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Note on construction of dual-trace factor in Yang-Mills theory

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Abstract: In this note we provide a new construction of BCJ dual-trace factor using the kinematic algebra proposed in arXiv:1105.2565 and arXiv:1212.6168. Different from the construction given in arXiv:1304.2978 based on the proposal of arXiv:1103.0312, the method used in this note exploits the adjoint representation of kinematic algebra and the use of inner product in dual space. The dual-trace factor defined in this way naturally satisfies cyclic symmetry condition but not KK-relation, just like the trace of U(N) Lie algebra satisfies cyclic symmetry condition, but not KK-relation. In other words the new construction naturally leads to formulation sharing more similarities with the color decomposition of Yang-Mills amplitude.

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1 Introduction

Since the discovery of color-kinematic (Bern-Carrasco-Johansson or BCJ) duality [1] of Yang-Mills theory in 2008, extensive studies have been carried out. At tree-level, BCJ duality states that Yang-Mills amplitudes at tree level can be written as [1]

$$\mathcal{A}_{\text{tot}} = \sum_i \frac{c_i n_i}{D_i}, \tag{1.1}$$

where the sum is taken over all possible Feynman-like diagrams constructed only from cubic vertices. For given cubic tree, D_i is the product of propagators, while c_i and n_i are the associated color and kinematic numerators. A crucial observation regarding to the BCJ-form (1.1) is that whenever the color numerators of three cubic trees satisfy Jacobi identity $c_i + c_j + c_k = 0$, so do the corresponding kinematic numerators

$$n_i + n_j + n_k = 0. \tag{1.2}$$

Furthermore when antisymmetry of the color algebra dictates that color numerators of two trees are related by $c_i \to -c_i$, so do their kinematic counterparts, $n_i \to -n_i$. Because of these properties, in BCJ-form c_i and n_i are treated on the same footing.

The color-kinematic duality provides relations between color-ordered Yang-Mills tree amplitudes such as Kleiss-Kuijf (KK) [2] and Bern-Carrasco-Johansson (BCJ) relations [1], both of which were proved in string theory [3–5] and in field theory [6–10]. BCJ relation is also crucial to the understanding of KLT relation [11] which at tree-level expresses gravity amplitudes in terms of products of color-ordered Yang-Mills amplitudes (See [12–15]). The duality is conjectured to hold at loop levels, where many nontrivial calculations have been carried out [16–25]. Among these applications of BCJ-forms, an important technical issue

DDM form: $\mathcal{A}_{\rm tot} = g^{n-2} \sum_{\sigma\in S_{n-2}} c_{1\vert\sigma(2,\cdots,n-1)\vert n} A(1,\sigma,n)$	Dual-DDM form: $\mathcal{A}_{\rm tot} = g^{n-2} \sum_{\sigma\in S_{n-2}} n_{1\vert\sigma(2,\cdots,n-1)\vert n} \widetilde{A}(1,\sigma,n)$
Trace form: $\mathcal{A}_{\rm tot} = g^{n-2} \sum_{\sigma\in S_{n-1}} {\rm Tr}(T^1 T^{\sigma_2}\cdots T^{\sigma_n}) A(1,\ldots,\sigma_n)$	Dual-trace form: $\mathcal{A}_{\rm tot} = g^{n-2} \sum_{\sigma\in S_{n-1}} \tau_{1\sigma_2\cdots\sigma_n} \widetilde{A}(1,\sigma_2,\ldots,\sigma_n)$

Table 1. Various formulations of tree amplitudes in Yang-Mills theory.

is how to construct the desired kinematic numerators n_i explicitly. There have been many works existing in the literature. The kinematic numerators can be constructed from the pure-spinor string method [26], at light-cone gauge from the algebra of area-preserving diffeomorphism [27, 28] or from the algebra of general diffeomorphism [29].

A very interesting implication of BCJ duality is that it suggests the interchangeability between $c_i \leftrightarrow n_i$. It is well known that the total tree-level Yang-Mills amplitude can be written either in the formulation discovered by Del Duca, Dixon, Maltoni [6] (DDM-form) or in the color decomposition formula (Trace-form) listed in the left column of table 1. Thus from color-kinematic duality, it is natural to guess that the total amplitude can as well be expressed in Dual-DDM and Dual-trace forms given in the right column of table 1. The problem faced is how to construct these dual formulations. For Dual-DDM form it is a little bit easier. The formulation was suggested in [30] and derived via the algebraic manipulation used in [31]. However, the derivation of Dual-trace form is not so straightforward.

To have a better idea for the construction of dual-trace factor τ, let us recall the relation between DDM-form and Trace-form. Note that the color dependence in DDM-form is introduced by structure constant

$$c_{1\vert\sigma(2,\ldots,n-1)\vert n} = F^{\sigma_1\sigma_2 x_1} F^{x_1\sigma_3 x_2}\ldots F^{x_{n-3}\sigma_{n-1}n}, \tag{1.3}$$

whereas in Trace-form this is carried by a trace ${\rm Tr}(T^{\sigma_1}\ldots T^{\sigma_n})$ of the generator of U(N) Lie algebra in fundamental representation. To establish the connection between these two forms, the following two properties are crucial:

$$\text{Orthogonality} : \quad {\rm Tr}(T^a T^b) = \delta^{ab} \Leftrightarrow F^{aij} = {\rm Tr}(T^a[T^i,T^j]), \tag{1.4}$$

$$\begin{aligned} \text{Completeness} : \quad & \sum_a {\rm Tr}(XT^a){\rm Tr}(T^a Y) = {\rm Tr}(XY) \\ & \sum_a {\rm Tr}(XT^a Y T^a) = {\rm Tr}(X){\rm Tr}(Y) \quad . \end{aligned} \tag{1.5}$$

Using orthogonality we can write $F^{aij} = {\rm Tr}([T^a,T^i]T^j) = {\rm Tr}(T^a[T^i,T^j])$, and then using the complete relation we can establish the following relation

$$\begin{aligned} c_{1\vert 2\ldots(n-1)\vert n} &= {\rm Tr}(T^{a_1}[T^{a_2},[\ldots,[T^{a_{n-1}},T^{a_n}]\ldots]]) \\ &= {\rm Tr}([[[T^{a_1},T^{a_2}],T^{a_3}],\ldots,T^{a_{n-1}}]T^{a_n}). \end{aligned} \tag{1.6}$$

To complete the derivation from the DDM form to the Trace-form we use the KK relation [2] between color-ordered amplitudes

$$A(1,\{\alpha\},n,\{\beta^T\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\alpha\bigcup\beta)} A(1,\sigma,n), \tag{1.7}$$

where the sum in (1.7) is over all permutations of the set $\{\alpha\}\bigcup\{\beta\}$ where relative ordering in both subsets α and β are kept.

Now two obstacles arise if we would like to replicate the above procedure to derive dual-color factor τ from BCJ numerator $n_{1|\sigma|n}$. Unlike U(N), we do not have orthogonality (1.4) and completeness relation (1.5) for kinematic algebra. An alternative solution was suggested in [32] which bypassed these obstacles. Suppose if we impose the condition

$$n_{1|\sigma_2\ldots\sigma_{n-1}|n} = \tau_{1[\sigma_2,[\ldots,[\sigma_{n-1},n]]]} \tag{1.8}$$

with the notation $\tau_{1[2,3]}$ understood as $\tau_{123} - \tau_{132}$, equation (1.8) provides the enough condition which allows us to derive the Dual-trace form from the Dual-DDM form, with the help of KK-relations between color-ordered amplitudes. However note that the number of independent BCJ numerator $n_{1|\sigma_2\ldots\sigma_{n-1}|n}$ is $(n-2)!$, while the number of independent dual-color factor τ is $n!$, even after imposing cyclic symmetry on τ, which reduces the number of independent τ to $(n-1)!$, there is still a mismatch. To fix the remaining degrees of freedom, KK-relation was imposed among τ as well in [32]. Based on this proposal, systematic construction of dual-color factor τ was carried out in [33]. A surprising feature of this construction is that the expressions of τ in terms of BCJ numerators n satisfy natural relabeling property, i.e., knowing the expression of just one τ, we can generate expressions of all others by relabeling the ordering of external particles.

Although the above proposal works, the imposition of KK-relations among dual color factors τ was not completely justified. This relation is apparently not satisfied by the trace of U(N) Lie algebra. The expression does not have nice local diagram picture and bear little similarity with the procedure we have recalled in previous paragraphs. In this note, we provide a new construction of the dual-trace factor which is explicitly in trace form. For this purpose, we need to find a way to generalize the orthogonality (1.4) and completeness (1.5) relations in the infinite dimensional kinematic algebra. With the generalization, our new construction can be carried out exactly the same way as was done in previous paragraphs.

The structure of this note is the following. In section 2, we provide a short review of the kinematic algebra and BCJ numerator. In section 3, we derive an explicit matrix representation of the kinematic algebra and its (singular) dual operator. In section 4 we present a new construction of dual-trace factor based on the matrix representation described in section 3 and show that the new construction does not respect KK-relation. We also provide a discussion on the relation between the construction presented in this note and the one in [33]. A brief summary is given in section 5.

2 A brief review of kinematic algebra

Before constructing dual-traces, let us review the kinematic algebra defined in [29]. A generator of the algebra of general diffeomorphism [29] is given by

$$T^{k,a} \equiv e^{ik\cdot x}\partial_a, \tag{2.1}$$

which satisfies the commutation relation

$$\begin{aligned}[T^{k_1,a}, T^{k_2,b}] &= (-i)(\delta_a{}^c k_{1b} - \delta_b{}^c k_{2a})\, e^{i(k_1+k_2)\cdot x}\partial_c \\ &= f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)}\, T^{(k_1+k_2,c)}.\end{aligned} \tag{2.2}$$

The structure constant $f^{(k_1,a),(k_2,b)}{}_{(k_1+k_2,c)}$ satisfies antisymmetry and Jacobi identity

$$f^{12}{}_3 = -f^{21}{}_3, \tag{2.3}$$

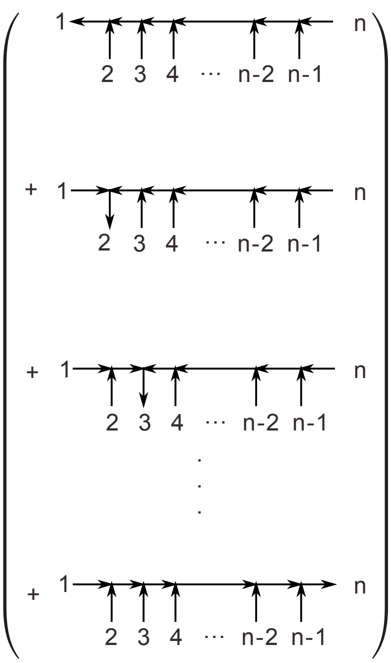
$$\begin{aligned} 0 &= f^{1_a,2_b}{}_{(1+2)^e} f^{(1+2)_e,3_c}{}_{(1+2+3)^d} + f^{2_b,3_c}{}_{(2+3)^e} f^{(2+3)_e,1_a}{}_{(1+2+3)^d} \\ &\quad + f^{3_c,1_a}{}_{(1+3)^e} f^{(1+3)_e,2_b}{}_{(1+2+3)^d}. \end{aligned} \tag{2.4}$$

For simplicity, we have used 1_a to denote upper index (k_1, a) and $(1+2)^e$ to denote the lower index $(k_1 + k_2, e)$.

Unlike $\mathrm{U}(N)$, which is semi-simple Lie algebra and has a lot of nice properties, the Lie algebra defined by $T^{k,a}$ is very nontrivial. Although we could not prove, it seems that we can not define the positive defined metric for this algebra. In other words, unlike the $\mathrm{U}(N)$ algebra, where one can use positive defined metric to raise or lower indices freely, here the upper and lower indices are distinct, and we shall keep track of their positions throughout discussions in this paper.

The BCJ numerator in dual-DDM form is given as

$$n_{1|2\ldots(n-1)|n} = \sum_{j=1}^{N} c_j \epsilon(q_j) \cdot$$



, (2.5)

where each term in the bracket is constructed using kinematic structure constants as coupling for each cubic vertex. The $\epsilon(q_j)$ is defined as $\prod_{t=1}^{n} \epsilon_t^{\mu_t}(q_{tj})$ where $\epsilon_t^{\mu_t}(q_{tj})$ is the polarization vector of the t-th external particle with gauge choice q_{tj}. The c_j's are coefficients solved by our averaging procedure given in [29]. The in-coming and out-going arrows represent upper and lower indices of structure constants at each vertex respectively.

3 The kinematic algebra and its dual space

In previous section we saw that the BCJ numerator is given by a summation over n distinct arrow configurations. For example, if the arrow carried by external leg n is the only out-going arrow, the expression is given by[1]

$$f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{n-2},n-1}{}_{n}, \tag{3.1}$$

To reproduce the above combination (3.1), we notice that

$$[T^1, T^2] = \sum_{\rho_2} f^{12}{}_{\rho_2} T^{\rho_2}, \qquad [[T^1, T^2], T^3] = \sum_{\rho_2,\rho_3} f^{12}{}_{\rho_2} f^{\rho_2 3}{}_{\rho_3} T^{\rho_3}, \dots \tag{3.2}$$

and similarly,

$$[[[T^1, T^2], T^3], \dots, T^{n-1}] = \sum_{\rho_i} f^{12}{}_{\rho_2} f^{\rho_2 3}{}_{\rho_3} \dots f^{\rho_{n-2}(n-1)}{}_{\rho_{n-1}} T^{\rho_{n-1}}, \tag{3.3}$$

which contains (3.1) up to a generator. To get rid of this generator, a natural way is to define inner product between the space generated by T and its dual space generated by M

$$\left\langle T^{k_1,a_1}, M_{k_2,a_2} \right\rangle = \delta^{a_1}_{a_2} \delta_{k_1,-k_2}, \tag{3.4}$$

With this definition, we have

$$\left\langle [[[T^1, T^2], T^3], \dots, T^{n-1}], M_n \right\rangle = f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{n-2},n-1}{}_{n} \tag{3.5}$$

where the momentum of leg n is $-(k_1 + k_2 + \dots + k_{n-1})$.

However, such inner product is not so satisfied, and yet we wish to obtain a suitable trace analogous to what we have in U(N) Lie algebra. To do so, we need to find a linear space $\mathcal{V}$ such that operators T^a, M_b both have representations in $\mathcal{V}$, and their inner product is equivalent to the trace in such space. As it turns out, it is hard to define such a dual algebra and find its proper representation in general space, because the Lie algebra defined by $T^{k,a}$ does not have nice properties as the Lie algebra of U(N) as mentioned in previous section. In fact, what we find is that there is a special space (the adjoint representation space of T) where we can define a singular matrix representation of M to achieve the goal (3.4).

To make the discussion simpler and easier to follow, we will consider a D-dimensional Minkowski space to be finite with volume V. Periodic boundary condition fixes momentum to be $k = \frac{2\pi}{L}(n_0, n_1, \dots, n_{D-1})$ with n_i an integer. With this discrete values we have

$$\langle p|x \rangle = \frac{e^{-ip\cdot x}}{\sqrt{V}}, \qquad \langle x|p \rangle = \frac{e^{ip\cdot x}}{\sqrt{V}} \tag{3.6}$$

thus

$$\begin{aligned} \delta_{k,k'} = \left\langle k|k' \right\rangle &= \int d^D x \left\langle k|x \right\rangle \left\langle x|k' \right\rangle \\ &= \int_V \frac{d^D x}{V} e^{-i(k-k')\cdot x} = \delta_{n_0,n_0'} \delta_{n_1,n_1'} \dots \delta_{n_{D-1},n_{D-1}'} \end{aligned} \tag{3.7}$$

[1]For simplicity we write the indices k_i, a_i as i.

and

$$\delta^D(x-y) = \langle x|y\rangle = \sum_k \langle x|k\rangle \langle k|y\rangle = \frac{1}{V}\sum_{n_i=-\infty}^{\infty} e^{ik\cdot(x-y)}. \tag{3.8}$$

Going back to the continuous limit is very easy and we just need to do following replacement for all formula below: $V\delta^D_{k,k'} \to \delta^D(k-k')$, and $\frac{1}{V}\sum_k \to \int d^Dk$ as the volume V goes to infinity.

With the above set-up, let us now define the representation space, where the label of a vector $|k,a\rangle$ is consisting of two parts: one carries the momentum information given by D arbitrary integers and another one, the direction a takes value $a = 0,1\ldots,D-1$. This space is, in fact, the adjoint representation space of kinematic algebra when momentum is discretized. The metric of adjoint space is defined as

$$\langle k_1,a_1|k_2,a_2\rangle = \delta_{k_1,k_2}\delta_{a_1,a_2} \tag{3.9}$$

In other words, the identity operator is given by

$$\widehat{I} = \sum_{k,a} |k,a\rangle \langle k,a| \tag{3.10}$$

where the $|k,a\rangle$ is column vector and $\langle k,a|$ is row vector. In this space, the matrix representation of operator $\widehat{T}^{(k_2,a_2)}$ is given by

$$\Big\langle k_1,a_1|\widehat{T}^{(k_2,a_2)}|k_3,a_3\Big\rangle \equiv f^{(k_1,a_1)(k_2,a_2)}{}_{(k_3,a_3)} = -i(\delta^{a_3}_{a_1}(k_1)_{a_2} - \delta^{a_3}_{a_2}(k_2)_{a_1})\delta^{k_3}_{k_1+k_2} \tag{3.11}$$

or in matrix form,

$$(\widehat{T}^{(k_2,a_2)})^{(k_1,a_1)}{}_{(k_3,a_3)} = f^{(k_1,a_1)(k_2,a_2)}{}_{(k_3,a_3)} \tag{3.12}$$

It is easy to show that the definition (3.11) indeed furnishes a representation by noticing that the action of $\widehat{T}$ over vectors is equal to

$$\widehat{T}^{(k_2,a_2)}|k_3,a_3\rangle = f^{(k_1,a_1)(k_2,a_2)}{}_{(k_3,a_3)}|k_1,a_1\rangle = (\widehat{T}^{(k_2,a_2)})^{(k_1,a_1)}{}_{(k_3,a_3)}|k_1,a_1\rangle \tag{3.13}$$

and

$$\langle k_1,a_1|\widehat{T}^{(k_2,a_2)} = \langle k_3,a_3| f^{(k_1,a_1)(k_2,a_2)}{}_{(k_3,a_3)} = \langle k_3,a_3|(\widehat{T}^{(k_2,a_2)})^{(k_1,a_1)}{}_{(k_3,a_3)}. \tag{3.14}$$

Thus we have (for simplicity, we have shorted the notation)

$$\begin{aligned}
[T^1,T^2]|p\rangle &= \sum_q T^1 f^{q2}{}_p|q\rangle - \sum_q T^2 f^{q1}{}_p|q\rangle = \sum_q\sum_t f^{q2}{}_p f^{t1}{}_q|t\rangle - \sum_q\sum_t f^{q1}{}_p f^{t2}{}_q|t\rangle \\
&= \sum_t\left(\sum_q f^{t1}{}_q f^{q2}{}_p + f^{2t}{}_q f^{q1}{}_p\right)|t\rangle = \sum_t\sum_q -f^{12}{}_q f^{qt}{}_p|t\rangle \\
&= \sum_q f^{12}{}_q\left(\sum_t f^{tq}{}_p|t\rangle\right) = \sum_q f^{12}{}_q T^q|p\rangle
\end{aligned}$$

and similarly

$$\langle p|\,[T^1,T^2] = \sum_q f^{p1}{}_q \langle q|\,T^2 - \sum_q f^{p2}{}_q \langle q|\,T^1 = \sum_t \langle t|\sum_q (f^{p1}{}_q f^{q2}{}_t - f^{p2}{}_q f^{q1}{}_t)$$
$$= \sum_t \langle t|\sum_q (-) f^{12}{}_q f^{qp}{}_t = \sum_q f^{12}{}_q \langle p|\,T^q$$

We claim that a matrix representation of M that serves our purpose is given by

$$\left\langle k_3,a_3|\widehat{M}_{p,b}|k_1,a_1\right\rangle = -i\delta_{k_1,0}\delta_{k_3,-p}\delta^b_{a_3}\frac{u_{a_1}}{k_3\cdot u} = i\delta_{k_1,0}\delta_{k_3,-p}\delta^b_{a_3}\frac{u_{a_1}}{p\cdot u} \tag{3.15}$$

where u is an arbitrary momentum. It is worth noticing that since the factor $\delta_{k_1,0}$, this matrix representation is degenerate. In other words, the operator $\widehat{M}$ does not give a well defined algebra. This is again related to the not-so-nice algebra defined by $\widehat{T}^{k,a}$. Although singular, the desired trace property (3.4) is satisfied. To check that, in adjoint representation we have

$$\sum_{k_1,a_1;k_3,a_3}\left\langle k_1,a_1|\widehat{T}^{(k_2,a_2)}|k_3,a_3\right\rangle\left\langle k_3,a_3|\widehat{M}_{p,b}|k_1,a_1\right\rangle$$
$$= \sum_{k_1,a_1;k_3,a_3} i(\delta^{a_3}_{a_1}(k_1)_{a_2} - \delta^{a_3}_{a_2}(k_2)_{a_1})\delta_{k_1+k_2,k_3} i\delta_{k_1,0}\delta_{k_3,-p}\delta^b_{a_3}\frac{u_{a_1}}{k_3\cdot u}$$
$$= \sum_{k_3,a_3}\delta^{a_3}_{a_2}\delta_{k_2,k_3}\delta_{k_3,p}\delta^b_{a_3}\frac{k_2\cdot u}{k_3\cdot u} = \delta_{k_2,-p}\delta^b_{a_2}, \tag{3.16}$$

which is what we want!

Before ending this section, there is one thing worth to discuss. In the definition of (3.15), there is a free parameter, i.e., the vector u. Different choices will give different expressions of τ's. This is much like the definition of BCJ numerator n_j in (1.1), which is unique up to some "gauge". For n_i, some gauge freedom is related to the definition of polarization vectors of external particles as seen in (2.5).[2] For τ, we can not see the direct relation between the freedom of u and the freedom of the choice of polarization vectors (see, for example, eq. (4.12)). More concretely, let us define the difference of two choices of u to be $\Delta\tau$, the consistent condition is equivalent to

$$0 = \sum_{\sigma\in S_{n-1}}\Delta\tau_{1\sigma_2\ldots\sigma_n}\widetilde{A}(1,\sigma_2,\ldots,\sigma_n) \tag{3.17}$$

Equation (3.17) has one trivial solution, i.e., if all $\tau_{1\sigma_2\ldots\sigma_n}$'s are same constant, by U(1) decoupling identity, equation (3.17) will be true. In other words, consistent condition (3.17) seems to indicate that at least partially the gauge freedom u is related to relations among color-ordered partial amplitudes $\widetilde{A}$. More discussions will be given in next section.

4 The algebraic construction of dual-trace factor

In this section we use the T and M in adjoint space to derive the dual-color decomposition formula. We show that the KK relation is not satisfied in general for this construction, and we comment on the connection between this construction and the construction given previously in [33].

[2] It is very likely that some gauges of n_i is not related to the choice of polarization vectors.

4.1 Construction of dual-trace factor

Having derived matrix representations of operators T and M, we can now construct the dual-trace factor τ. For a BCJ numerator with n legs, let us first consider the cubic chain graph at the bottom of (2.5) with out-going arrow assigned to leg n. In this case we simply have

$$\mathcal{I}_n = \mathrm{Tr}_{\mathcal{V}}([[[T^1, T^2], T^3], \cdots, T^{n-1}]M_n) = f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{n-2},n-1}{}_n \tag{4.1}$$

For cases where the out-going arrow is assigned to leg i, with $3 \le i \le n-2$, the expression is given by

$$\mathcal{I}_i = \left(f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{i-2},i-1}{}_{\rho_{i-1}}\right) f^{\rho_i,\rho_{i-1}}{}_i \left(f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \dots f^{i+1,\rho_{i+1}}{}_{\rho_i}\right) \tag{4.2}$$

(Note that the ordering is different from that of (4.1).) To write it into trace form, noticing that

$$\begin{aligned}
\mathbb{A}_i &\equiv \sum_\rho f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{i-2},i-1}{}_{\rho_{i-1}} T^{\rho_{i-1}} = [[[T^1, T^2], T^3], \dots, T^{i-1}] \\
\mathbb{B}_i &\equiv \sum_\rho f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \dots f^{i+1,\rho_{i+1}}{}_{\rho_i} T^{\rho_i} \\
&= (-)^{n-i-1} \sum_\rho f^{n,n-1}{}_{\rho_{n-2}} f^{\rho_{n-2},n-2}{}_{\rho_{n-3}} \dots f^{\rho_{i+1},i+1}{}_{\rho_i} T^{\rho_i} \\
&= (-)^{n-i-1} [[[T^n, T^{n-1}], T^{n-2}] \dots, T^{i+1}],
\end{aligned}$$

so we have

$$\begin{aligned}
[\mathbb{B}_i, \mathbb{A}_i] = \sum_\sigma \left(f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{i-2},i-1}{}_{\rho_{i-1}}\right) f^{\rho_i,\rho_{i-1}}{}_\sigma \cdot \\
\cdot T^\sigma \left(f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \dots f^{i+1,\rho_{i+1}}{}_{\rho_i}\right).
\end{aligned} \tag{4.3}$$

Therefore a cubic chain $\mathcal{I}_i$ with the out-going arrow assigned to leg i contributes as

$$\begin{aligned}
\mathcal{I}_i &= \mathrm{Tr}_{\mathcal{V}}([\mathbb{B}_i, \mathbb{A}_i]M_i) = \mathrm{Tr}_{\mathcal{V}}([\mathbb{A}_i, M_i]\mathbb{B}_i) \\
&= \mathrm{Tr}_{\mathcal{V}}([(-)^{n-i-1}[[[T^n, T^{n-1}], T^{n-2}] \dots, T^{i+1}], [[[T^1, T^2], T^3], \dots, T^{i-1}]]M_i).
\end{aligned}$$

Note that using the identity

$$\mathrm{Tr}_{\mathcal{V}}([A,B]C) = \mathrm{Tr}_{\mathcal{V}}(A[B,C]) = -\mathrm{Tr}_{\mathcal{V}}([B,A]C) = -\mathrm{Tr}_{\mathcal{V}}(B[A,C]) = \mathrm{Tr}_{\mathcal{V}}(B[C,A]) \tag{4.4}$$

repeatedly, we can transform $\mathcal{I}_i$ into a standard form

$$\mathcal{I}_i = \mathrm{Tr}_{\mathcal{V}}([[[[[[T^1, T^2], T^3], \dots, T^{i-1}], M_i], T^{i+1}], \dots, T^{n-1}]T_n). \tag{4.5}$$

For the case where $(n-1)$-th leg carries the out-going arrow, we have

$$\begin{aligned}
\mathbb{A}_{n-1} &= [[[T^1, T^2], T^3] \dots, T^{n-2}] = \sum_\rho f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{n-3},n-2}{}_{\rho_{n-2}} T^{\rho_{n-2}}, \\
[T^n, T^{\rho_{n-2}}] &= f^{n,\rho_{n-2}}{}_\sigma T^\sigma
\end{aligned}$$

thus

$$\begin{aligned}\mathcal{I}_{n-1} &= f^{1,2}{}_{\rho_2} f^{\rho_2,3}{}_{\rho_3} \dots f^{\rho_{n-3},n-2}{}_{\rho_{n-2}} f^{n,\rho_{n-2}}{}_{n-1} = \mathrm{Tr}_{\mathcal{V}}([T^n, \mathbb{A}_{n-1}] M_{n-1}) \\ &= \mathrm{Tr}_{\mathcal{V}}([[[[T^1,T^2],T^3]\dots,T^{n-2}],M^{n-1}]T^n) \end{aligned} \tag{4.6}$$

In the case where leg 1 carries the out-going arrow we have

$$\begin{aligned}\mathcal{I}_1 &= f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \cdots f^{2,\rho_2}{}_1 = \mathrm{Tr}_{\mathcal{V}}((-)^{n-2}[[[T^n,T^{n-1}],T^{n-2}]\dots,T^2]M_1) \\ &= \mathrm{Tr}_{\mathcal{V}}([[[[M_1,T^2],T^3],\dots,T^{n-2}],T^{n-1}]T^n), \end{aligned} \tag{4.7}$$

and similarly, when leg 2 carries the out-going arrow,

$$\begin{aligned}\mathbb{B}_2 &= f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \cdots f^{3,\rho_3}{}_{\rho_2} T^{\rho_2} = (-)^{n-3}[[[T^n,T^{n-1}],T^{n-2}]\dots,T^3] \\ [T^{\rho_2},T^1] &= f^{\rho_2,1}{}_{\sigma} T^{\sigma}\end{aligned}$$

thus

$$\begin{aligned}\mathcal{I}_2 &= f^{n-1,n}{}_{\rho_{n-2}} f^{n-2,\rho_{n-2}}{}_{\rho_{n-3}} \cdots f^{3,\rho_3}{}_{\rho_2} f^{\rho_2,1}{}_2 \\ &= \mathrm{Tr}_{\mathcal{V}}((-)^{n-3}[[[[T^n,T^{n-1}],T^{n-2}]\dots,T^3],T^1]M_2) \\ &= \mathrm{Tr}_{\mathcal{V}}([[[[T^1,M_2],T^3],\dots,T^{n-2}],T^{n-1}]T^n) \end{aligned} \tag{4.8}$$

With above calculations for BCJ numerator, the Dual-DDM form can be written as[3]

$$\mathcal{A}_{\rm tot} = \sum_{\sigma\in S_{n-2}} \sum_{j=1}^{N} c_j \epsilon(q_j) \cdot \left\{ \sum_{i=1}^{n} \mathcal{I}_i(1,\sigma,n) \right\} \widetilde{A}(1,\sigma_2,\dots,\sigma_{n-1},n). \tag{4.9}$$

where

$$\begin{aligned}\mathcal{I}_i(1,\sigma,n) &\equiv \mathrm{Tr}_{\mathcal{V}}([[[\mathcal{T}^1,\mathcal{T}^{\sigma_2}],\mathcal{T}^{\sigma_3}],\dots,\mathcal{T}^{\sigma_{n-1}}]\mathcal{T}^n) \\ &= \sum_{\sigma\in OP(\{\alpha\},\{\beta\})} \mathrm{Tr}_{\mathcal{V}}(-1)^{n-2-r}(\mathcal{T}_i^1\mathcal{T}_i^{\alpha_1}\cdots\mathcal{T}_i^{\alpha_r}\mathcal{T}_i^n\mathcal{T}_i^{\beta_{n-r-2}}\cdots\mathcal{T}_i^{\beta_1}), \end{aligned} \tag{4.10}$$

where we have summed over all possible splittings of σ into two subsets $\{\alpha\}$ and $\{\beta\}$ such that σ can be reconstructed by the union of $\{\alpha\}$ and $\{\beta\}$ with arbitrary relative ordering between α and β. In (4.10), we have expanded the first line into sum of traces with following convention: $\mathcal{T}_i^j := T^j$ (for $j \neq i$) and $\mathcal{T}_i^j := M_j$ (for $j = i$). Now comparing expression (4.9) with DDM-form in table 1 and expression (4.10) with (1.6), we see a striking similarity. Since the color-ordering partial amplitude $\widetilde{A}$ satisfies the KK-relation [29], the Dual-DDM form will automatically gives the Dual-trace form

$$\mathcal{A}_{\rm tot} = \sum_{\sigma\in S_{n-1}} \tau_{1,\sigma_2\cdots\sigma_n} \widetilde{A}(1,\sigma_2,\cdots,\sigma_n), \tag{4.11}$$

[3]Here we have neglected the coupling constants g^{n-2} for convenience.

with the dual-trace factors τ defined by

$$\begin{aligned}\tau_{1\sigma_2\ldots\sigma_n} &\equiv \left(\sum_{j=1}^{N} c_j\epsilon(q_j)\right)\cdot\left[\sum_{i=1}^{n}\mathrm{Tr}_{\mathcal{V}}(\mathcal{T}_i^1\mathcal{T}_i^{\sigma_2}\cdots\mathcal{T}_i^{\sigma_{n-1}}\mathcal{T}_i^{\sigma_n})\right]\\ &= \left(\sum_{j=1}^{N} c_j\epsilon(q_j)\right)\cdot\left[\sum_{i=1}^{n}\mathrm{Tr}_{\mathcal{V}}(T^1T^{\sigma_2}\cdots M_i\cdots T^{\sigma_{n-1}}T^{\sigma_n})\right].\end{aligned}\tag{4.12}$$

The dual-trace factors naturally satisfy the cyclic symmetry

$$\tau_{12\ldots n} = \tau_{n1\ldots(n-1)}\tag{4.13}$$

as well as

$$n_{1|2\ldots(n-1)|n} = \tau_{1[2,[\ldots,[n-1,n]\ldots]]}\tag{4.14}$$

by our construction.

Having the above definition (4.12) for dual-trace factor τ, we give the explicit expression using the matrix representation. Taking

$$\mathrm{Tr}_{\mathcal{V}}(T^1T^2\ldots T^{i-1}M_iT^{i+1}\ldots T^n)\tag{4.15}$$

as an example, after inserting completeness relation we have

$$\begin{aligned}&\sum_{\rho}\left\langle\rho_0,b_0|T^{k_1,a_1}|\rho_1,b_1\right\rangle\left\langle\rho_1,b_1|T^{k_2,a_2}|\rho_2,b_2\right\rangle\ldots\left\langle\rho_{i-2},b_{i-2}|T^{k_{i-1},a_{i-1}}|\rho_{i-1},b_{i-1}\right\rangle\\ &\qquad\langle\rho_{i-1},b_{i-1}|M_{k_i,a_i}|\rho_i,b_i\rangle\left\langle\rho_i,b_i|T^{k_{i+1},a_{i+1}}|\rho_{i+1},b_{i+1}\right\rangle\ldots\left\langle\rho_{n-1},b_{n-1}|T^{k_n,a_n}|\rho_0,b_0\right\rangle\\ &=\sum_{\rho}f^{(\rho_0,b_0)(k_1,a_1)}{}_{(\rho_1,b_1)}f^{(\rho_1,b_1)(k_2,a_2)}{}_{(\rho_2,b_2)}\cdots f^{(\rho_{i-2},b_{i-2})(k_{i-1},a_{i-1})}{}_{(\rho_{i-1},b_{i-1})}\\ &\qquad(-i)\delta^0_{\rho_i}\delta^{k_i}_{\rho_{i-1}}\delta^{a_i}_{b_{i-1}}\frac{u_{b_i}}{k_i\cdot u}f^{(\rho_i,b_i)(k_{i+1},a_{i+1})}{}_{(\rho_{i+1},b_{i+1})}\cdots f^{(\rho_{n-1},b_{n-1})(k_n,a_n)}{}_{(\rho_0,b_0)}\end{aligned}\tag{4.16}$$

where we have inserted the expression (3.15). Carrying out the sums over $\rho_{i-1},\rho_i,b_{i-1}$ we have

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{V}}(T^1T^2\ldots T^{i-1}M_iT^{i+1}\ldots T^n) = \mathrm{Tr}_{\mathcal{V}}(M_iT^{i+1}\ldots T^nT^1T^2\ldots T^{i-1})\\ &=\sum_{b_i}\frac{-iu_{b_i}}{k_i\cdot u}f^{(0,b_i)(k_{i+1},a_{i+1})}{}_{(\rho_{i+1},b_{i+1})}\cdots f^{(\rho_{n-1},b_{n-1})(k_n,a_n)}{}_{(\rho_0,b_0)}f^{(\rho_0,b_0)(k_1,a_1)}{}_{(\rho_1,b_1)}\\ &\qquad f^{(\rho_1,b_1)(k_2,a_2)}{}_{(\rho_2,b_2)}\cdots f^{(\rho_{i-2},b_{i-2})(k_{i-1},a_{i-1})}{}_{(k_i,a_i)}\end{aligned}\tag{4.17}$$

This expression has the following interpretation. It is the expression of an $(n+1)$-leg chain $0(i+1)(i+2)\ldots(n)(1)\ldots(i-1)(i)$ with leg i carrying the out-going arrows. The leg 0 has zero momentum and its polarization vector is given by $\frac{-iu_{b_i}}{k_i\cdot u}$ (so it is not physical polarization), thus in some sense, the momentum u_i has the meaning of gauge choice for the auxiliary particle 0 (but not for real external particles).

Moreover, since the choice of u_{b_i} introduced by this construction does not change the contraction of structure constants of the form (4.2), thus all the BCJ numerators n_i in DDM form will not be affected by the choice of u. Thus this choice of gauge is different from the generalized gauge transformations [1] which change the BCJ numerators but preserve the Jacobi-like identity.

4.2 KK-relation

Since the construction of the dual-trace factor in our previous work [33] satisfies KK relations, it is natural to wonder whether the construction (4.12) also satisfies KK relation. As we will point out, the construction (4.12) does not satisfy KK relation in general.

Let us start with the simplest example, i.e., the three-point case where the KK-relation is reduced to $\tau_{123} = -\tau_{132}$. By our construction, we have

$$\tau_{123} = \epsilon_1\epsilon_2\epsilon_3(\mathrm{Tr}_{\mathcal{V}}(T^1T^2M_3) + \mathrm{Tr}_{\mathcal{V}}(T^1M_2T^3) + \mathrm{Tr}_{\mathcal{V}}(M_1T^2T^3)). \tag{4.18}$$

and

$$\tau_{132} = \epsilon_1\epsilon_2\epsilon_3(\mathrm{Tr}_{\mathcal{V}}(T^1T^3M_2) + \mathrm{Tr}_{\mathcal{V}}(T^1M_3T^2) + \mathrm{Tr}_{\mathcal{V}}(M_1T^3T^2)). \tag{4.19}$$

With a little bit of algebra, we found

$$\begin{aligned}\mathrm{Tr}_{\mathcal{V}}(T^1T^2M_3) &= \sum_{a_i,a_j,a_k,k_i,k_j,k_k} (-i)\left(\delta^{a_j}_{a_i}(k_i)_{a_1} - \delta^{a_j}_{a_1}(k_1)_{a_i}\right)\delta^{k_j}_{k_i+k_1}(-i)\cdot \\ &\qquad\cdot\left(\delta^{a_k}_{a_j}(k_j)_{a_2} - \delta^{a_k}_{a_2}(k_2)_{a_j}\right)\delta^{k_k}_{k_j+k_2}\, i\delta_{k_i,0}\delta_{k_k,-k_3}\delta^{a_3}_{a_k}\frac{u_{a_i}}{k_3\cdot u} \\ &= i\left(\delta^{a_3}_{a_1}(k_1)_{a_2} - \delta^{a_3}_{a_2}(k_2)_{a_1}\right)\frac{k_1\cdot u}{k_3\cdot u},\end{aligned} \tag{4.20}$$

and similarly for other traces. Thus

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{V}}(T^1T^2M_3) + \mathrm{Tr}_{\mathcal{V}}(T^1M_2T^3) + \mathrm{Tr}_{\mathcal{V}}(M_1T^2T^3) \\ &\quad= i\left(\delta^{a_3}_{a_1}(k_1)_{a_2} - \delta^{a_3}_{a_2}(k_2)_{a_1}\right)\frac{k_1\cdot u}{k_3\cdot u} + i\left(\delta^{a_2}_{a_3}(k_3)_{a_1} - \delta^{a_2}_{a_1}(k_1)_{a_3}\right)\frac{k_3\cdot u}{k_2\cdot u} \\ &\qquad+ i\left(\delta^{a_1}_{a_2}(k_2)_{a_3} - \delta^{a_1}_{a_3}(k_3)_{a_2}\right)\frac{k_2\cdot u}{k_1\cdot u}.\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{V}}(T^1T^3M_2) + \mathrm{Tr}_{\mathcal{V}}(T^1M_3T^2) + \mathrm{Tr}_{\mathcal{V}}(M_1T^3T^2) \\ &\quad= i\left(\delta^{a_2}_{a_1}(k_1)_{a_3} - \delta^{a_2}_{a_3}(k_3)_{a_1}\right)\frac{k_1\cdot u}{k_2\cdot u} + i\left(\delta^{a_3}_{a_2}(k_2)_{a_1} - \delta^{a_3}_{a_1}(k_1)_{a_2}\right)\frac{k_2\cdot u}{k_3\cdot u} \\ &\qquad+ i\left(\delta^{a_1}_{a_3}(k_3)_{a_2} - \delta^{a_1}_{a_2}(k_2)_{a_3}\right)\frac{k_3\cdot u}{k_1\cdot u}.\end{aligned}$$

If we leave the polarization vectors apart, $\tau_{123} = -\tau_{132}$ implies

$$\frac{k_1\cdot u}{k_2\cdot u} = -\frac{k_3\cdot u}{k_2\cdot u},\quad \frac{k_2\cdot u}{k_3\cdot u} = -\frac{k_1\cdot u}{k_3\cdot u},\quad \frac{k_3\cdot u}{k_1\cdot u} = -\frac{k_2\cdot u}{k_1\cdot u}. \tag{4.21}$$

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which does not have solution when momentum conservation and on-shell condition of external legs are taken into account. Thus we see that even in the simplest case, KK relation does not hold.

In general, KK relation in Yang-Mills theory is caused by the antisymmetry of structure constants. The construction in this paper contain both T and M in the adjoint space. Though the adjoint representation of T have antisymmetry, the M does not. Thus it is not so surprising that KK-relation in general does not hold. However, we must emphasize that in above discussion, we concentrate only on the algebraic structure and have not paid attention to the helicity configuration and the choice of polarization vectors of external legs. It is possible (but unlikely) one may achieve KK-relation by particular choice of gauge. Since this situation is more complicated we will not discuss it further in this note.

4.3 The relation between the two different constructions

In [33], we discussed another construction of dual-trace factor τ also expressed in terms of structure constants, which not only satisfies cyclic symmetry (4.13) and (4.14) but also KK-relation. We note that this solution is unique when both KK-relation and (4.14) are imposed, therefore the construction presented in this note is different from that given in [33]. In this section we would like to discuss the connection between these two constructions.

Let us start with three-point case. The dual-trace factor $\widetilde{\tau}$ defined in [33] gives[4]

$$\widetilde{\tau}_{123} = \frac{1}{2} n_{123}. \tag{4.22}$$

Using the result in this note, we can write BCJ numerator in trace form, thus we have

$$\begin{aligned}\widetilde{\tau}_{123} &= \frac{1}{2} n_{123} \\ &= \frac{1}{2} \sum_i c_i \epsilon_i \cdot \left(\mathrm{Tr}_{\mathcal{V}}([T^1, T^2] M_3) + \mathrm{Tr}_{\mathcal{V}}([T^1, M^2] T^3) + \mathrm{Tr}_{\mathcal{V}}([M_1, T^2] M^3) \right) \\ &= \frac{1}{2} (\tau_{123} - \tau_{132}). \end{aligned} \tag{4.23}$$

Putting this result back into the dual-color decomposition formula, we have

$$\begin{aligned}\mathcal{A}_{\mathrm{tot}}(1,2,3) &= \widetilde{\tau}_{123} \widetilde{A}(1,2,3) + \widetilde{\tau}_{1,3,2} \widetilde{A}(1,3,2) \\ &= \frac{1}{2}(\tau_{123} - \tau_{321}) \widetilde{A}(1,2,3) + \frac{1}{2}(\tau_{132} - \tau_{231}) \widetilde{A}(1,3,2) \\ &= \tau_{123} \widetilde{A}(1,2,3) + \tau_{132} \widetilde{A}(1,3,2), \end{aligned} \tag{4.24}$$

where we have used the KK-relation for color-scalar amplitudes [31] $\widetilde{A}(1,2,3) = -\widetilde{A}(1,3,2)$. Thus we can see that the two different constructions yield the same dual-trace forms up to terms cancelled by KK-relation between scalar amplitudes.

[4] In this section, we shorten our notation of the numerator factors of dual-DDM form as $n_{1|2,\ldots,n-1|n} \equiv n_{12\ldots(n-1)n}$. We just need to remember the first element and the last element are fixed in a given representation.

For the four-point case, the dual-trace factor given in [33] is

$$\widetilde{\tau}_{1234} = \frac{1}{3} n_{1234} - \frac{1}{6} n_{1324}, \tag{4.25}$$

Substituting n_{1234} and n_{1324} by traces τ defined in this note, we have

$$\widetilde{\tau}_{1234} = \frac{1}{3}(\tau_{1234} - \tau_{3124} - \tau_{2134} + \tau_{3214}) - \frac{1}{6}(\tau_{1324} - \tau_{2134} - \tau_{3124} + \tau_{2314}). \tag{4.26}$$

The total amplitude using $\widetilde{\tau}$ is

$$\begin{aligned}
&\mathcal{A}_{\rm tot}(1,2,3,4) = \\
&= \left[\frac{1}{3}(\tau_{1234} - \tau_{3124} - \tau_{2134} + \tau_{3214}) - \frac{1}{6}(\tau_{1324} - \tau_{2134} - \tau_{3124} + \tau_{2314})\right] \widetilde{A}(1,2,3,4) \\
&\quad + \left[\frac{1}{3}(\tau_{1243} - \tau_{4123} - \tau_{2143} + \tau_{4213}) - \frac{1}{6}(\tau_{1423} - \tau_{2143} - \tau_{4123} + \tau_{2413})\right] \widetilde{A}(1,2,4,3) \\
&\quad + \left[\frac{1}{3}(\tau_{1324} - \tau_{2134} - \tau_{3124} + \tau_{2314}) - \frac{1}{6}(\tau_{1234} - \tau_{3124} - \tau_{2134} + \tau_{3214})\right] \widetilde{A}(1,3,2,4) \\
&\quad + \left[\frac{1}{3}(\tau_{1342} - \tau_{4132} - \tau_{3142} + \tau_{4312}) - \frac{1}{6}(\tau_{1432} - \tau_{3142} - \tau_{4132} + \tau_{3412})\right] \widetilde{A}(1,3,4,2) \\
&\quad + \left[\frac{1}{3}(\tau_{1423} - \tau_{2143} - \tau_{4123} + \tau_{2413}) - \frac{1}{6}(\tau_{1243} - \tau_{4123} - \tau_{2143} + \tau_{4213})\right] \widetilde{A}(1,4,2,3) \\
&\quad + \left[\frac{1}{3}(\tau_{1432} - \tau_{3142} - \tau_{4132} + \tau_{3412}) - \frac{1}{6}(\tau_{1342} - \tau_{4132} - \tau_{3142} + \tau_{4312})\right] \widetilde{A}(1,4,3,2).
\end{aligned}$$

Using cyclic symmetry, we can collect terms with the same τ, for example, terms that carry τ_{1234} collectively yield

$$\begin{aligned}
\tau_{1234}\Bigg[&\widetilde{A}(1,2,3,4) - \frac{1}{6}\widetilde{A}(1,2,4,3) - \frac{1}{6}\widetilde{A}(1,3,2,4) \\
&- \frac{1}{6}\widetilde{A}(1,3,4,2) - \frac{1}{6}\widetilde{A}(1,4,2,3) + \frac{1}{3}\widetilde{A}(1,4,3,2)\Bigg].
\end{aligned} \tag{4.27}$$

Using the following KK-relations

$$\begin{aligned}
\widetilde{A}(1,4,2,3) + \widetilde{A}(1,2,4,3) + \widetilde{A}(1,2,3,4) &= 0, \\
\widetilde{A}(1,2,3,4) + \widetilde{A}(1,3,2,4) + \widetilde{A}(1,3,4,2) &= 0, \\
\widetilde{A}(1,4,3,2) &= \widetilde{A}(1,2,3,4),
\end{aligned} \tag{4.28}$$

the contributions collapse to $\tau_{1234}\widetilde{A}(1,2,3,4)$. Terms with other orderings can be obtained similarly and we reach

$$\mathcal{A}_{\rm tot}(1,2,3,4) = \sum_{\sigma\in S_3} \widetilde{\tau}_{1,\sigma}\widetilde{A}(1,\sigma) = \sum_{\sigma\in S_3} \tau_{1,\sigma}\widetilde{A}(1,\sigma). \tag{4.29}$$

From the simple examples above, we see how to establish the relation between the $\widetilde{\tau}$ in [33] and the τ in this note. Although the algorithm is clear, it is not so easy to write down such relation explicitly for general n. We will leave the solution for future investigation.

5 Conclusion

In this note, we have given another construction of dual-trace factor τ using the adjoint representation of generators of kinematic algebra and their dual operators. This new construction bears strong similarity to the construction of trace factor from $\mathrm{U}(N)$ Lie group algebra and respects cyclic symmetry naturally. However, it does not satisfy KK-relation in general. In addition, we have shown that the new construction and the old one given in [33] can be connected nontrivially in the lower-point examples.

Having the dual-trace form for the tree-level amplitude of Yang-Mills theory, it is natural to ask the generalization to loop levels. For one-loop amplitude, schematically, the color decomposition for $\mathrm{U}(N_c)$ gauge theory can be written as [34]

$$\begin{aligned}\mathcal{A}_n^{\text{full}}\left(\{k_i,\lambda_i,a_i\}\right) &=\\ &= g^n \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{\sigma\in S_n/S_{n;m}} \mathrm{Gr}_{n-m,m}\left(\sigma\right)\, A_{n-m,m}(\sigma_1,\sigma_2,\ldots,\sigma_{n-m};\sigma_{n-m+1},\ldots,\sigma_n)\,,\end{aligned} \tag{5.1}$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x, S_n is the set of all permutations of n objects, and $S_{n;m}$ is the subset leaving $\mathrm{Gr}_{n-m,m}$ invariant. The color structure part $\mathrm{Gr}_{n-m,m}\left(\sigma\right)$ has two different forms: the single trace part for primitive amplitude is (For convenience we abbreviate $\mathrm{Tr}\left(T^{a_1}\cdots T^{a_n}\right)$ as $\mathrm{Tr}\left(a_1,\cdots,a_n\right)$)

$$\mathrm{Gr}_{n,0} = N_c \mathrm{Tr}\left(a_1,\cdots,a_n\right)\,,$$

and the double trace part for other partial amplitudes

$$\mathrm{Gr}_{n-m,m} = \mathrm{Tr}\left(a_1,\cdots,n-m\right)\,\mathrm{Tr}\left(n-m+1,\cdots,n\right)\,.$$

Form (5.1) has been derived in [6] by noticing that full Yang-Mills amplitude can be written as

$$\mathcal{A}_n^{\text{full}}\left(\{k_i,\lambda_i,a_i\}\right) = g^n \sum_{\sigma\in S_{n-1}/\mathcal{R}} \mathrm{tr}(F^{a_{\sigma_1}}F^{a_{\sigma_2}}\ldots F^{a_{\sigma_n}})A_{n,0}(\sigma_1,\sigma_2,\ldots,\sigma_n)\,, \tag{5.2}$$

where $S_{n-1}\equiv S_n/Z_n$ is the group of noncyclic permutation and $\mathcal{R}$ is the reflection $\mathcal{R}(1,2,\ldots,n)=(n,\ldots,2,1)$. Form (5.2) looks like the familiar color-decomposition form for tree-level amplitude. However, there is one crucial difference: the matrix F is given by the adjoint representation, i.e., $(F^a)_{bc}\equiv if^{bac}$. In other words, the distinguish of form (5.2) for tree-level amplitude and one-loop amplitude is If F^a is given by fundamental representation or adjoint representation.

Having the relation between the form (5.2) and the form (5.1), it is very naturally to suspect that we could establish the dual trace form of (5.1) by starting from a dual form of (5.2) along the line in this note. In other words, we need to find a similar expression like (2.5). More accurately, we need to find the systematically construction of BCJ numerator n_i for loop levels. This is a difficult task and has been actively studied by several groups including us.

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