = \delta Z_{i0} = f_{ii}\dot\varphi^i. \tag{6.18}$$

All massive terms are functionals of $[M^n]$, for some integer n. Since g is diagonal in the Bianchi type I space, we can show that the off-diagonal terms in $M^2(= g^{-1}Z)$ vanish with respect to the perturbation of the Stükelberg field. To be more specific, we can show that

$$M^2 = \begin{bmatrix} N_2^2[1+2\dot{\varphi}^0+(\dot{\varphi}^0)^2]-\sum_{i=1}^3 f_{ii}(\dot{\varphi}^i)^2, & 0, & 0, & 0 \\ 0, & A^2, & 0, & 0 \\ 0, & 0, & B^2, & 0 \\ 0, & 0, & 0, & B^2 \end{bmatrix}, \tag{6.19}$$

when the Stükelberg field is perturbed away from the unitary gauge condition. Therefore, the perturbation of the matrix M can only affect the M_{00} component. The result is the linear perturbation given by

$$\frac{\delta M_{00}}{M_{00}} = \dot{\varphi}^0 = \delta\dot{\phi}^0. \tag{6.20}$$

Note that the massive terms only depend on the $\dot{\phi}^a$. Hence, the time-dependent perturbation of the Stükelberg field ϕ^a will effectively go through the perturbation of $\dot{\phi}^a$. The result shown above simply establishes the claim that the perturbation of the Stükelberg field is equivalent to the perturbation of the M_{00} component of the matrix M. Therefore, the perturbation of the Stükelberg field equation is equivalent to the perturbation equation $s_{00} \sim \delta\mathcal{L}_M/\delta M_{00} \sim \delta\mathcal{L}_M/\delta\dot{\phi}^0 = 0$. Hence, the effect of the perturbation of the Stükelberg field has the same contribution as the perturbation of f_{00}, with the only difference being that $\delta\dot{\phi}^0$ replaces δf_{00}. This result indeed supports the proof shown earlier that the perturbation of the Stükelberg field will not affect the stability of the massive gravity theory.

On the other hand, the Stükelberg field equation is, from (2.24),

$$D_\nu(s^{\mu\nu} f_{ab}\partial_\mu\phi^a) = \frac{1}{2}s^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^c(\partial_{\phi^0}f_{ac})\delta_{b0}. \tag{6.21}$$

Therefore, the perturbation equation against the background solution $s_{\mu\nu} = 0$ and $\phi^a = x^a$ is

$$D_\nu((\delta s^{\mu\nu})f_{\mu b}) = \frac{1}{2}\delta s^{\mu\nu}(\partial_{\phi^0}f_{\mu\nu})\delta_{b0}. \tag{6.22}$$

Hence, the solution to the $\delta s_{\mu\nu} = 0$ perturbation equation is also a solution to the perturbation of the Stükelberg field equation.

Both approaches show that the perturbation of the Stükelberg field equation is equivalent to the perturbation of the $s_{00} = 0$ equation. Hence, the perturbation of the Stükelberg field has the same contribution as the perturbation of f_{00}, with the only difference being that $\delta\dot{\phi}^0$ replaces $\delta f_{00} = \delta N_2$. Therefore, the perturbation of the Stükelberg field equation will effectively be decoupled from the matrix determinant of $\mathcal{D}_1$ defined above.

B. Effect of scalar fields

Motivated by the work in Ref. [26], we would like to discuss the effect of an additional scalar field ϕ coupled to the system (2.8). We would like to understand how the inclusion of a scalar field will affect the stability of the expanding solutions. To be more specific, the new action proposed will be

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[R - 2\Lambda - \omega\partial_\mu\phi\partial^\mu\phi - 2V(\phi) + m_g^2(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4)]. \tag{6.23}$$

Note that we have set the units with the Planck mass $M_p = 1$ and $m_g = 1$ for convenience. In addition, $\omega = -1$ will represent the action with a phantom field [29]. Note that there have been discussions on the nonlinear massive gravity theory with a functional of the scalar field representing the massive gravity [9,18]. Nonetheless, the effect of an additional scalar field could be an interesting research subject all by itself.

The field equations of the model (6.23) can be shown as

$$3(\dot{\alpha}_1^2 - \dot{\sigma}_1^2) = \Lambda_1 + V + \omega\frac{\dot{\phi}^2}{2}, \tag{6.24}$$

$$2\ddot{\alpha}_1 + 3\dot{\alpha}_1^2 + 3\dot{\sigma}_1^2 = \Lambda_1 + V - \omega\frac{\dot{\phi}^2}{2}, \tag{6.25}$$

$$\dot{\sigma}_1 = k\exp[-3\alpha_1], \tag{6.26}$$

$$\ddot{\phi} = -3\dot{\alpha}_1\dot{\phi} - \frac{\partial_\phi V}{\omega}. \tag{6.27}$$

Note that we have also integrated the σ_1 equation to obtain the final expression for the σ_1 equation as shown in Eq. (6.26).

It is easy to show that $\phi = \phi_0 =$ constant is a trivial solution to ϕ. In this case, the inclusion of the scalar field potential will only contribute another cosmological constant V to the metric equation. Note also that the massive terms will be decoupled from the metric perturbation equations.

We will present a general discussion without obtaining a specific set of solutions to the field equations at this moment. Our interest here is to discuss the effect due to the presence of the scalar field in the evolutionary universe. Therefore, we will simply assume that there exists a set of nontrivial solutions with the inclusion of ϕ. As a result, we can write the metric perturbation equations and ϕ perturbation equation as a matrix equation, with $\delta\phi = C_\phi\exp[\kappa t]$. We can also perturb Eqs. (6.24), (6.26), and (6.27) and write the result as a matrix equation:

$$\hat{\mathcal{D}}_2\begin{pmatrix} C_\alpha \\ C_\sigma \\ C_\phi \end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{pmatrix} C_\alpha \\ C_\sigma \\ C_\phi \end{pmatrix} = 0, \tag{6.28}$$

with $\mathcal{D}_2$ given by

$$\mathcal{D}_2 = \begin{bmatrix} 6\dot{\alpha}_1\kappa, & -6\dot{\sigma}_1\kappa, & -\omega\dot{\phi}\kappa - V' \\ 3\dot{\sigma}_1, & \kappa, & 0 \\ 3\dot{\phi}\kappa, & 0, & \kappa^2 + 3\dot{\alpha}_1\kappa + \omega V'' \end{bmatrix}. \quad (6.29)$$

In order to admit a nontrivial solution to the perturbation equation, $\mathcal{D}_2$ has to be singular, i.e., $\det \mathcal{D}_2 = 0$. As a result, $\det \mathcal{D}_2 = 0$ can be shown as

$$\det \mathcal{D}_2 = 3\kappa\mathcal{A}, \quad (6.30)$$

$$\mathcal{A} = 2\dot{\alpha}_1\kappa^3 + (\omega\dot{\phi}^2 + 6\dot{\alpha}_1^2 + 6\dot{\sigma}_1^2)\kappa^2 + (\dot{\phi}V' + 2\omega\dot{\alpha}_1 V'' + 18\dot{\alpha}_1\dot{\sigma}_1^2)\kappa + 6\omega V''\dot{\sigma}_1^2. \quad (6.31)$$

Note that $\dot{\alpha}_1 > 0$ for the expanding solution. It is then easy to show that if the constant coefficient $6\omega V''\dot{\sigma}_1^2$ of the polynomial function $\mathcal{A}$ is negative, there is at least a positive root to the equation $\mathcal{D}_2 = 0$. Therefore, once $\omega V'' < 0$, the solution we found will be an unstable solution. If we assume $V(\phi) = \exp[\lambda_1\phi]$ with coupling constant λ_1, then an unstable mode will definitely exist when a phantom field is present. This conclusion is also consistent with the investigations shown in Ref. [26]. We also note that, in particular, there will be no unstable mode when $V = 0$. Therefore, the stability of the system with a phantom field is critical for the presence of a scalar potential term.

C. Isotropic fiducial metric and global stability analysis

In the literature, earlier works in finding expanding solutions associated with the massive gravity are all based on the model with an isotropic fiducial metric. Anisotropic physical metrics are the only way to generalize the massive theories in earlier work. Since the massive theories remain ghost-free for an arbitrary fiducial metric, we start to wonder what will happen if the anisotropic fiducial metric is introduced.

Once a general analysis is performed in the presence of an arbitrary fiducial metric, we soon realize, as indicated by Eq. (2.25), that the fiducial metric equation $s^{\mu\nu} = 0$ is also a solution to the Stückelberg equation (2.26). As a result, the existence of the anisotropic expanding solutions to $s^{\mu\nu} = 0$ will require that physical metric fields $(g_{00}, g_{11}, g_{22}, g_{33})$ and the corresponding fiducial metric fields $(f_{00}, f_{11}, f_{22}, f_{33})$ have to evolve harmonically such that γ, A and B remain constant all the time. This is also the key point to make the massive terms act as an effective cosmological constant.

For a comparison with earlier solutions, we can take the isotropic fiducial limit of our solutions to obtain the solutions found, for example, in Ref. [12]. Indeed, we can show that the Stückelberg equation (2.26) turns out to be

$$\gamma\left(\frac{d}{dt} + 3H\right)\left(\frac{1}{1+\dot{f}}\frac{\partial\mathcal{L}_M}{\partial\gamma}\right) = [\alpha_2' - 2\sigma_2']A\frac{\partial\mathcal{L}_M}{\partial A} + [\alpha_2' + \sigma_2']B\frac{\partial\mathcal{L}_M}{\partial B}. \quad (6.32)$$

Here $'$ in $F'(\phi^0) = \partial_{\phi^0}F(\phi^0)$ denotes the differentiation of any function $F(\phi^0)$ with respect to its argument ϕ^0. Note that this equation can also be derived as the variational equation of $f(t)$:

$$\sqrt{g}\frac{\partial\mathcal{L}_M}{\partial f} - \frac{d}{dt}\left(\sqrt{g}\frac{\partial\mathcal{L}_M}{\partial\dot{f}}\right) = 0. \quad (6.33)$$

Indeed, by replacing $\phi^0 = x^0$, the leading order of this equation reduces to

$$\gamma\left(\frac{d}{dt} + 3H\right)\left(\frac{\partial\mathcal{L}_M}{\partial\gamma}\right) = [\dot{\alpha}_2 - 2\dot{\sigma}_2]A\frac{\partial\mathcal{L}_M}{\partial A} + [\dot{\alpha}_2 + \dot{\sigma}_2]B\frac{\partial\mathcal{L}_M}{\partial B}. \quad (6.34)$$

With the variational equations of γ, A and B given by Eqs. (5.4), (5.5), and (5.6), the ϕ^0 equation becomes

$$\dot{\alpha}_1[3\gamma_1 - 2\gamma_2(A+2B) + \gamma_3 B(2A+B)] - 2(A-B)\left[\dot{\sigma}_1(\gamma_2 - \gamma_3 B) - \frac{\dot{\sigma}_2}{N_2}(\gamma_1 - \gamma_2 B)\right] - \frac{\dot{\alpha}_2}{N_2}[\gamma_1(A+2B) - 2\gamma_2 B(2A+B) + 3\gamma_3 AB^2] = 0. \quad (6.35)$$

In particular, in the isotropic fiducial limit $\sigma_2 = 0$, the above equation reduces to

$$(\gamma_1 - 2\gamma_2 B + \gamma_3 B^2)(H + 2\Sigma - H_f A) + 2[\gamma_1 - \gamma_2(A+B) + \gamma_3 AB](H - \Sigma - H_f B) = 0, \quad (6.36)$$

which is exactly Eq. (7) in Ref. [12]. In addition, the fixed point solutions found in Ref. [12] are also under the condition $\sigma_1 =$ constant. Therefore, the fixed point solutions they found in fact agree with our solutions in the isotropic limit.

Indeed, the condition that σ_1 and σ_2 are both constants enforces that the B field acts as an A field dynamically in this limit. Hence, our approach shown in this paper also reveals clearly the reason why the expanding solutions can only be found under the condition $\dot{\sigma}_1 = 0$. This is in fact the only solution that can be found when we also adopt the isotropic fiducial metric that is incompatible with the Bianchi type I physical metric. Note also that the situation is similar for the model studied in the second paper in Ref. [13]. The fixed point solutions are found in the limit that the physical metric becomes compatible with the fiducial metric. In addition, the Friedmann-Robertson-Walker physical metric was introduced with a flat Minkowski fiducial metric in Ref. [9]. The solution was then shown to be $\alpha_1 =$ constant. This also agrees with our result if we take the limit $\alpha_2 = \sigma_2 = 0$.

Note that our solution is derived from the constraint equations $\partial\mathcal{L}_M/\partial\gamma = \partial\mathcal{L}_M/\partial A = \partial\mathcal{L}_M/\partial B = 0$. If we replace the γ equation $\partial\mathcal{L}_M/\partial\gamma = 0$ by the ϕ^0 variational equation (6.34) as was done in Ref. [12], the constraint equations involved will then be $\partial \mathcal{L}_M/\partial A = \partial \mathcal{L}_M/\partial B = 0$ and the ϕ^0 equation given by Eq. (6.35) or Eq. (6.34). Hence, the ϕ^0 equation will reduce to

$$\left(\frac{d}{dt} + 3H\right)\left(\frac{\partial \mathcal{L}_M}{\partial \gamma}\right) = 0. \tag{6.37}$$

As a result, the ϕ^0 equation can be integrated to give

$$\left(\frac{\partial \mathcal{L}_M}{\partial \gamma}\right) = k_1 \exp[-3\alpha_1] \rightarrow 0, \tag{6.38}$$

with an integration constant k_1. Hence, the solution to the new ϕ^0 equation will approach the γ equation in time infinity for any expanding solutions with $\dot{\alpha}_1 > 0$. Therefore, we prove that the fixed point solution we found is in fact a set of global attractor solutions for all expanding solutions. This is exactly the reason why the phase flow diagram indicates that the fixed point solutions found in Ref. [12] are attractor solutions.

VII. CONCLUSIONS

In summary, we have studied the cosmological implications of a ghost-free nonlinear massive gravity theory with a cosmological constant. In particular, we showed that the massive terms will serve as an effective cosmological constant for a large class of metric spaces with a similar and compatible fiducial metric associated with the massive terms. A specific solution is also solved as an example for this model under the Bianchi type I space and a compatible Bianchi type I fiducial metric. We have also shown that the stability analysis indicates that this set of solutions tend to be stable. Nonetheless, we have also shown that the presence of a phantom field will, in general, make the anisotropically expanding solution unstable. In addition, a heuristic derivation of the field equations has also been shown in Sec. II. The analysis of the universal properties associated with a general reference metric shown in Sec. III could also be very useful in the study of the physics related to the massive gravity. We hope the material presented here will shed light on a deeper understanding of massive gravity.

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