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Mathematical and Computer Modelling 41 (2005) 501–522

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Day-To-Day Vehicular Flow Dynamics in Intelligent Transportation Network

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(Received April 2003; revised and accepted March 2004)

Abstract—In this paper, a flow evolution model is developed by using the dynamical system approach for a vehicular network equipped with predicted travel information. The concerned system variables, path flow, and predicted minimal travel time of an origin-destination (OD) pair, are measured on the peak-hour-average base for each day. The time change rates of these two variables are formulated as a system of ordinary differential equations under the assumption of daily learning and adaptive processes for commuters. By incorporating the total perceived travel time loss (or saving) into the proposed models, time change rates of path flows are generated with a flow-related manner to prevent path flow dynamics from being insensitive to traffic congestion which had been formulated in the existing studies. Heterogeneous models with various user adjusting sensitivities and predicted travel information are also presented. Equilibrium solutions of the proposed network dynamics satisfy the Wardrop user equilibria and are proved to be asymptotically stable by using the stability theorem of Lyapunov. The issue of existence and uniqueness of solutions is proved both on the lemma of Lipschitz condition and the fundamental theorem of ordinary differential equations. In addition, some simple examples are demonstrated to show the asymptotic behaviors of the proposed models numerically. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Day-to-day network dynamics, Nonlinear dynamical system, Lyapunov function, Dynamic traffic assignment.

NOMENCLATURE

t	time index; unit: day	h_w^t	the sum of h_p^t , $\forall p \in P_w$; unit: vehicle/hour
W	the full set of OD pairs with \bar{W} OD pairs	\mathbf{h}^t	the full vector of path flows at day t
P	the full set of paths with \bar{P} paths	f_a^t	the nonnegative peak volume of link a at day t , $f_a^t = \sum_p \delta_{ap} h_p^t$, where $\delta_{ap} = 1$ if link a belongs to path p , otherwise $\delta_{ap} = 0$; unit: vehicle/hour
A	the full set of links with \bar{A} links		
P_w	the set of paths connecting OD pair w with \bar{P}_w paths		
h_p^t	the nonnegative peak-flow of path p at day t ; unit: vehicle/hour		

This research was supported in part by the Ministry of Education of Taiwan, R.O.C. under Grant EX-91-E-FA06-4-4 and in part by the National Science Council of Taiwan, R.O.C. under Grant NSC-91-2211-E-009-050.

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k_a	the capacity of link a ; unit: vehicle/hour	D_w	the travel demand of OD pair w over whole time period of interest; unit: vehicle/hour
$c_a(f_a^t)$	the unit average travel time on link a at day t , a smooth and strict monotone function of f_a^t ; unit: hour	α_p	the positive and path-specific parameter to denote the propensity of path flow dynamics; unit: vehicle/hour ²
c_p^t	the unit average travel time on path p at day t , without considering node travel time, $c_p^t = \sum_a \delta_{ap} c_a(f_a^t)$; unit: hour	β_w	the positive and OD-specific parameter to denote the sensitivity of predicted travel time dynamics; unit: hour ² /vehicle
c_w^t	the unit average travel time on OD pair w at day t predicted and provided by ATIS; unit: hour	\dot{x}	the ordinary derivative of x with respect to t
c^t	the full vector of c_w^t at day t	\mathbf{x}'	the transpose of vector (or matrix) \mathbf{x}

1. INTRODUCTION

The ability to predict how the information prepared by advanced traveler information systems (ATIS) influences the time trajectory of network flows is essential in the era of intelligent transportation systems (ITS). By providing road users with travel information and so allowing them to make better travel decisions, ATIS promises to enhance the utilization of existing transportation infrastructure and, consequently, to mitigate traffic congestion. The focus of this paper is to develop an analytical approach capturing the effects of travel information on network flow evolutions, especially the day-to-day interactions among system variables, network performances, and travel information. Network flow dynamics had been studied analytically [1–6] or simulation-oriented [7–15]. Some other researchers were concerned about formulations and solution methods of dynamic traffic assignment problem to compute the flow pattern under the conditions of system optimum [16–17], dynamic user equilibrium [18], or dynamic user optimum [19–21]. These studies did not simulate the evolutions of network flows but only a unique flow solution following some optimal criteria.

Conversely, the flow evolution model is able to capture the transition states of the system and finally approaches to steady state [22] if the system is well-behaved with a long enough evolution time. They are classified into two broad categories according to the time scales of system adjustment, the fluctuations of the system variables within each single day (intraday dynamics) and between subsequent “days” or more generally observation periods of similar characteristics (interday dynamics). Chang and Mahmassani’s serial experiments [7–10] on route choice behavior of commuters had indicated that the learning and adaptive process of route choice may take weeks, partly because of the dynamic feedbacks from the traffic system, and, indeed, it can be lengthened by complex switching that resulted from the provision of better information. Friesz *et al.* [4] addressed Mahmassani’s design theoretically by introducing a tatonnement process to model the transition of disequilibria from one state to another. Day-to-day dynamic models are particularly useful in capturing traffic patterns when perturbations of the traffic system create disequilibria that might adjust toward equilibria. If the duration of disruption or the time it takes for the system to reach equilibrium are such that the system stays longer in a nonequilibrium state rather than in an equilibrium one, it is important to catch the interday transition process of traffic diversion. Issues about interday transformation of peak-period commuter trips are examined in this research. A formulation of path flow alteration is presented though it is partly similar to Friesz *et al.* [4] but with new conceptual and mathematical refinements to offer a congestion-sensitive evolution behavior of network flows.

To sum up, the purpose of this paper is to develop an analytical approach capable of both modeling the interacted relationships between the aggregate user behaviors and ATIS information provision and characterizing the theoretical issues of the existence, uniqueness, and stability

for the proposed model with a thoroughly mathematical foundation. The achievements in the paper have never been reached before especially in a multiclass users scenario. As pointed out by Friesz *et al.* [23,24], the application of the proposed model mainly concentrates on the dynamic network design problem. The dynamic network design problem is to determine a temporal management (or control) plan which recognizes that perturbations generated from the new management (or control) treatments bring about disequilibria which adjust toward equilibrium. Therefore, a model of this type describes the time evolution of system states in a disequilibrium adjustment framework and forms the basic feasible solutions of the dynamic network design problem.

The remainder of the paper is organized as follows. In the next section, we briefly mention the assumptions common throughout the paper. The major developments for network dynamics are described in Section 3. The proof of existence, uniqueness, and stability of proposed models is provided in Section 4. Numerical examples are demonstrated in Section 5, and the conclusions of this research are outlined in Section 6.

2. ASSUMPTIONS

The basic assumption in the paper is the daily learning and adaptive processes of how system behaviors interact with the travel information predicted and provided by ATIS. Travel information about the upcoming day's travel time from origin to destination is provided with users and compared with the actual travel time experienced by all path users. Then, travelers who experienced travel time less than predicted travel time should have encountered a (pseudo) travel time saving. On the other hand, users who experienced travel time that is more than the predicted travel time encounter a (pseudo) travel time loss. These deviations result in the path flows adjustments of the next day. Each link cost function is assumed to be a smooth and strict monotone function of link flow and is used to estimate the actual travel time for all paths. In addition, travel demand is presumably fixed in this study, without the loss of generality, under the assumption of no structural changes from competing transportation facilities over the whole period of interest. The authors are concerned that the variations of path flow and path travel time are much more sensitive than that of the OD demand if the travel information provided by ATIS is the only perturbation of transportation system. Therefore, by recording actual traffic volumes, ATIS evaluates the difference between the fixed travel demand and the sum of the path flows for each OD pair. In order to reflect the relative scarcity (or surplus) of transportation facilities, it is assumed that the predicted travel time of an OD pair adjusts accordingly as the difference between the fixed travel demand and the sum of the corresponding path flows is detected. Moreover, it is assumed also that travel demands of all OD pairs do not jointly violate any capacity constraints of all the links.

Some notation (see Nomenclature) based on the typical equilibrium models of commuter route choice are employed and augmented to meet our concerns. In particular, we take all vectors to be column vectors. Vectors and matrices are expressed in boldface.

3. MODELING NETWORK DYNAMICS

The contents of this section are distributed into two parts. The first is path flow dynamics that interact with ATIS, which are formulated in Section 3.1. The second is travel time evolutions of OD pairs that are predicted by ATIS, which are analyzed in Section 3.2. The whole network dynamics is a combination of these two components.

3.1. Path Flow Dynamics with ATIS Information

The basic behavioral assumption of least travel time seeking has been mentioned in Section 2. Before giving a mathematical relationship between path flow dynamics and this supposition, we

define the value of total perceived travel time loss (or saving) for path $p \in P_w$ at day t as

$$\text{PTTL}_{p,w}^t \equiv h_p^t (c_p^t - c_w^t). \quad (1)$$

The perceived travel time loss (or saving) is measured by multiplying path flow with the difference between users' average experienced travel time for a path estimated by link cost functions and the predicted travel time provided by ATIS for the corresponding OD pair. This quantity can be viewed as an estimation of the total travel time loss (or saving) perceived by travelers driving path p at day t . The meaning of positive (negative) $\text{PTTL}_{p,w}^t$ is that the average path travel time the users actually underwent is greater (less) than the travel time predicted by ATIS. Travelers might be motivated to change their route due to this difference. Alternatively, $\text{PTTL}_{p,w}^t$ can be denoted as a measurement of path performance for the previous time point that results in the consequent shift of path flow at next time point. In addition, the congestion effect is embedded in $\text{PTTL}_{p,w}^t$ implicitly by including the current state of path flow. It means that the values of $\text{PTTL}_{p,w}^t$ will be different at various congestion levels even under the condition that the difference between the average path travel time experienced by the users and the travel time predicted by ATIS is equal. The predicted travel time of OD pair w at day t , c_w^t , is updated by another dynamic process that will be presented in Section 3.2. To continue our development, it is useful to postulate that future path flow is established through the tuning of the present state at a rate proportional to the value of $\text{PTTL}_{p,w}^t$. That is,

$$h_p^{t+\Delta t} \equiv h_p^t - \alpha_p (\text{PTTL}_{p,w}^t) \Delta t, \quad (2)$$

where

$$0 < \alpha_p < \inf_{c_p^t - c_w^t > 0} \left(\frac{1}{c_p^t - c_w^t} \right) \quad (3a)$$

and

$$\sum_{w \in W} \sum_{p \in P_w} (-\delta_{ap} \alpha_p \text{PTTL}_{p,w}^t) \leq k_a - \sum_{\forall p \in P} \delta_{ap} h_p^t, \quad (3b)$$

for all OD pair $w \in W$, $p \in P_w$, $a \in A$, at day t . Inequalities (3a) and (3b) ensure to avoid non-feasibilities of nonnegative flow and link capacity constraints. It is obvious that inequalities (3a) and (3b) are naturally satisfied if α_p is carefully calibrated from the empirical data. By taking the limit of (2) as Δt approaches to zero and substituting (1) into (2), our path flow dynamics follow immediately as

$$\dot{h}_p^t = \frac{dh_p^t}{dt} = -\alpha_p h_p^t (c_p^t - c_w^t), \quad (4)$$

for all OD pair $w \in W$, $p \in P_w$, at day t . The physical meaning of (4) is that the time change rate of the flow for path p at day t is equal to the negative product of the propensity of path flow shift and the value of $\text{PTTL}_{p,w}^t$.

3.2. Travel Time Dynamics Predicted by ATIS

To forward the network traffic to a steady status is the major intent of ATIS. This goal is realized by successfully capturing travel time dynamics and delivering information to users. In this section, we will focus on the formulation of predicted travel time dynamics. To start the task, we first define the excess travel demand of an OD pair w at day t as

$$\text{ETD}_w^t \equiv D_w - h_w^t, \quad (5)$$

where D_w denotes the time-invariant travel demand of an OD pair w over whole time period of interest. ETD_w^t is considered to be the difference between the travel demand and the sum of the corresponding path flows for an OD pair as denoted in [4] and [23]. Positive (negative) excess

travel demand means that the rate at which users desire to depart from an origin to a destination is greater than (less than or equal to) that where such movements are actually occurring. Then, predicted travel time is adjusted due to this deviation. Consequently, the predicted travel time dynamics for all OD pairs $w \in W$ at day t can be written as

$$c_w^{t+\Delta t} \equiv c_w^t + \beta_w (D_w - h_w^t) \Delta t, \quad (6)$$

where

$$0 < \beta_w < \min \left\{ \inf_{\forall (D_w - h_w^t) < 0} \left(\frac{\bar{c}_w - c_w^t}{D_w - h_w^t} \right), \inf_{\forall (D_w - h_w^t) > 0} \left(\frac{\hat{c}_w - c_w^t}{D_w - h_w^t} \right) \right\}, \quad (7)$$

if we further suppose that the travel time prediction of the next time point for an OD pair is transformed from the current status at a rate scaled to ETD_w^t . Inequality (7) guarantees that all predicted travel times stay in the feasible region without greater than \hat{c}_w , the travel time at maximal flow, or less than \bar{c}_w , the travel time at free flow, for OD pair w . Then, we define

$$\bar{c}_w \equiv \min_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = 0) \right) \quad (7a)$$

and

$$\hat{c}_w \equiv \max_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = k_a) \right). \quad (7b)$$

After taking the limit of (6), the predicted travel time dynamics can be written in differential form as

$$\dot{c}_w^t = \frac{dc_w^t}{dt} = \beta_w (D_w - h_w^t), \quad (8)$$

for all OD pairs $w \in W$, at day t . Equation (8) reveals that the time change rate of the predicted travel time for OD pair w at day t is equal to the product of the sensitivity of predicted travel time dynamics and ETD_w^t . The whole version of network dynamics under operations of ATIS is accomplished as

$$\begin{aligned} \dot{h}_p^t &= -\alpha_p h_p^t (c_p^t - c_w^t), \\ \dot{c}_w^t &= \beta_w (D_w - h_w^t), \\ 0 < \alpha_p < \inf_{c_p^t - c_w^t > 0} \left(\frac{1}{c_p^t - c_w^t} \right), \quad &\text{and} \quad \sum_{w \in W} \sum_{\substack{p \in P_w \\ c_p^t - c_w^t < 0}} (-\delta_{ap} \alpha_p \text{PTTL}_{p,w}^t) \leq k_a - \sum_{p \in P} \delta_{ap} h_p^t, \\ 0 < \beta_w < \min \left\{ \inf_{\forall (D_w - h_w^t) < 0} \left(\frac{\bar{c}_w - c_w^t}{D_w - h_w^t} \right), \inf_{\forall (D_w - h_w^t) > 0} \left(\frac{\hat{c}_w - c_w^t}{D_w - h_w^t} \right) \right\}, \end{aligned} \quad (9)$$

for all OD pairs $w \in W$, $p \in P_w$, $a \in A$, at day t .

3.3. Multiclass Users Model

In this section, road users are classified into n subgroups to consider the effects of heterogeneous adjustments due to the various sensitivities of multiclass users on path flow dynamics. That is to say, the parameter α_p in (4) is not only path-specific but also user-specific. Consequently, there are n path flow dynamics for path $p \in P_w$, at day t and (4) is augmented as

$$\dot{h}_{ip}^t = \frac{dh_{ip}^t}{dt} = -\alpha_{ip} h_{ip}^t (c_p^t - c_w^t), \quad (10)$$

for all OD pair $w \in W$, $p \in P_w$, $i = 1, 2, \dots, n$, at day t . The physical meaning of (10) is that the time change rate of the flow for user class i using path p , at day t is equal to the

negative product of the propensity of path flow shift for user class i and the value of perceived travel time loss (saving) for user class i , $\text{PTTL}_{ip,w}^t$. If we further suppose that there are two options for ATIS to provide predicted OD travel time, i.e., a homogeneous forecast for all users or individual predictions according to multiclass users, the predicted OD travel time dynamics should be reformulated as

$$\dot{c}_w^t = \frac{dc_w^t}{dt} = \beta_w (D_w - h_w^t) \quad (11)$$

or

$$\dot{c}_{iw}^t = \frac{dc_{iw}^t}{dt} = \beta_{iw} (D_{iw} - h_{iw}^t), \quad (12)$$

for all OD pairs $w \in W$, $i = 1, 2, \dots, n$, at day t , respectively. In (11), the definition of h_w^t is the same as mentioned before but calculated in a different way and expressed as

$$h_w^t = \sum_{p \in P_w} \sum_i h_{ip}^t. \quad (13)$$

All components in (12) are reindexed from (8) by adding a subscript i for the corresponding user class in particular with

$$h_{iw}^t = \sum_{p \in P_w} h_{ip}^t. \quad (14)$$

Equation (12) tells us that the time change rate of the predicted travel time for user class i traveling on OD pair w , at day t is equal to the product of the sensitivity of predicted travel time dynamics of user class i and the value of excess travel demand of user class i , ETD_{iw}^t . Now, the multiclass users dynamical systems for all OD pairs $w \in W$, $p \in P_w$, $i = 1, 2, \dots, n$, at day t are proposed as

$$\begin{aligned} \dot{h}_{ip}^t &= -\alpha_{ip} h_{ip}^t (c_p^t - c_w^t), \\ \dot{c}_w^t &= \beta_w (D_w - h_w^t) \\ 0 < \alpha_{ip} &< \inf_{c_p^t - c_w^t > 0} \left(\frac{1}{c_p^t - c_w^t} \right), \quad \text{and} \quad \sum_{w \in W} \sum_{i \in N} \sum_{\substack{p \in P_w \\ c_p^t - c_w^t < 0}} (-\delta_{ap} \alpha_{ip} \text{PTTL}_{ip,w}^t) \leq k_a - \sum_{\substack{p \in P \\ i \in N}} \delta_{ap} h_{ip}^t, \\ 0 < \beta_w &< \min \left\{ \inf_{\forall (D_w - h_w^t) < 0} \left(\frac{\bar{c}_w - c_w^t}{D_w - h_w^t} \right), \inf_{\forall (D_w - h_w^t) > 0} \left(\frac{\bar{c}_w - c_w^t}{D_w - h_w^t} \right) \right\}, \end{aligned} \quad (15)$$

for the case of providing a uniform prediction of OD travel time by ATIS and as

$$\begin{aligned} \dot{h}_{ip}^t &= -\alpha_{ip} h_{ip}^t (c_p^t - c_{iw}^t), \\ \dot{c}_{iw}^t &= \beta_{iw} (D_{iw} - h_{iw}^t), \\ 0 < \alpha_{ip} &< \inf_{c_p^t - c_{iw}^t > 0} \left(\frac{1}{c_p^t - c_{iw}^t} \right), \quad \text{and} \quad \sum_{w \in W} \sum_{i \in N} \sum_{\substack{p \in P_w \\ c_p^t - c_{iw}^t < 0}} (-\delta_{ap} \alpha_{ip} \text{PTTL}_{ip,w}^t) \leq k_a - \sum_{\substack{p \in P \\ i \in N}} \delta_{ap} h_{ip}^t \\ 0 < \beta_{iw} &< \min \left\{ \inf_{\forall (D_{iw} - h_{iw}^t) < 0} \left(\frac{\bar{c}_w - c_{iw}^t}{D_{iw} - h_{iw}^t} \right), \inf_{\forall (D_{iw} - h_{iw}^t) > 0} \left(\frac{\bar{c}_w - c_{iw}^t}{D_{iw} - h_{iw}^t} \right) \right\}, \end{aligned} \quad (16)$$

for the case of individually supplying user-definite OD travel time, respectively.

Empirically, it is easy to be accomplished for such a hybrid system by providing a multiaccess information inquiry system. However, we are concerned about the asymptotic behavior of the steady state and whether it is in complete accord with the well-known Wardrop's user equilibrium. These topics are the major components in Section 4.

4. EXISTENCE, UNIQUENESS, AND STABILITY

A dynamical system is able to demonstrate the future states unambiguously when the evolution rules and initial situations are specified. The analysis of stability guides us to find out the network dynamics of where to go and when to be tranquil after perturbations are stirred up. In this section, we briefly illustrate the issue of existence and uniqueness of the proposed uniform-user model, i.e., (9), by the fundamental theorem of ordinary differential equations in Section 4.1. The steady state of (9) is analyzed to be in agreement with Wardrop's user equilibrium in Section 4.2, followed by the proof of the associated stability theorems using Lyapunov's direct method in Section 4.3. For the part of a multiclass model, brief statements based on the results of uniform-user model are provided in the latter part of these three sections. Time indices, day t , and inequalities (3) and (7) are omitted for conciseness in the subsequent sections, then, (9) is rewritten as

$$\begin{aligned}\dot{h}_p &= -\alpha_p h_p (c_p - c_w), \\ \dot{c}_w &= \beta_w (D_w - h_w).\end{aligned}\tag{17}$$

4.1. Existence and Uniqueness

A dynamical system is a way of describing the time passage of all the points for a given space E . Mathematically, the space E might be a Euclidean space, R , or a subset of R . For the network dynamics mentioned in Section 3, the set of possible nonnegative path flows and predicted travel time is clearly a convex subset of $R_+^{\bar{P}+\bar{W}}$, denoted as S even the constraints of path capacity are added. We have the following theorem in [26].

THEOREM 1. Suppose that $G \in C^1(E)$ and that $G(\mathbf{x})$ satisfies the global Lipschitz condition,

$$|G(\mathbf{x}) - G(\mathbf{y})| \leq L |\mathbf{x} - \mathbf{y}|,\tag{18}$$

for all $\mathbf{x}, \mathbf{y} \in E$. Then, for $\mathbf{x}_0 \in R^n$, the initial value problem,

$$\dot{\mathbf{x}} = G(\mathbf{x}), \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0,\tag{19}$$

has a unique solution $\mathbf{x}(t)$ defined, for all $t \in R$. The existence and uniqueness of (17) can be claimed if the $G(\mathbf{x})$ in (17) $\in C^1(R_+^{\bar{P}+\bar{W}})$ and satisfies the global Lipschitz condition.

The assumptions of the smooth and strict monotone function of link travel time ensure that the $G(\mathbf{x})$ in (17) is $C^1(R_+^{\bar{P}+\bar{W}})$. Then, the proof of the global Lipschitz of $G(\mathbf{x})$ is given by the following lemma.

For convenience, $G : S \rightarrow R_+^{\bar{P}+\bar{W}}$ in (17) is reindexed as

$$G(\mathbf{h}', \mathbf{c}') \equiv \begin{cases} g_j(\mathbf{h}', \mathbf{c}') = -\alpha_j h_j (c_j - c_w), & \forall w \in W, \quad j \in \{1, 2, \dots, \bar{P}\} \text{ if path } j \in P_w, \\ g_j(\mathbf{h}', \mathbf{c}') = \beta_w [D_w - h_w], & \forall w \in W, \quad j \in \{1 + \bar{P}, 2 + \bar{P}, \dots, w + \bar{P}\}, \end{cases}\tag{20}$$

and the norm

$$|\mathbf{y}| = \sum_{l=1}^{\bar{P}+\bar{W}} |y_l|$$

is used $\forall \mathbf{y} \in R_+^{\bar{P}+\bar{W}}$. Now, we give the lemma of the Lipschitz condition for (20) and prove it.

LEMMA. $G(\mathbf{h}', \mathbf{c}')$ defined in (20) satisfies the global Lipschitz condition with a Lipschitz constant,

$$L = \max \left\{ \max_{\forall w} \{\beta_w\}, L_p \right\},$$

where

$$L_p = \max_{1 \leq j \leq \bar{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left(\left(\widehat{c}_w - \check{c}_w \right) + h_j \frac{\partial c_j}{\partial h_j} \Big|_{h_j = \widehat{h}_j} \right) \right), \right. \\ \left. \sup_{j, l \neq j} \left(\alpha_j h_j \frac{\partial c_j}{\partial h_l} \Big|_{h_l = \widehat{h}_l} \right), \sup_j (\alpha_j \widehat{h}_j) \right\}. \quad (21)$$

PROOF. From the assumptions and definitions of the link cost function mentioned in previous sections, the $G(\mathbf{h}', \mathbf{c}')$ in (20) is obviously continuously differentiable on S . Let $\mathbf{u} = \mathbf{x} - \mathbf{y}$ and \mathbf{y}, \mathbf{x} are two given points in S , i.e., $\mathbf{y}' = (\mathbf{h}'_y, \mathbf{c}'_y)$ and $\mathbf{x}' = (\mathbf{h}'_x, \mathbf{c}'_x)$. Since convexity of S , the points $\mathbf{v} = \mathbf{y} + b\mathbf{u}$ are also in S , for all $0 \leq b \leq 1$. Defining the function $H : [0, 1] \rightarrow R_+^{\bar{P}+\bar{W}}$ by

$$H(b) = G(\mathbf{v}), \quad (22)$$

and by the chain rule, we have

$$\frac{dH(b)}{db} = \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \frac{\partial g_j(\mathbf{v})}{\partial v_l} \frac{dv_l}{db} = \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \frac{\partial g_j(\mathbf{v})}{\partial v_l} (x_l - y_l),$$

and hence,

$$\left| \frac{dH(b)}{db} \right| \leq \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| \left| \frac{dv_l}{db} \right| = \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| |x_l - y_l|. \quad (23)$$

There are three conditions in (23) that keep $\left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right|$ from vanishing for the path flow dynamics, i.e., $\forall g_j, 1 \leq j \leq \bar{P}$. If we let $\mathbf{v}' = (\mathbf{h}'_v, \mathbf{c}'_v)$, they are

$$\left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = \left| \alpha_j \left((c_j - c_w) + h_j \frac{\partial c_j}{\partial h_l} \right) \right|, \quad \text{if } l = j \text{ and } 1 \leq l \leq \bar{P}, \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = \left| \alpha_j h_j \frac{\partial c_j}{\partial h_l} \right|, \quad \text{if } l \neq j, \text{ path } l \text{ overlaps partly with path } j \text{ and } 1 \leq l \leq \bar{P}, \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = |\alpha_j h_j|, \quad \forall 1 + \bar{P} \leq l \leq \bar{P} + \bar{W}, \quad l = \bar{P} + w, \quad j \in P_w. \quad (24)$$

The upper bound of $\left| \frac{\partial g_j}{\partial v_l} \right|$ in condition (24) can be decided by

$$\max_{1 \leq j \leq \bar{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left((c_j - c_w) + h_j \frac{\partial c_j}{\partial h_j} \right) \right), \sup_{j, l \neq j} \left(\alpha_j h_j \frac{\partial c_j}{\partial h_l} \right), \sup_j (\alpha_j h_j) \right\}. \quad (27)$$

We know that h_p and c_w are bounded globally by constraints (3a), (3b), (7), (7a) and (7b). Hence, (25) is also bounded as the boundary conditions of (3a), (3b), (7), (7a) and (7b) take place. That is to say, the upper bound of $\left| \frac{\partial g_j}{\partial v_l} \right|$ in condition (24) is shown as

$$L_p = \max_{1 \leq j \leq \bar{P}} \left| \frac{\partial g_j}{\partial v_l} \right| = \max_{1 \leq j \leq \bar{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left(\left(\widehat{c}_w - \check{c}_w \right) + h_j \frac{\partial c_j}{\partial h_j} \Big|_{h_j = \widehat{h}_j} \right) \right), \right. \\ \left. \sup_{j, l \neq j} \left(\alpha_j \widehat{h}_j \frac{\partial c_j}{\partial h_l} \Big|_{h_l = \widehat{h}_l} \right), \sup_j (\alpha_j \widehat{h}_j) \right\}, \quad (26)$$

where

$$\widehat{h}_p \equiv \min_{\forall a \in p} \{k_a\}. \quad (27)$$

By similar treatments, there is only one condition that $|\frac{\partial g_j(\mathbf{v})}{\partial v_l}|, \forall g_j$ and $\bar{P} + 1 \leq j \leq \bar{P} + \bar{W}$ in (23) is not equal to zero. It can be expressed as

$$\left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = \beta_w, \quad \forall 1 \leq l \leq \bar{P}, \text{ if path } l \in P_w. \quad (28)$$

The results of (27) and (28) lead us to set

$$L = \max \left\{ \max_{\forall w} \{ \beta_w \}, L_p \right\} \quad (29)$$

and hence,

$$\begin{aligned} \left| \frac{dH(b)}{db} \right| &\leq \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| \left| \frac{dv_l}{db} \right| \\ &= \sum_{j=1}^{\bar{P}+\bar{W}} \sum_{l=1}^{\bar{P}+\bar{W}} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| |x_l - y_l| \leq L \sum_{l=1}^{\bar{P}+\bar{W}} |x_l - y_l| = L |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

Now, from the relation $G(\mathbf{x}) - G(\mathbf{y}) = H(1) - H(0) = \int_0^1 H'(z) dz$, we find that $|G(\mathbf{x}) - G(\mathbf{y})| \leq \int_0^1 |H'(z)| dz \leq L |\mathbf{x} - \mathbf{y}|$. Thus, $G(\mathbf{h}', \mathbf{c}')$ in (20) satisfies the global Lipschitz condition and the corresponding Lipschitz constant L can be determined by (29). After proving the above lemma, the global existence and uniqueness of (17) is standard and we refer readers to [26] for a proof of the fundamental global theorem.

For the cases of multiclass users, (20) is amplified as

$$G(\mathbf{h}', \mathbf{c}') \equiv \begin{cases} g_j(\mathbf{h}', \mathbf{c}') = -\alpha_{ip} h_{ip} (c_p - c_w), & \forall w, p \in P_w, \quad j = \bar{P}(i-1) + p, \\ g_j(\mathbf{h}', \mathbf{c}') = \beta_w [D_w - h_w], & \forall w \in W, \quad j = n\bar{P} + w, \end{cases} \quad (30)$$

and

$$G(\mathbf{h}', \mathbf{c}') \equiv \begin{cases} g_j(\mathbf{h}', \mathbf{c}') = -\alpha_{ip} h_{ip} (c_p - c_{iw}), & \forall w, p \in P_w, \quad j = \bar{P}(i-1) + p, \\ g_j(\mathbf{h}', \mathbf{c}') = \beta_{iw} [D_{iw} - h_{iw}], & \forall w \in W, \quad j = n\bar{P} + \bar{W}(i-1) + w, \end{cases} \quad (31)$$

where $i = 1, 2, \dots, n$ to fit (15) and (16), respectively. The dimension of $G(\mathbf{h}', \mathbf{c}')$ and $(\mathbf{h}', \mathbf{c}')$ in (30) are the same and denoted as $n\bar{P} + \bar{W}$ and in (31) as $n(\bar{P} + \bar{W})$. By the similar treatments, condition (24) for path flow dynamics with $1 \leq j \leq n\bar{P}$ is modified as

$$\begin{aligned} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= \left| \alpha_{ip} \left((c_p - c_w) + h_{ip} \frac{\partial c_p}{\partial h_{ip}} \right) \right|, \quad \forall l \in [1, n\bar{P}], \quad \text{if } l = \bar{P}(i-1) + p, \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= \left| \alpha_{ip} h_{ip} \frac{\partial c_p}{\partial h_{ez}} \right|, \quad \forall l \in [1, n\bar{P}], \\ &\quad \text{if } l \neq \bar{P}(i-1) + p \text{ and } l = \bar{P}(e-1) + z, \quad (32) \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= |\alpha_{ip} h_{ip}|, \quad \forall l \in [1 + n\bar{P}, n\bar{P} + \bar{W}], \quad p \in P_w, \\ &\quad \text{if } l = n\bar{P} + w, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= \left| \alpha_{ip} \left((c_p - c_{iw}) + h_{ip} \frac{\partial c_p}{\partial h_{ip}} \right) \right|, \quad \forall l \in [1, n\bar{P}], \quad \text{if } l = \bar{P}(i-1) + p, \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= \left| \alpha_{ip} h_{ip} \frac{\partial c_p}{\partial h_{ez}} \right|, \quad \forall l \in [1, n\bar{P}], \\ &\quad \text{if } l \neq \bar{P}(i-1) + p \text{ and } l = \bar{P}(e-1) + z, \quad (33) \\ \left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| &= |\alpha_{ip} h_{ip}|, \quad \forall l \in [1 + n\bar{P}, n\bar{P} + n\bar{W}], \quad p \in P_w, \\ &\quad \text{if } l = n\bar{P} + \bar{W}(i-1) + w, \end{aligned}$$

for the two cases of multiclass users model, respectively. The corresponding nonzero parts of partial derivative of predicted travel time dynamics can be similarly derived as

$$\left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = \beta_w, \quad \forall 1 \leq l \leq n\bar{P}, \quad \text{if } l = \bar{P}(i-1) + p \text{ and } p \in P_w \quad (34)$$

$\forall g_j, j \in [1 + n\bar{P}, n\bar{P} + \bar{W}]$ in (31), and

$$\left| \frac{\partial g_j(\mathbf{v})}{\partial v_l} \right| = \beta_{iw}, \quad \forall 1 \leq l \leq n\bar{P}, \quad \text{if } l = \bar{P}(i-1) + p \text{ and } p \in P_w. \quad (35)$$

$\forall g_j, j \in [1 + n\bar{P}, n\bar{P} + n\bar{W}]$ in (31). Eventually, the Lipschitz constant can be expressed as $L = \max\{\max_{\forall w}\{\beta_w\}, L_{Ip}\}$, where

$$L_{Ip} = \max_{\substack{1 \leq i, e \leq n \\ 1 \leq p, z \leq \bar{P}}} \left\{ \sup_{i, p \in P_w} \left(\alpha_{ip} \left(\left(\widehat{c}_w - \check{c}_w \right) + h_{ip} \left. \frac{\partial c_p}{\partial h_{ip}} \right|_{h_{ip} = \widehat{h}_{ip}} \right) \right), \right. \\ \left. \sup_{\substack{i, p, e, z \\ l \neq \bar{P}(i-1)+p \\ l = \bar{P}(e-1)+z}} \left(\alpha_{ip} \widehat{h}_{ip} \left. \frac{\partial c_p}{\partial h_{ez}} \right|_{h_{ez} = \widehat{h}_{ez}} \right), \sup_{i, p} \left(\alpha_{ip} \widehat{h}_{ip} \right) \right\}, \quad (36)$$

for (30) and as $L = \max\{\max_{\forall i, w}\{\beta_{iw}\}, L_{Ip}\}$, where

$$L_{Ip} = \max_{\substack{1 \leq i, e \leq n \\ 1 \leq p, z \leq \bar{P}}} \left\{ \sup_{i, p \in P_w} \left(\alpha_{ip} \left(\left(\widehat{c}_{iw} - \check{c}_{iw} \right) + h_{ip} \left. \frac{\partial c_p}{\partial h_{ip}} \right|_{h_{ip} = \widehat{h}_{ip}} \right) \right), \right. \\ \left. \sup_{\substack{i, p, e, z \\ l \neq \bar{P}(i-1)+p \\ l = \bar{P}(e-1)+z}} \left(\alpha_{ip} \widehat{h}_{ip} \left. \frac{\partial c_p}{\partial h_{ez}} \right|_{h_{ez} = \widehat{h}_{ez}} \right), \sup_{i, p} \left(\alpha_{ip} \widehat{h}_{ip} \right) \right\}, \quad (37)$$

for (31). The same definitions of (7a), (7b), and (27) are employed here for \check{c}_{iw} , \widehat{c}_{iw} , and \widehat{h}_{ip} .

4.2. Analysis of Steady State

It is useful to recall the definition of Wardrop's static user equilibrium in our terms before elaborating on the steady state of the proposed network dynamics. If the symbol, “ $\overline{\quad}$ ”, is used to denote steady-state or equilibrium point, the Wardrop's user equilibrium can be described as

$$\begin{aligned} \bar{h}_p &> 0 \rightarrow c_p = \bar{c}_w, \\ c_p &> \bar{c}_w \rightarrow \bar{h}_p = 0, \quad \forall w \in W \text{ and } p \in P_w. \\ \bar{h}_w &= D_w, \end{aligned} \quad (38)$$

Condition (38) states a condition that is stable only when no traveler can improve his travel time by unilaterally changing paths. All path travel times of the same OD pair are equal and minimal at this status.

The steady state of (17) implies $\dot{h}_p = 0$ and $\dot{c}_w = 0$, for all OD pair $w \in W$, $p \in P_w$. After some algebraic reasoning, we get the conditions below jointly equivalent to the steady state of path flow dynamics

$$\begin{aligned} h_p > 0 \rightarrow c_p = c_w, & \quad \text{or} \quad h_p < 0 \rightarrow c_p = c_w, \\ c_p > c_w \rightarrow h_p = 0, & \quad \text{or} \quad c_p < c_w \rightarrow h_p = 0, \\ c_w > 0 \rightarrow D_w = h_w, & \end{aligned} \quad (39)$$

for all OD pair $w \in W$, $p \in P_w$. The second subcase of the first part in condition (39) never happens because of the violating nonnegative flow constraint. The second subcase of the second relationship in condition (39) will never happen if initial conditions with positive path flows are provided. For the positive nature of the predicted travel time even at zero path flow level, the last equilibrium state in condition (39) is held on evidently. Finally, we abstract the critical components in condition (39) as

$$\begin{aligned} h_p > 0 \rightarrow c_p = c_w, \\ c_p > c_w \rightarrow h_p = 0, \\ D_w = h_w, \end{aligned} \quad (40)$$

for all OD pair $w \in W$, $p \in P_w$. It is easy to infer that the actual average travel time is equal to the predicted travel time by ATIS and is minimal among all paths of an OD pair simultaneously in condition (40). The travel demand of an OD pair is equal to the sum of corresponding path flows. Based on these results, we can claim that the steady state of (17) is identical to Wardrop's user equilibrium.

Similarly, the equilibrium state of multiclass-user models are derived as

$$\begin{aligned} h_{ip} > 0 \rightarrow c_p = c_{iw}, \\ c_p > c_{iw} \rightarrow h_{ip} = 0, \\ D_w = h_w, \end{aligned} \quad (41)$$

in (15) and as

$$\begin{aligned} h_{ip} > 0 \rightarrow c_p = c_{iw}, \\ c_p > c_{iw} \rightarrow h_{ip} = 0, \\ D_{iw} = h_{iw}, \end{aligned} \quad (42)$$

in (16), for all OD pair $w \in W$, $p \in P_w$, $i = 1, 2, \dots, n$, respectively. In condition (41), h_w is defined in (13). Because c_p is not user-specific, we have that path travel times with positive path flow are equal to the predicted travel time of OD pair w simultaneously and this path travel time is the minimal for all paths connecting OD pair w . Travel demand of an OD pair is equal to the sum of flows distributed over all user classes and corresponding paths. In condition (42), similarly the predicted travel times of OD pair w for all user classes are the same and equal to the minimal path travel times simultaneously. The demand of an OD pair is divided into n parts for n user classes and each part is equal to the sum of corresponding path flows.

4.3. Analysis of Stability

In this section, our interest is in showing that (9) is asymptotically stable. The definitions and theorem of stability in the sense of Lyapunov are employed as the following statements [27].

DEFINITION 1. Let \bar{v} be a steady state of a dynamical system, $G \in C^1(E)$. A function $L : E \rightarrow \mathbb{R}$ is called a strict Lyapunov function for \bar{v} if the following conditions are satisfied.

- (1) $L(\bar{v}) = 0$, and $L(v) > 0$, $\forall v \neq \bar{v}$, $v \in E$.
- (2) $\dot{L}(v) < 0$, $\forall v \neq \bar{v}$, $v \in E$.

THEOREM 2. Let $\bar{\mathbf{v}}$ be a steady state of $\dot{\mathbf{v}} = G(\mathbf{v})$. If there exists a strict Lyapunov function $\forall \mathbf{v} \neq \bar{\mathbf{v}}, \mathbf{v} \in E$, then, $\bar{\mathbf{v}}$ is asymptotically stable.

Accordingly, the stability theorem of proposed dynamical system is illustrated as Theorem 3.

THEOREM 3. Let $\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}' = (\bar{h}_1, \bar{h}_2, \dots, \bar{h}_p, \bar{c}_1, \bar{c}_2, \dots, \bar{c}_w)$ be a steady state of (9) and $\begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}$ is asymptotically stable.

For the sake of conciseness, the following definitions are introduced to rewrite (17) in vector and matrix form. Let

$$\hat{\mathbf{h}} \equiv \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_p \end{pmatrix} \quad (43)$$

and rearrange it into several diagonal matrices as

$$\hat{\mathbf{h}}_r \equiv \begin{pmatrix} \mathbf{h}_+ & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_- \end{pmatrix}, \quad (44)$$

where the components of \mathbf{h}_+ and \mathbf{h}_- are path flows with $(h_p - \bar{h}_p)(c_p - c_w) \geq 0$ and $(h_p - \bar{h}_p)(c_p - c_w) < 0$, respectively. Moreover, let

$$\bar{\mathbf{h}}_\varepsilon \equiv \begin{pmatrix} \bar{\mathbf{h}}_{\varepsilon+} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{h}}_{\varepsilon-} \end{pmatrix}, \quad (45)$$

where $\bar{\mathbf{h}}_{\varepsilon+}$ and $\bar{\mathbf{h}}_{\varepsilon-}$ are two diagonal matrices and there exists $\varepsilon_p^+ \in R^+$ and $\varepsilon_p^- \in R^+$, such that the elements of $\bar{\mathbf{h}}_{\varepsilon+}$ and $\bar{\mathbf{h}}_{\varepsilon-}$ are $\bar{h}_p^* \varepsilon_p^+$ and $\bar{h}_p^* \varepsilon_p^-$, respectively with

$$\frac{h_p}{\bar{h}_p^* \varepsilon_p^+} > 1, \quad \forall h_p \in \mathbf{h}_+, \quad (46)$$

and

$$\frac{h_p}{\bar{h}_p^* \varepsilon_p^-} < 1, \quad \forall h_p \in \mathbf{h}_-, \quad (47)$$

where $\bar{h}_p^* = \sup_{\forall h_p \in h} (\bar{h}_p)$ and \bar{h}_p is the steady state of h_p . The zero matrix with suitable dimension is denoted by $\mathbf{0}$. Furthermore, if we let $\mathbf{s} \equiv \begin{pmatrix} \Delta \mathbf{h} \\ \mathbf{c} \end{pmatrix}$, $\mathbf{M}(\mathbf{s}) \equiv \begin{pmatrix} \Delta' \mathbf{c}_a(\Delta \mathbf{h}) \\ \mathbf{O} \end{pmatrix}$, $\mathbf{\Omega} \equiv \begin{pmatrix} \mathbf{0} & -\mathbf{\Gamma} \\ \mathbf{\Gamma}' & \mathbf{0} \end{pmatrix}$, and $\phi \equiv \begin{pmatrix} \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta} \end{pmatrix}$, where \mathbf{I} , $\mathbf{c}_a(\Delta \mathbf{h})$, Δ , \mathbf{O} , and $\mathbf{\Gamma}$ denote identity matrix, full link-cost vector, link-path incident matrix, full OD pair demand vector, and path-OD pair incident matrix, respectively. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are diagonal matrices with all α_p and β_w as diagonal elements, respectively. All the elements of the mentioned vectors and matrices are ordered in accordance with $\hat{\mathbf{h}}_r$. Then, we rewrite (17) as follows,

$$\begin{pmatrix} \dot{\mathbf{h}} \\ \dot{\mathbf{c}} \end{pmatrix} = - \begin{pmatrix} \hat{\mathbf{h}}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \phi \begin{pmatrix} \Delta' \mathbf{c}_a(\Delta \mathbf{h}) - \mathbf{\Gamma} \mathbf{c} \\ \mathbf{\Gamma}' \mathbf{h} - \mathbf{O} \end{pmatrix} = - \begin{pmatrix} \hat{\mathbf{h}}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \phi \left(\mathbf{M}(\mathbf{s}) + \mathbf{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} \right). \quad (48)$$

Now, we are ready to prove Theorem 3.

PROOF. Let $L : E \rightarrow R$ be a C^1 map and

$$L \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} \equiv \frac{1}{2} \left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix} \right)' \begin{pmatrix} \bar{\mathbf{h}}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1} \phi^{-1} \left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix} \right), \quad (49)$$

where $\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}$ is an equilibrium point of the proposed model. It is obvious that $L\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right) = 0$ and $L\left(\begin{pmatrix} h \\ c \end{pmatrix}\right) > 0$, if $\forall \begin{pmatrix} h \\ c \end{pmatrix} \neq \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}$ and $\begin{pmatrix} h \\ c \end{pmatrix} \in E$.

$$\begin{aligned} \dot{L}\left(\begin{pmatrix} h \\ c \end{pmatrix}\right) &= \left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \begin{pmatrix} \bar{h}_\epsilon & 0 \\ 0 & I \end{pmatrix}^{-1} \phi^{-1} \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix} \\ &= -\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \begin{pmatrix} \bar{h}_r & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{h}_\epsilon & 0 \\ 0 & I \end{pmatrix}^{-1} \phi^{-1} \phi \left(M(s) + \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right) \\ &= -\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \begin{pmatrix} \bar{h}_r(\bar{h}_\epsilon)^{-1} & 0 \\ 0 & I \end{pmatrix} \left(M(s) + \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right) \\ &= -\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \begin{pmatrix} h_+(\bar{h}_\epsilon+)^{-1} & 0 & 0 \\ 0 & h_-(\bar{h}_\epsilon-)^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} \left(M(s) + \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right) \end{aligned}$$

Because $c_a(\Delta h)$ is a strict monotone function, we have $(s - \bar{s})' \left(\begin{pmatrix} c_a(\Delta h) \\ -O \end{pmatrix} - \begin{pmatrix} c_a(\Delta \bar{h}) \\ -O \end{pmatrix} \right) > 0$, $\forall s \neq \bar{s} = \begin{pmatrix} \Delta \bar{h} \\ \bar{c} \end{pmatrix}$ and $s \in E$, i.e., $(s - \bar{s})' \left(\begin{pmatrix} c_a(\Delta h) \\ -O \end{pmatrix} \right) > (s - \bar{s})' \left(\begin{pmatrix} c_a(\Delta \bar{h}) \\ -O \end{pmatrix} \right)$, and this implies $\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \left(\begin{pmatrix} \Delta' c_a(\Delta h) \\ -O \end{pmatrix} \right) > \left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \left(\begin{pmatrix} \Delta' c_a(\Delta \bar{h}) \\ -O \end{pmatrix} \right)$. So, we have

$$\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' M(s) > \left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' M(\bar{s}).$$

From the analysis of the steady state, we also have

$$\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \left(M(\bar{s}) + \Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)\right) = 0 \quad \text{and} \quad \left(\begin{pmatrix} h \\ c \end{pmatrix}\right)' \left(M(\bar{s}) + \Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)\right) \geq 0,$$

this implies $\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' M(\bar{s}) \geq \left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)$. Hence, we have

$$\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' M(s) > \left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix} - \begin{pmatrix} h \\ c \end{pmatrix}\right)' \Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right). \quad (50)$$

By adding $\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)$ to both sides of inequality (50) gives

$$\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \left(M(s) + \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right) > \left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix} - \begin{pmatrix} h \\ c \end{pmatrix}\right)' \left(\Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right) - \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right),$$

i.e.,

$$\left(\begin{pmatrix} h \\ c \end{pmatrix} - \begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix}\right)' \left(M(s) + \Omega\left(\begin{pmatrix} h \\ c \end{pmatrix}\right)\right) > \left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix} - \begin{pmatrix} h \\ c \end{pmatrix}\right)' \Omega\left(\begin{pmatrix} \bar{h} \\ \bar{c} \end{pmatrix} - \begin{pmatrix} h \\ c \end{pmatrix}\right).$$

For further calculating, we have

$$\begin{aligned} \left(\begin{pmatrix} \bar{h} - h \\ \bar{c} - c \end{pmatrix}\right)' \begin{pmatrix} 0 & -\Gamma \\ \Gamma' & 0 \end{pmatrix} \left(\begin{pmatrix} \bar{h} - h \\ \bar{c} - c \end{pmatrix}\right) &= \left(\begin{pmatrix} (\bar{c} - c)' \Gamma' \\ (\bar{h} - h)' (-\Gamma) \end{pmatrix}\right)' \begin{pmatrix} (\bar{h} - h) \\ (\bar{c} - c) \end{pmatrix} \\ &= (\bar{c} - c)' \Gamma' (\bar{h} - h) - (\bar{h} - h)' \Gamma (\bar{c} - c) \\ &= 0, \end{aligned}$$

i.e., $\left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}\right)' \left(\mathbf{M}(\mathbf{s}) + \boldsymbol{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}\right) > 0$ and equivalently, $-\left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}\right)' \left(\mathbf{M}(\mathbf{s}) + \boldsymbol{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}\right) < 0$. Now, recall $\dot{L} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} = -\left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}\right)' \begin{pmatrix} \mathbf{h}_+ (\bar{\mathbf{h}}_{\varepsilon+})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_- (\bar{\mathbf{h}}_{\varepsilon-})^{-1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{pmatrix} \left(\mathbf{M}(\mathbf{s}) + \boldsymbol{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}\right)$, by the definitions of \mathbf{h}_+ , \mathbf{h}_- , $\bar{\mathbf{h}}_{\varepsilon+}$, and $\bar{\mathbf{h}}_{\varepsilon-}$, in (44)–(47), we have

$$\begin{aligned} \left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}\right)' \begin{pmatrix} \mathbf{h}_+ (\bar{\mathbf{h}}_{\varepsilon+})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_- (\bar{\mathbf{h}}_{\varepsilon-})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \left(\mathbf{M}(\mathbf{s}) + \boldsymbol{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}\right) \\ > \left(\begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \bar{\mathbf{h}} \\ \bar{\mathbf{c}} \end{pmatrix}\right)' \left(\mathbf{M}(\mathbf{s}) + \boldsymbol{\Omega} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix}\right). \end{aligned}$$

Accordingly, we have $\dot{L} \begin{pmatrix} \mathbf{h} \\ \mathbf{c} \end{pmatrix} < 0$.

Hence, (49) is affirmed as a strict Lyapunov function of dynamical system (9). Then, asymptotic stability is immediate from Theorem 2.

For the multiclass users model shown in (16), we replace (44) and (45) with

$$\hat{\mathbf{h}}_{ir} \equiv \begin{pmatrix} \mathbf{h}_{i+} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{i-} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{h}}_{Nr} \equiv \begin{pmatrix} \hat{\mathbf{h}}_{1r} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{h}}_{nr} \end{pmatrix} \quad (51)$$

and

$$\bar{\mathbf{h}}_{N\varepsilon} \equiv \begin{pmatrix} \bar{\mathbf{h}}_{1\varepsilon} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{h}}_{n\varepsilon} \end{pmatrix}, \quad \text{where } \bar{\mathbf{h}}_{i\varepsilon} \equiv \begin{pmatrix} \bar{\mathbf{h}}_{i\varepsilon+} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{h}}_{i\varepsilon-} \end{pmatrix}, \quad (52)$$

where $\hat{\mathbf{h}}_{ir}$, $\bar{\mathbf{h}}_{i\varepsilon+}$, and $\bar{\mathbf{h}}_{i\varepsilon-}$ denote diagonal matrices of user class i defined by the same rule as $\hat{\mathbf{h}}_r$, $\bar{\mathbf{h}}_{\varepsilon+}$, and $\bar{\mathbf{h}}_{\varepsilon-}$ in (44) and (45), respectively. Now, the whole system dynamics can be rewritten as

$$\begin{aligned} \dot{\mathbf{s}}_N &= - \begin{pmatrix} \hat{\mathbf{h}}_{Nr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix} \phi_N \begin{pmatrix} \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) - \boldsymbol{\Gamma} \mathbf{c}_1 \\ \vdots \\ \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) - \boldsymbol{\Gamma} \mathbf{c}_n \\ \boldsymbol{\Gamma}' \mathbf{h}_1 - \mathbf{O}_1 \\ \vdots \\ \boldsymbol{\Gamma}' \mathbf{h}_n - \mathbf{O}_n \end{pmatrix} \\ &= - \begin{pmatrix} \hat{\mathbf{h}}_{Nr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix} \phi_N (\mathbf{M}_N(\mathbf{s}_N) + \boldsymbol{\Omega}_N \mathbf{s}_N), \end{aligned}$$

where

$$\mathbf{s}_N \equiv \begin{pmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_n \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_n \end{pmatrix}, \quad \mathbf{M}_N(\mathbf{s}_N) \equiv \begin{pmatrix} \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) \\ \vdots \\ \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) \\ -\mathbf{O}_1 \\ -\mathbf{O}_n \end{pmatrix},$$

$$\Omega_N \equiv \begin{pmatrix} 0 & \cdots & 0 & -\Gamma_1 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & -\Gamma_n \\ \Gamma'_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \Gamma'_n & 0 & \cdots & 0 \end{pmatrix}, \quad \text{with } \Gamma_i = \Gamma,$$

$$\phi_N \equiv \begin{pmatrix} \alpha_1 & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \alpha_n & 0 & \vdots \\ \vdots & & 0 & \beta_1 & \ddots \\ & \ddots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & \beta_n \end{pmatrix}$$

and α_i and β_i are diagonal matrices with their elements to be α_{ip} and β_{iw} , respectively, \mathbf{h}_i , \mathbf{c}_i , \mathbf{O}_i denote path flows, predicted OD travel times and OD demands, respectively of user class i . Then, a strict Lyapunov function of the multiclass user dynamical system described as (53) can be proposed as

$$\mathbf{L}_N(\mathbf{s}_N) \equiv \frac{1}{2}(\mathbf{s}_N - \bar{\mathbf{s}}_N)' \begin{pmatrix} \bar{\mathbf{h}}_{N\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix}^{-1} \phi_N^{-1}(\mathbf{s}_N - \bar{\mathbf{s}}_N). \quad (54)$$

It is easy to check (53) by satisfying the two conditions in Definition 1 in a similar way used in the proof of Theorem 3. Finally, the multi-class user dynamical system described as (15) can be reformulated as

$$\dot{\mathbf{s}}_N = - \begin{pmatrix} \hat{\mathbf{h}}_{Nr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix} \phi_N \begin{pmatrix} \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) - \Gamma \mathbf{c} \\ \vdots \\ \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) - \Gamma \mathbf{c} \\ \left(\sum_{i=1}^n \Gamma' \mathbf{h}_i \right) - \mathbf{O} \end{pmatrix} \\ = - \begin{pmatrix} \hat{\mathbf{h}}_{Nr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix} \phi_N (\mathbf{M}_N(\mathbf{s}_N) + \Omega_N \mathbf{s}_N), \quad (55)$$

with the same function form of (53) to be a strict Lyapunov function but newly defined

$$\mathbf{s}_N \equiv \begin{pmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_n \\ \mathbf{c} \end{pmatrix}, \quad \mathbf{M}_N(\mathbf{s}_N) \equiv \begin{pmatrix} \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) \\ \vdots \\ \Delta' \mathbf{c}_a \left(\sum_{i=1}^n \Delta \mathbf{h}_i \right) \\ -\mathbf{O} \end{pmatrix},$$

$$\Omega_N \equiv \begin{pmatrix} 0 & \cdots & 0 & -\Gamma_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\Gamma_n \\ \Gamma'_1 & \cdots & \Gamma'_n & 0 \end{pmatrix}, \quad \text{with } \Gamma_i = \Gamma, \quad \text{and} \quad \phi_N \equiv \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_n & 0 \\ 0 & \cdots & 0 & \beta \end{pmatrix}.$$

5. NUMERICAL EXAMPLES

A simple network with four nodes and five links illustrated as Figure 1 is used to show the numerical results of the proposed models. There is only one OD pair $w = \{\text{node 1, node 4}\}$ which

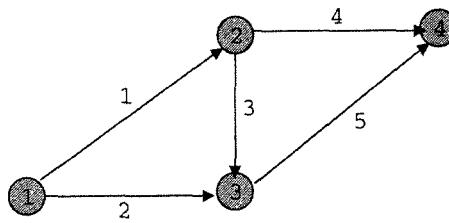


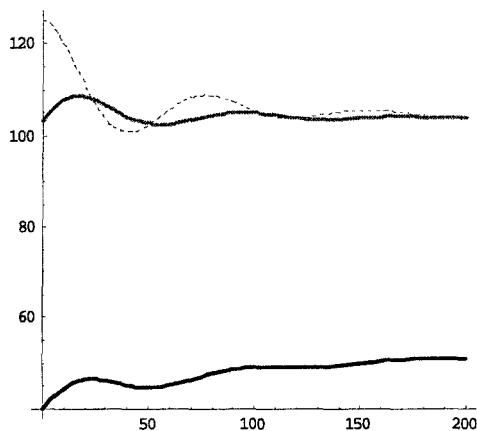
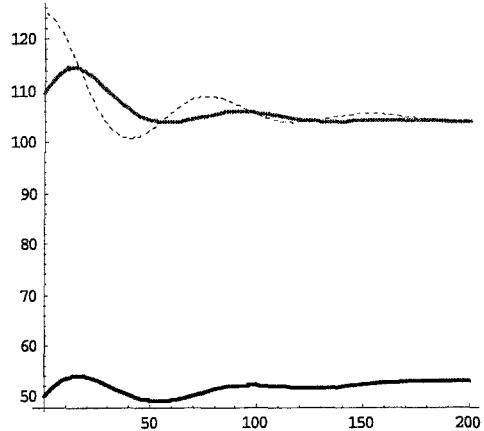
Figure 1. Graph of numerical example.

Table 1. Parameters of link cost function.

Links	A_a	B_a	k_a
1	40	20	80
2	60	30	80
3	20	10	120
4	50	25	25
5	30	15	80

Table 2. Results of numerical example for homogeneous user model (M1).

	Flow			Travel Time		
	Initial	State at $t = 200$	Steady State	Initial	State at $t = 200$	Steady State
Path 1	40	51.06	56.16	103.29	103.84	103.79
Path 2	50	53.13	56.95	109.58	104.05	103.79
Path 3	30	15.69	6.89	116.76	107.91	103.80
Link 1	70	66.75	63.05	51.72	49.69	47.72
Link 2	50	53.13	56.95	64.58	65.84	67.70
Link 3	30	15.69	6.89	20.04	20.01	20.00
Link 4	40	51.06	56.16	51.56	55.15	56.07
Link 5	80	68.82	63.84	45.00	38.22	36.08
Predicted OD travel time by ATIS			125.00	104.25	103.79	

Figure 2. Evolutions of path 1 flow dynamics and unit $PTTL_{path 1,w}^t$ (M1). (Dashed line: predicted OD travel time by ATIS; gray line: travel time of path 1; black line: flow of path 1.)Figure 3. Evolutions of path 2 flow dynamics and unit $PTTL_{path 2,w}^t$ (M1). (Dashed line: predicted OD travel time by ATIS; gray line: travel time of path 2; black line: flow of path 2.)

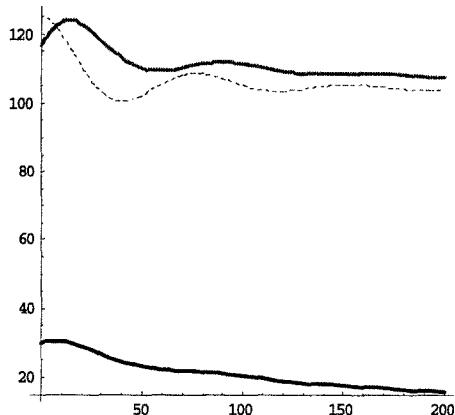


Figure 4. Evolutions of path 3 flow dynamics and unit $PTTL_{path 3,w}^t$ (M1). (Dashed line: predicted OD travel time by ATIS; gray line: travel time of path 3; black line: flow of path 3.)

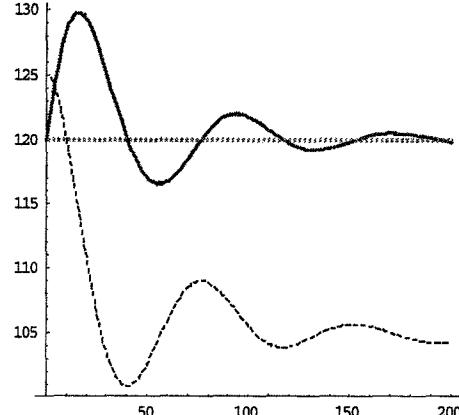


Figure 5. Evolutions of predicted OD travel time by ATIS and ETD_w^t (M1). (Dashed line: predicted OD travel time by ATIS; dotted line: OD demand; gray line: sum of path flows.)

Table 3. Results of numerical example for M2. Two-class users model with identical predicted OD travel time for two user classes, i.e., (15) with $n = 2$; C1 denotes user class one and C2 user class two.

		Flow			Travel Time		
		Initial	State at $t = 200$	Steady State	Initial	State at $t = 200$	Steady State
Path 1	C1	20	21.95	22.23	103.29	103.79	103.79
	C2	20	31.84	33.94			
Path 2	C1	25	25.81	26.08	109.58	103.80	103.79
	C2	25	29.30	30.88			
Path 3	C1	15	9.55	6.62	116.76	105.72	103.79
	C2	15	1.57	0.25			
Link 1	C1	35	31.50	28.85	51.72	48.67	47.71
	C2	35	33.41	34.19			
Link 2	C1	25	25.81	26.08	64.58	66.76	67.71
	C2	25	29.30	30.88			
Link 3	C1	15	9.55	6.62	20.04	20.00	20.00
	C2	15	1.57	0.25			
Link 4	C1	20	21.95	22.23	51.56	55.11	56.08
	C2	20	31.84	33.94			
Link 5	C1	40	35.36	32.70	45.00	37.05	36.08
	C2	40	30.87	31.13			
Predicted OD travel time by ATIS				125.00	103.88	103.79	

is connected by three paths denoted as path 1 = {link 1, link 4}, path 2 = {link 2, link 5}, and path 3 = {link 1, link 3, link 5}, respectively. The parameters of the link cost functions are set in Table 1 with the same function form as $c_a(f_a^t) = A_a + B_a(f_a^t/k_a)^4$.

The following three examples (M1, M2, and M3) were solved using the high-order Runge-Kutta numerical method with the OD demand fixed as 120. The initial conditions, assumed identical parameters of the propensity of path flow dynamics, and parameters to denote the sensitivity of predicted travel time dynamics for the homogeneous user model (M1), i.e., (9), are given as $(h_1^0, h_2^0, h_3^0, c_1^0, \alpha, \beta) = (40, 50, 30, 125, -0.0006, 0.1)$. Table 2 shows the dynamics of flows, travel times, and predicted OD travel times by ATIS at three different states for the first example.

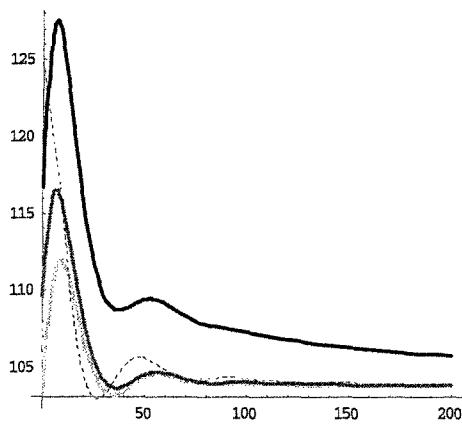


Figure 6. Travel time evolutions of M2. (Dashed line: predicted OD travel time; light gray line: path 1; medium gray line: path 2; black line: path 3.)

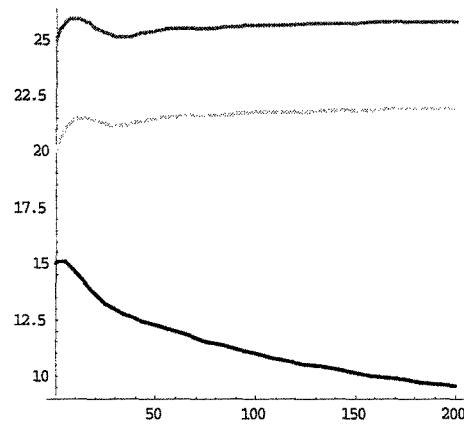


Figure 7. Path flow evolutions of user class one in M2. (Light gray line: path 1; medium gray line: path 2; black line: path 3.)

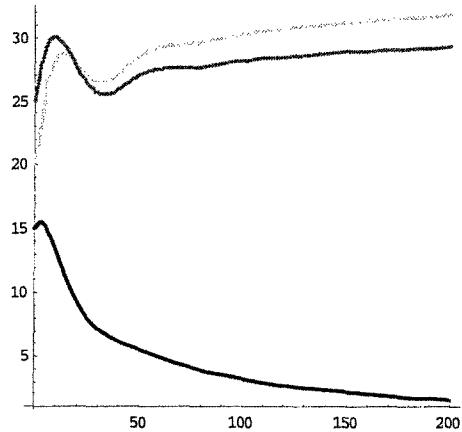


Figure 8. Path flow evolutions of user class two in M2. (Dashed line: predicted OD travel time; light gray line: path 1; medium gray line: path 2; black line: path 3.)

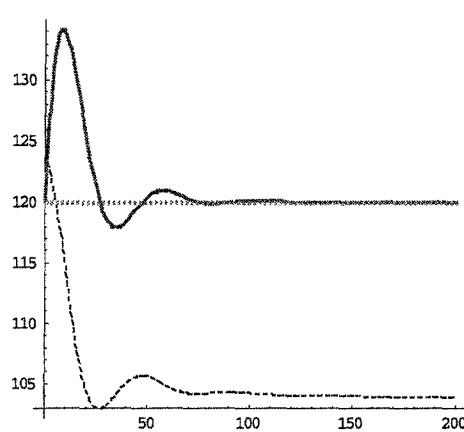


Figure 9. Evolutions of predicted OD travel time by ATIS and ETD_w^t (M2). (Dashed line: predicted OD travel time by ATIS; dotted line: OD demand; gray line: sum of path flows.)

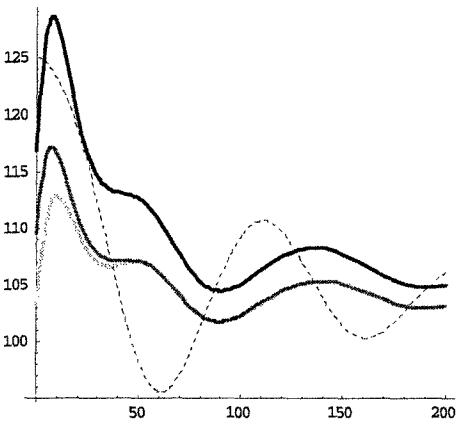


Figure 10. Evolutions of path travel times and C1 predicted OD travel times for M3. (Dashed line: C1 predicted OD travel time; light gray line: path 1; medium gray line: path 2; black line: path 3.)

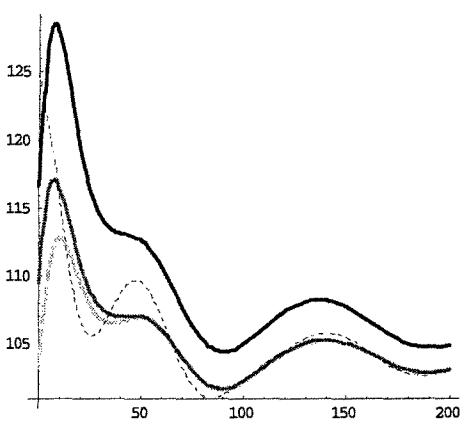


Figure 11. Evolutions of path travel times and C2 predicted OD travel times for M3. (Dashed line: C2 predicted OD travel time; light gray line: path 1; medium gray line: path 2; black line: path 3.)

Table 4. Results of numerical example for M3. Two-class users model with user-specific predicted OD travel time for two user classes, i.e., (16) with $n=2$; C1 denotes user class one and C2 user class two.

		Flow			Travel Time		
		Initial	State at $t = 200$	Steady State	Initial	State at $t = 200$	Steady State
Path 1	C1	20	22.62	24.56	103.29	103.04	103.78
	C2	20	30.40	31.60			
Path 2	C1	25	26.56	28.71	109.58	103.09	103.78
	C2	25	27.80	28.24			
Path 3	C1	15	9.66	6.72	116.76	104.91	103.78
	C2	15	1.36	0.15			
Link 1	C1	35	32.28	31.28	51.72	48.21	47.71
	C2	35	31.76	31.75			
Link 2	C1	25	26.56	28.71	64.58	66.40	67.70
	C2	25	27.80	28.24			
Link 3	C1	15	9.66	6.72	20.04	20.00	20.00
	C2	15	1.36	0.15			
Link 4	C1	20	22.62	24.56	51.56	54.82	56.07
	C2	20	30.40	31.60			
Link 5	C1	40	36.22	35.43	45.00	36.69	36.08
	C2	40	29.16	28.39			
Predicted OD travel time by ATIS				C1	125.00	103.88	103.79
				C2	125.00	102.99	103.79

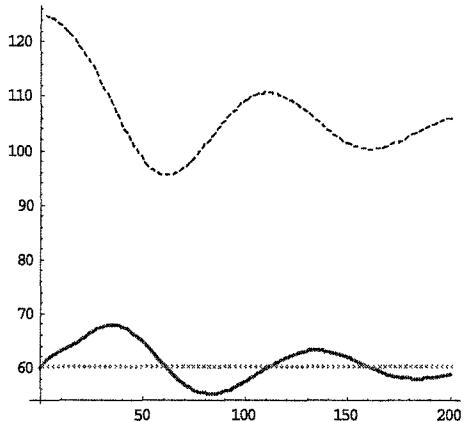


Figure 12. Evolutions of C1 predicted OD travel time by ATIS and ETD_{1w}^t (M3). (Dashed line: C1 predicted OD travel time by ATIS; dotted line: OD demand; gray line: sum of path flows for C1.)

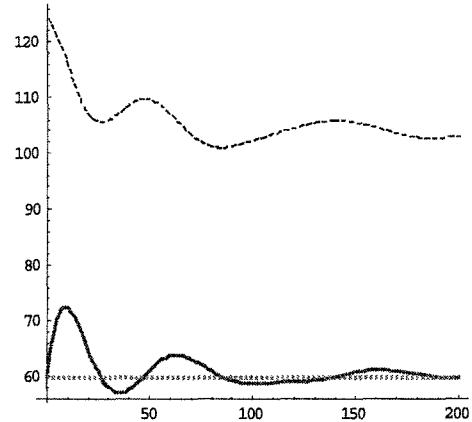


Figure 13. Evolutions of C2 predicted OD travel time by ATIS and ETD_{2w}^t (M3). (Dashed line: C2 predicted OD travel time by ATIS; dotted line: OD demand; gray line: sum of path flows for C2.)

It is clear that the steady state satisfies the Wardrop's user equilibrium and the predicted OD travel time is equal to the path travel times of which path flows are positive simultaneously. Numerical results by the evolutions of network dynamics illustrated from Figures 2-5 also show that the path flow increases (decreases) as path travel time is less (more) than the predicted OD

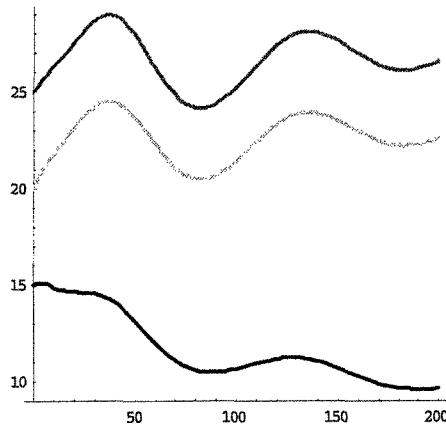


Figure 14. Path flow evolutions of C1 in M3. (Light gray line: path 1; medium gray line: path 2; black line: path 3.)

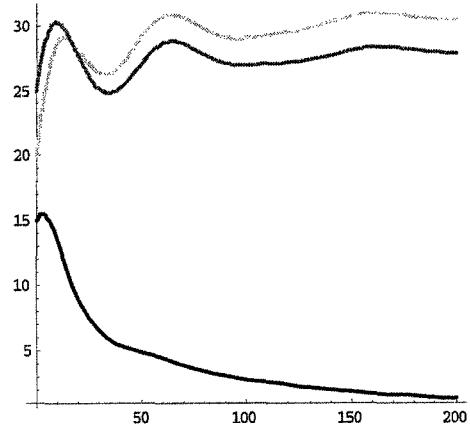


Figure 15. Path flow evolutions of C2 in M3. (Light gray line: path 1; medium gray line: path 2; black line: path 3.)

travel time and predicted OD travel time increases (decreases) as OD demand is more (less) than the sum of path flows.

The second example is set for the multiclass user model with identical predicted OD travel time by ATIS (M2), i.e., (15) as $n = 2$. Parameters of the propensity of path flow dynamics for the two classes are assumed to be different but the same within a user class. They are -0.0006 and -0.003 for class one and class two, respectively. The other inputs of the second example are the same as the previous case, and the outputs reveal that the steady state is qualified as a Wardrop's user equilibrium. The embedded mechanisms of adjustments are also sustained by the results in Table 3 and Figures 6–9. However, two additional facts are found that sensible users occupy better routes sooner than less sensible users in the evolution process until equilibrium is reached, and the ratio of first-class users to second-class users for three paths at a steady state is different from that at the initial status. The latter one can be interpreted as the result of the former and evidently due to various parameters of the propensity of path flow dynamics. Moreover, we remind readers that the shares of OD demand for two user classes might be changed in the processes.

Finally, the third case is prepared for the two-class user model with the user-specific predicted OD travel time by ATIS (M3), i.e., (16) as $n = 2$. All inputs are the same as the second example but each user class is provided with a dedicated OD travel time predicted by ATIS and an equal OD demand, i.e., $D_{11} = D_{21} = 60$. Unsurprisingly, the outcome, shown in Table 4 and Figures 10–15, follows the asymptotic behaviors claimed in the previous section. It is also observed that the adjustment speeds in M3 are less than that in M2 from the initial conditions to the 200th time state. It is in that the path flow dynamics are limited by a fixed and smaller OD demand for user class two in M3. However, it seems to be reasonable that the OD demand ratio of user class one to user class two should not be overly distorted in the evolution processes even though the total OD demand is kept unchanged. This point should be a necessary check when M2 is implemented into the empirical study.

6. CONCLUSIONS

In this paper, the authors deal with further developments of vehicular network dynamics in a day-to-day time scale by using a nonlinear dynamical system approach. Incorporating the total perceived travel time loss (or saving) into the proposed models, time change rates of path flows are generated on a flow-weighted base to prevent the path flow dynamics from being insensible of flow level which have been formulated in previous studies [4,24]. Issues of heterogeneous users and corresponding means of providing travel information are also considered in multiclass users models

by dividing path users into several classes according to the sensitivity of path flow dynamics due to the total perceived travel time loss (or saving) for each user class. The equilibrium solutions of presented models are analyzed mathematically to be the Wardropian equilibria and proved to be asymptotically stable in the sense of Lyapunov. The lemma of the Lipschitz condition for the proposed dynamical system is a key in the proof of existence and uniqueness by the way of the fundamental theorem of differential equations. Based on these results, the proposed models build an analytical linkage between the Wardrop's user equilibrium and the empirical adaptability of route preference under the operations of intelligent transportation systems.

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