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Process capability assessment for index C_{pk} based on bayesian approach

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Abstract. Process capability indices have been proposed in the manufacturing industry to provide numerical measures on process reproduction capability, which are effective tools for quality assurance and guidance for process improvement. In process capability analysis, the usual practice for testing capability indices from sample data are based on traditional distribution frequency approach. Bayesian statistical techniques are an alternative to the frequency approach. Shiau, Chiang and Hung (1999) applied Bayesian method to index C_{pm} and the index C_{pk} but under the restriction that the process mean μ equals to the midpoint of the two specification limits, m. We note that this restriction is a rather impractical assumption for most factory applications, since in this case C_{pk} will reduce to C_p. In this paper, we consider testing the most popular capability index C_{pk} for general situation – no restriction on the process mean based on Bayesian approach. The results obtained are more general and practical for real applications. We derive the posterior probability, p, for which the process under investigation is capable and propose accordingly a Bayesian procedure for capability testing. To make this Bayesian procedure practical for in-plant applications, we tabulate the minimum values of $\hat{C}_{pk}$ for which the posterior probability p reaches desirable confidence levels with various pre-specified capability levels.

Key words: Bayesian approach, posterior distribution, process capability indices, posterior probability

1 Introduction

Process capability indices (PCI), C_p, C_{pk}, C_{pm} and C_{pmk} have been proposed in the manufacturing industry and the service industry providing numerical measures on whether a process is capable of reproducing items within specification limits preset in the factory (see Kane (1986), Chan, Cheng and Spiring (1988), Pearn, Kotz and Johnson (1992), Kotz and Lovelace (1998)). These indices have been defined as:

$$C_p = \frac{USL - LSL}{6\sigma}, C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}.$$

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean, σ is the process standard deviation (overall process variation), and T is the target value. The index C_p considers the overall process variability relative to the manufacturing tolerance, reflecting product quality consistency. The index C_{pk} takes the magnitude of process variance as well as process departure from target value, and has been regarded as a yield-based index since it providing lower bounds on process yield. The index C_{pm} emphasizes on measuring the ability of the process to cluster around the target, which therefore reflects the degrees of process targeting (centering). Since the design of C_{pm} is based on the average process loss relative to the manufacturing tolerance, the index C_{pm} provides an upper bound on the average process loss, which has been alternatively called the Taguchi index. The index C_{pmk} is constructed from combining the modifications to C_p that produced C_{pk} and C_{pm}, which inherits the merits of both indices.

Process yield is currently defined as the percentage of the processed product units passing the inspections. Units are inspected according to specification limits placed on various key product characteristics and sorted into two categories: passed (conforming) and rejected (defectives). Thus, yield is one of the commonly understood basic criteria for interpretations of the process capability. Suppose a proportion conforming items is the primary concern, then most natural measure is the proportion itself called the yield, which we refer to as X defined as:

$$Y = \int_{LSL}^{USL} dF(x) = F(USL) - F(LSL), \tag{1}$$

where $F(x)$ is the cumulative distribution function of the measured characteristic X, USL and LSL are the upper and the lower specification limits respectively. As the index C_{pk} provides a lower bound on the process yield, a widely used criterion for measuring process, it has become the most popular capability index used in the industry. Existing methods for testing the capability indices have focused on using the traditional but long time been widely used distribution frequency approaches. The usual practice of judging process capability by evaluating the point estimates of process capability indices is highly unreliable, as there is no assessment on the error distributions of these estimates. A point estimate to the index is not very useful in making reliable decision. Interval estimation approach, in fact, is more appropriate and widely accepted. But the frequency distributions of these estimates are usually complicated that it is very difficult to obtain exact interval estimates. A process is usually defined as a capable process if its capability exceeds a pre-specified value w. A reliable approach for testing process capability is to establish an interval estimate, for which we can assert with a reasonable degree of certainty that it contains the true PCI value. However, the construction of such an interval estimate is not trivial, since the distributions of the commonly used PCI estimators are usually quite complicated.

An alternative is to use the Bayesian approach, which essentially specifies a prior distribution for the parameter of interest, and to obtain the posterior distribution of the parameter and then infer about the parameter by only using its posterior distribution given the observations. It is not difficult to obtain the posterior distribution when a prior distribution is given, even in the case where the form of the posterior distribution is complicated, as one could always use numerical methods or Monte Carlo methods to obtain an approximate but quite accurate interval estimate. This is the advantage of the Bayesian approach over the traditional distribution frequency approach (Kalos and Whitlock (1986)).

In this paper, we consider testing the most popular capability index C_{pk} using Bayesian approach. We obtain the posterior probability p for which the process under investigation is capable, and propose accordingly a Bayesian procedure for capability testing. To make this Bayesian procedure practical for in-plant applications, we tabulate the minimum values of $\hat{C}_{pk}$ for which the posterior probability p reaches various desirable confidence levels. An application example to the oil-hydraulic cylinders manufacturing process is presented to illustrate the applicability of the proposed approach. Finally, some concluding remarks are made in Section 7.

2 Distribution frequency approach for C_{pk}

Utilizing the identity $\min\{a,b\} = (a+b) - |a-b|/2$, the definition of the index C_{pk} can be alternatively written as:

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma}, \tag{2}$$

where $d = (USL - LSL)/2$ is half of the length of the specification interval, $m = (USL + LSL)/2$ is the mid-point between the lower and the upper specification limits. The natural estimator $\hat{C}_{pk}$ is obtained by replacing the process mean μ and the process standard deviation σ by their conventional estimators $\bar{x}$ and s, which may be obtained from a process that is demonstrably stable (under statistical control).

$$\hat{C}_{pk} = \frac{d - |\bar{x} - m|}{3s} = \left\{1 - \frac{|\bar{x} - m|}{d}\right\}\hat{C}_p, \tag{3}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $s = [\sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)]^{1/2}$. Under the assumption of normality, Kotz and Johnson (1993) obtained the r-th moment, and the first two moments as well as the mean and the variance of $\hat{C}_{pk}$. In addition, numerous methods for constructing approximate confidence intervals of C_{pk} have been proposed in the literature. Examples include Chou, Owen and Borrego (1990), Zhang, Stenback and Wardrop (1990), Franklin and Wasserman (1991), Kushler and Hurley (1992), Nagata and Nagahata (1994), Tang, Than and Ang (1997), Hoffman (2001), and many others.

Kotz and Johnson (2002) presented a thorough review for the development of process capability indices during the years 1992 to 2000. Furthermore, from the estimated $\hat{C}_{pk}$ defined in (1), since $\hat{C}_p$ is distributed as $(\mathrm{n}-1)^{1/2}C_p(\chi^{-1}_{\mathrm{n}-1})$, and $n^{1/2}|\bar{x}-m|/\sigma$ is distributed as the folded normal distribution with parameter $n^{1/2}|\mu-m|/\sigma$ (see Leone, Nelson and Nottingham (1961) for details about this distribution). Thus, $\hat{C}_{pk}$ is a mixture of χ^{-1}_{n-1} and the folded normal distribution (Pearn, Kotz and Johnson (1992)). The probability density function of $\hat{C}_{pk}$ can be obtained as (Pearn, Chen and Lin (1999)), where $D=(n-1)^{1/2}d/\sigma$, $a=[(n-1)/n]^{1/2}$.

$$f_{\hat{C}_{pk}}(y)=\begin{cases} 4A_n\sum_{\ell=0}^{\infty}P_\ell(\lambda)B_\ell\times\frac{D^{n+2\ell}}{a^{2\ell+1}}\int_0^{\infty}(1-yz)^{2\ell}z^{n-1} \\ \times\exp\left\{-\frac{D^2}{18a^2}\left(a^2z^2+9(1-yz)^2\right)\right\}dz, \quad y\le 0, \\ 4A_n\sum_{\ell=0}^{\infty}P_\ell(\lambda)B_\ell\times\frac{D^{n+2\ell}}{a^{2\ell+1}}\int_0^{1/y}(1-yz)^{2\ell}z^{n-1} \\ \times\exp\left\{-\frac{D^2}{18a^2}\left(a^2z^2+9(1-yz)^2\right)\right\}dz, \quad y>0, \end{cases} \tag{4}$$

$$P_\ell(\lambda)=\frac{e^{-(\lambda/2)}(\lambda/2)^\ell}{\ell!},A_n=\frac{1}{3^{n-1}2^{n/2}\Gamma((n-1)/2)},B_\ell=\frac{1}{2^\ell\Gamma((2\ell+1)/2)}.$$

Using the integration technique similar to that presented in Vännman (1997), Pearn and Lin (2003) first obtain an exact and explicit form of the cumulative distribution function of the natural estimator $\hat{C}_{pk}$, under the assumption of normality. The cumulative distribution function of $\hat{C}_{pk}$ is expressed in terms of a mixture of the chi-square distribution and the normal distribution:

$$F_{\hat{C}_{pk}}(y)=1-\int_0^{b\sqrt{n}}G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9ny^2}\right)\left[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})\right]dt, \tag{5}$$

for $y>0$, where $b=d/\sigma$, $\xi=(\mu-m)/\sigma$, $G(\cdot)$ is the cumulative distribution function of the chi-square distribution with degree of freedom $n-1$, χ^2_{n-1}, and $\phi(\cdot)$ is the probability density function of the standard normal distribution $N(0,1)$. Based on the cumulative distribution function of $\hat{C}_{pk}$, Pearn and Lin (2003) implemented the statistical theory of the hypotheses testing, and developed a simple but practical procedure accompanied with convenient tabulated critical values, for engineers/practitioners to use for decisions making in their factory applications. Formulae for computing the power of the corresponding test are also obtained. Pearn and Shu (2003) further developed an efficient algorithm with Matlab computer program to find the exact (rather than just approximate) lower confidence bounds conveying critical information regarding the true process capability. An illustrative application of the lower confidence bound to the power distribution switch was given. Their investigations are all based on traditional distribution frequency approaches.

3 Bayesian approach for C_{pk}

Cheng and Spiring (1989) proposed a Bayesian procedure for assessing process capability index C_p. Shiau, Hung and Chiang (1999) derived the posterior distributions for C_p^2, C_{pm}^2 under the restriction that process mean μ equals to the target value T, and C_{pk}^2 under the restriction that the process mean μ equals to the midpoint of the two specification limits, m, with respect to the two priors (a non-informative and a Gamma prior). Shiau, Chiang and Hung (1999) applied Bayesian method to index C_{pm} relaxing the restriction on $\mu = T$. Wu and Pearn (2003) generalized this Bayesian procedure for testing process capability index C_{pm} to cases where data are collected over time as multiple samples. Shiau, Chiang and Hung (1999) also applied a similar Bayesian approach for testing the index C_{pk} but under the restriction $\mu = m$. We note that this restriction is a rather impractical assumption for most factory applications, since in this case C_{pk} will reduce to C_p. In the following, we consider a Bayesian procedure for the capability index C_{pk} for general situation – no restriction on the process mean. Thus, the results obtained are more general and practical for real applications. A $100p\%$ credible interval is the Bayesian analogue of the classical $100p\%$ confidence interval, where p is the confidence level for the interval. The credible interval covers $100p\%$ of the posterior distribution of the parameter (Berger (1980)). Assuming that the measures $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ are random sample taken from independent and identically distributed (*i.i.d.*) $N(\mu, \sigma^2)$, a normal distribution with mean μ and variance σ^2. Then, the likelihood function for μ and σ is

$$L(\mu, \sigma|\mathbf{x}) = \left(2\pi\sigma^2\right)^{-n/2} \times \exp\left\{-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right\} \quad (6)$$

The most important problem in Bayesian inference is how to specify an appropriate prior distribution. If prior information about the parameters is available, it should be incorporated in the prior density. If we have no prior information, we want a prior with minimal influence on the inference. There are mainly two types of priors: informative and non-informative. Ideally, a Bayesian should subjectively elicit a prior on the basis of available information, expert opinion or past experience. Informative prior distributions summarize the evidence about the parameters concerned from many sources and often have a considerable impact on the results. For an example of informative priors, conjugate priors, although being widely used, can only be justified if enough information is available to believe that the true prior distribution belongs to the specified family; otherwise, the main justification for using conjugate prior is their mathematical tractability.

On the other hand, non-informative prior, Bayesian analysis often leads to the procedures with approximate frequency validity while retaining the Bayesian flavor, thus allowing certain amount of reconciliation between the two conflicting paradigms of statistics and providing with mutual justification. Box and Tiao (1973) define a non-informative prior as prior, which provides little information relative to the experiment. Bernardo and Smith (1993) use a similar definition, they say that non-informative prior have minimal effect relative to the idea, on the final inference. And Kass and Wasserman (1996) stated two interpretations of non-informative priors.

Therefore, the first step for the Bayesian approach is to find an appropriate prior. Usually, when there is only a little or no prior information is available, or only one parameter of interest, one of the most widely used non-informative priors is the so-called reference prior, which is a non-informative prior that maximizes the difference between information (entropy) on the parameter provided by the prior and by the posterior. In other words, the reference prior allows the prior to provide information about the parameter as little as possible (see Bernardo and Smith (1993) for more details). For this reason, in this paper we adopt the following non-informative reference prior,

$$\pi(\mu, \sigma) = 1/\sigma, \quad 0 < \sigma < \infty. \tag{7}$$

We note that the parameter space of the prior is infinite, hence the reference prior is improper, which means that it does not integrate to one. However, it is not always a serious problem, since the prior incorporate with ordinary likelihood will lead to proper posterior. Furthermore, the credible interval obtained from a non-informative prior has a more precise coverage probability than that obtained from any other priors. The posterior probability density function (PDF), $f(\mu, \sigma|\mathbf{x})$ of (μ, σ) may be expressed as the following:

$$f(\mu, \sigma|\mathbf{x}) \propto L(\mu, \sigma|\mathbf{x}) \times \pi(\mu, \sigma) \propto \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right)$$

Since

$$\int_0^\infty \int_{-\infty}^\infty \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^\infty \sigma^{-(n+1)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\int_{-\infty}^\infty \exp\left(-\frac{n(\mu - \bar{x})^2}{2\sigma^2}\right) d\mu\right] d\sigma = \sqrt{\frac{\pi}{2n}}\Gamma(\alpha)\beta^\alpha$$

And in order to satisfy the integration property, probability over PDF is 1, so that

$$f(\mu, \sigma|\mathbf{x}) = \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right) \tag{8}$$

where $\alpha = (n-1)/2$, $\beta = [\sum_{i=1}^{n}(x_i - \bar{x})^2/2]^{-1} = [(n-1)s^2/2]^{-1}$.

Subsequently, we consider the quantity Pr{process is capable $|\mathbf{x}$} in the Bayesian approach. Since the index C_{pk} is our focus in this paper, so we are interested in finding the posterior probability $p = \Pr\{C_{pk} > w|\mathbf{x}\}$ for some fixed positive number w.

4 The posterior probability

Given a pre-specified capability level $w > 0$, the posterior probability based on index C_{pk} that a process is capable can be derived in the following way. From equation (8), we have the posterior probability density function (PDF) $f(\mu, \sigma|\mathbf{x})$ of (μ, σ) as the following, where

$$\alpha = (n-1)/2, \beta = \left[\sum_{i=1}^{n}(x_i - \bar{x})^2/2\right]^{-1} = [(n-1)s^2/2]^{-1},$$

$$f(\mu, \sigma|\mathbf{x}) = \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right).$$

Therefore, given a pre-specified capability level $w > 0$, the posterior probability based on index C_{pk} that a process is capable is given as

$$p=\Pr\{C_{pk}>w|\mathbf{x}\}=\Pr\left\{\frac{d-|\mu-m|}{3\sigma}>w\middle|\mathbf{x}\right\}=\Pr\{|\mu-m|<d-3\sigma w|\mathbf{x}\}$$
$$=\int_0^\infty\int_{m-d+3\sigma\omega}^{m+d-3\sigma\omega}f(\mu,\sigma|\mathbf{x})d\mu d\sigma=\int_0^\infty\int_{m-d+3\sigma\omega}^{m+d-3\sigma\omega}\frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha}\sigma^{-(n+1)}$$
$$\times\exp\left(-\frac{\sum_{i=1}^n(x_i-\mu)^2}{2\sigma^2}\right)d\mu d\sigma=\int_0^\infty\frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha}\sigma^{-(n+1)}$$
$$\times\exp\left(-\frac{1}{\beta\sigma^2}\right)\int_{m-d+3\sigma w}^{m+d-3\sigma w}\exp\left(-\frac{n(\mu-\bar{x})^2}{2\sigma^2}\right)d\mu d\sigma$$
$$=\int_0^\infty\frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right)$$
$$\times\left[\Phi\left(\frac{m+d-3\sigma w-\bar{x}}{\sigma/\sqrt{n}}\right)-\Phi\left(\frac{m-d+3\sigma w-\bar{x}}{\sigma/\sqrt{n}}\right)\right]d\sigma$$
$$=\int_0^\infty\frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right)$$
$$\times\left[\Phi\left(\frac{d-(\bar{x}-m)}{s/\sqrt{n}}\times\frac{s}{\sigma}-3\sqrt{n}w\right)+\Phi\left(\frac{d-(m-\bar{x})}{\mathrm{s}/\sqrt{n}}\times\frac{s}{\sigma}-3\sqrt{n}w\right)-1\right]d\sigma$$

Next, we consider the two cases for derivation of posterior probability p as follows:

CASE I: $\bar{x}\geq m$

If $\bar{x}\geq m$, then $\hat{C}_{pk}=\frac{d-(\bar{x}-m)}{3s}$ and $\frac{d-(m-\bar{x})}{3s}=\frac{d+(\bar{x}-m)}{3s}=\hat{C}_{pk}+\frac{2}{3}\delta$ where $\delta=\frac{|\bar{x}-m|}{s}$. Thus,

$$p=\Pr\{C_{pk}>w|\mathbf{x}\}=\int_0^\infty\frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right)\times\left[\Phi\left(3\sqrt{n}\hat{C}_{pk}\times\frac{s}{\sigma}-3\sqrt{n}w\right)\right.$$
$$\left.+\Phi\left(3\sqrt{n}\left(\hat{C}_{pk}+\frac{2}{3}\delta\right)\times\frac{s}{\sigma}-3\sqrt{n}w\right)-1\right]d\sigma$$

CASE II: $m>\bar{x}$

If $m>\bar{x}$, then $\frac{d-(\bar{x}-m)}{3s}=\frac{d+(m-\bar{x})}{3s}=\hat{C}_{pk}+\frac{2}{3}\delta$ and $\frac{d-(m-\bar{x})}{3s}=\hat{C}_{pk}$ where $\delta=\frac{|\bar{x}-m|}{s}$. Thus,

$$p=\Pr\{C_{pk}>w|\mathbf{x}\}=\int_0^\infty\frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right)$$
$$\times\left[\Phi\left(3\sqrt{n}\left(\hat{C}_{pk}+\frac{2}{3}\delta\right)\times\frac{s}{\sigma}-3\sqrt{n}w\right)\right.$$
$$\left.+\Phi\left(3\sqrt{n}\hat{C}_{pk}\times\frac{s}{\sigma}-3\sqrt{n}w\right)-1\right]d\sigma$$

From both cases,

$$p = \Pr\{C_{pk} > w|\mathbf{x}\} = \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{n}\hat{C}_{pk} \times \frac{s}{\sigma} - 3\sqrt{n}w\right)\right.$$
$$\left. + \Phi\left(3\sqrt{n}\left(\hat{C}_{pk} + \frac{2}{3}\delta\right) \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - 1\right] d\sigma$$

By changing the variable, let $y = \beta\sigma^2$, then $dy = 2\beta\sigma\, d\sigma$, and $\frac{s}{\sigma} = \sqrt{\frac{2}{(n-1)y}}$. Therefore, the posterior probability p may be rewritten as:

$$p = \Pr\{\text{the process is capable}|\mathbf{x}\} = \Pr\{C_{pk} > w|\mathbf{x}\}$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha)y^{\alpha+1}}\exp\left(-\frac{1}{y}\right) \times \left[\Phi\left(3\sqrt{n}\hat{C}_{pk} \times \sqrt{\frac{2}{(n-1)y}} - 3\sqrt{n}w\right)\right.$$
$$\left. +\Phi\left(3\sqrt{n}\left(\hat{C}_{pk} + \frac{2}{3}\delta\right) \times \sqrt{\frac{2}{(n-1)y}} - 3\sqrt{n}w\right) - 1\right] dy$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha)y^{\alpha+1}}\exp\left(-\frac{1}{y}\right) \times \left\{\Phi\left[3\sqrt{n}\left(\hat{C}_{pk} \times \sqrt{\frac{2}{(n-1)y}} - w\right)\right]\right.$$
$$\left. +\Phi\left[3\sqrt{n}\left((\hat{C}_{pk} + \frac{2}{3}\delta) \times \sqrt{\frac{2}{(n-1)y}} - w\right)\right] - 1\right\} dy$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha)y^{\alpha+1}}\exp\left(-\frac{1}{y}\right) \times \{\Phi[b_1(y)] + \Phi[b_2(y)] - 1\}\, dy, \qquad (9)$$

where $\alpha = (n-1)/2$, $\delta = \frac{|\bar{x}-m|}{s}$,

$$b_1(y) = 3\sqrt{n}\left(\hat{C}_{pk} \times \sqrt{\frac{2}{(n-1)y}} - w\right), b_2(y)$$
$$= 3\sqrt{n}\left((\hat{C}_{pk} + \frac{2}{3}\delta) \times \sqrt{\frac{2}{(n-1)y}} - w\right),$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Note that the posterior probability p depends on n, w, δ and $\hat{C}_{pk}$.

5 Bayesian procedure for testing C_{pk}

As we can see it is rather complicated to compute posterior probability p in (9) without advanced computer programming skills. However, by noticing that there is a one-to-one correspondence between p and C^* when n and w are given, and by the fact that $\hat{C}_{pk}$ can be calculated from the process data, we find that the minimum value of C^* required to ensure the posterior probability p reaching a certain desirable level, can be useful in assessing process capability. Denote this minimum value of C^* for probability p as $C^*(p)$. Thus, we can find the value of $C^*(p)$ satisfies equation (9) for various p, where p is a number between 0 and 1, say 0.95, for 95% confidence interval, which means that the posterior probability that the credible interval contains the true value of C_{pk} is p. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.00 or

1.33. From expression (9) we have the probability $p = \Pr\{C_{pk} > w|\mathbf{x}\}$ based on the observed process data. Moreover, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\hat{C}_{pk} > C^*(p)$. Throughout this paper it is assumed that the process measurements are independent and identically distributed from a normal distribution, and the process is under statistical control. We remark that estimation of these capability indices is meaningful only when the process is under statistical control.

To make this Bayesian procedure practical for in-plant applications, we calculate the values of $C^*(p)$ for various values of $n=$ 10(5)160 and $\delta=$ 0(0.5)2.0 with posterior probability $p=$ 0.90, 0.95, and 0.99 and $w=$ 1.00, 1.33, 1.50, 2.00. Tables 1(a)-1(c) summarize the values of $C^*(p)$ with $w=$ 1.00, for $p=$ 0.90, 0.95, and 0.99, respectively. Tables 2(a)-2(c) summarize the values of $C^*(p)$ with $w=$ 1.33, for $p=$ 0.90, 0.95, and 0.99, respectively. Tables 3(a)-3(c) summarize the values of $C^*(p)$ with $w=$ 1.50, for $p=$ 0.90, 0.95, and 0.99, respectively. And the values of $C^*(p)$ with $w=$ 2.00, for $p=$ 0.90, 0.95, and 0.99 are displayed in Tables 4(a)-4(c), respectively. Interested readers may visit the following website for details of those tables: http://www.nctu.edu.tw/~qtqm/paper/cpktables/. For example, if $w=$ 1.33 is the minimum capability requirement, then for $p=$ 0.95, $n=$ 100, $\delta=$ 0.5, $C^*(p)$ = 1.5173 by checking Table 2(b). Thus, the value $\hat{C}_{pk}$ calculated from sample data must satisfy $\hat{C}_{pk} \geq 1.5173$ to conclude that $C_{pk} \geq 1.33$ (process is capable). From these tables we observe that for each fixed p and n the value of $C^*(p)$ decreases as δ increases. Figures 1–4 display the value of $C^*(p)$ versus $\delta= |\bar{x} - m|/s$ for sample size $n=$ 10(10)50 from top to bottom in plots, with $w=$ 1.00 and $p=$ 0.95. This phenomenon can be explained by the following argument. For a fixed $\hat{C}_{pk}$, since

$$\hat{C}_{pk} = \frac{d - |\bar{x} - m|}{3s} = \frac{d/s - \delta}{3}, \tag{10}$$

then s becomes smaller when δ becomes larger, and a smaller s means that it is plausible that the underlying process is tighter (i.e. with smaller σ). Since the estimation is usually more accurate for data drawn from a tighter process, it is then plausible that the estimate $\hat{C}_{pk}$ is more accurate with a smaller s. In this case the required minimum value is smaller, so we need only a smaller $C^*(p)$ to account for the smaller uncertainty in the estimation. Intuitively, if the estimation error in our estimate is potentially large, then it is reasonable that we need a large $C^*(p)$ to be able to claim that the process is capable, and this means that the corresponding minimum value $C^*(p)$ should be large as well. Thus the value of $C^*(p)$ decreases as δ increases. Another observation from the tables is that the value of $C^*(p)$ decreases as n increases for fixed δ and p. This can also be seen from the same argument as above, a larger n implies that $\hat{C}_{pk}$ is more accurate.

6 Capability testing with applications

In current practice, a process is called "Inadequate" if $C_{pk} <$ 1.00; it indicates that the process is not adequate with respect to the production tolerances (specifications), either process variation (σ^2) needs to be reduced or process mean (μ) needs to be shifted closer to the target value T. A process is called

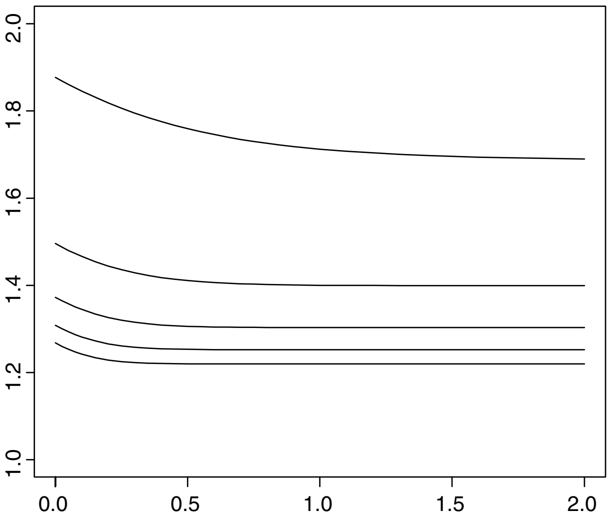

Fig. 1. Plots of $C^*(p)$ versus δ for $w = 1.00$, $p = 0.95$, and $n = 10, 20$, 30, 40, and 50 (top to bottom in plot)

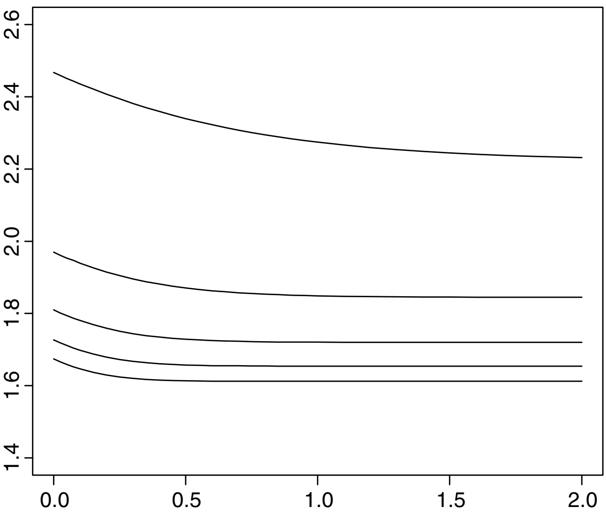

Fig. 2. Plots of $C^*(p)$ versus δ for $w = 1.33$, $p = 0.95$, and $n = 10, 20, 30, 40$, and 50 (top to bottom in plot)

"Capable" if $1.00 \leq C_{pk} < 1.33$; it indicates that caution needs to be taken regarding to process distribution, some process control is required. A process is called "Satisfactory" if $1.33 \leq C_{pk} < 1.50$; it indicates that process quality is satisfactory, material substitution may be allowed, and no stringent quality control is required. A process is called "Excellent" if $1.50 \leq C_{pk} < 2.00$; it indicates that process quality exceeds satisfactory. Finally, a process is called "Super" if $C_{pk} \geq 2.00$. Many companies have recently adopted criteria for

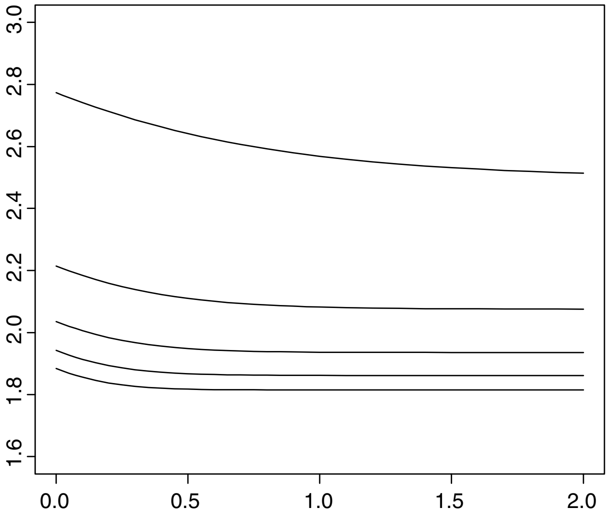

Fig. 3. Plots of $C^*(p)$ versus δ for $w = 1.50$, $p = 0.95$, and $n = 10, 20, 30, 40$, and 50 (top to bottom in plot)

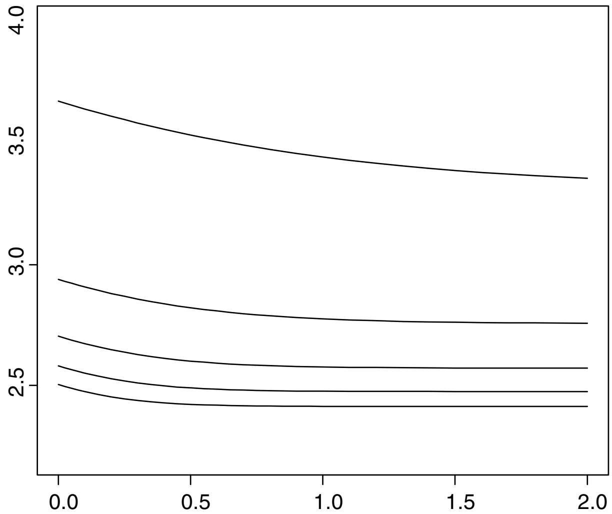

Fig. 4. Plots of $C^*(p)$ versus δ for $w = 2.00$, $p = 0.95$, and $n = 10, 20, 30, 40$, and 50 (top to bottom in plot)

evaluating their processes that include process capability objectives more stringent. For example, Motorola's "Six Sigma" program essentially requires the process capability at least 2.0 to accommodate the possible 1.5σ process shift (see Harry (1988)). Table 1 summarizes some commonly used capability requirements and fractions of nonconformities (in ppm) corresponding to process conditions.

We now describe a Bayesian procedure in the following. A $100p\%$ credible interval means the posterior probability that the true PCI lies in this interval

Table 1. Some commonly used capability requirements and nonconformities corresponding to process conditions

Process conditions	C_{pk} values	Non-conformities
Incapable	$C_{pk} < 1.00$	> 2700 ppm
Capable	$1.00 \leq C_{pk} < 1.33$	< 2700 ppm
Satisfactory	$1.33 \leq C_{pk} < 1.50$	< 66 ppm
Excellent	$1.50 \leq C_{pk} < 2.00$	< 6.795 ppm
Super	$2.00 \leq C_{pk}$	< 0.002 ppm

is p. Let p be a high probability, say, 0.95. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.33. Next, from the process data, we compute (or check the tables) the lower bound of the credible interval for the C_{pk} index. Thus, if $\hat{C}_{pk} > C^*(p)$, then we say that the process is capable in a Bayesian sense. Otherwise, we do not have sufficient information to conclude that the process meets the preset capability requirement, and then we tend to believe that the process is incapable in this case.

To illustrate how we apply the proposed procedure to actual data collected from the factory. We consider the following example taken from a company engaged mainly in making oil-hydraulic cylinder components and oil-hydraulic cylinder (oil-hydraulic propeller) assembly. Oil-hydraulic equipment is required for automation and oil-hydraulic cylinders are the main component of such equipment. The pistons are one of the most critical parts of oil-hydraulic cylinders. A typical piston for the oil-hydraulic cylinders has a 20 mm inner diameter. When the oil goes through the oil-hydraulic cylinder, it can exert pressure and make the piston move. The two points C are the grooves on the piston that must be fitted with the U-shaped oil seal to prevent the oil from leaking when the piston move. If the oil leaks, it affects the efficiency and performance of the oil-hydraulic cylinder. There are two points called A and two points called B, which are the prominent parts of the piston holding two U-shaped oil seals to make them assuming the pressure from the oil-hydraulic cylinder. Since it is the U-shaped oil seals, and is not the main body of the piston in direct contact with the tube of the oil-hydraulic cylinder, then it is essential to make the piston grooves (called point C) complying with the required manufacturing specifications.

The manufacturing specifications for the grooves of the piston are set to the followings: $USL = 13.25$ mm, $LSL = 13.15$ mm, target value $T = 13.20$ mm. The capability requirement for this particular model of oil-hydraulic cylinder was defined as "Satisfactory" if $C_{pk} > 1.33$. The process has been justified to be well in-controlled, and is near normally distributed. The collected sample data (a total of 150 observations) are displayed in Table 2. The sample mean $\bar{x} = 13.201$ and sample standard deviation s = 0.00969 are first calculated. For $n = 150$, we calculate the value of the estimator $\hat{C}_{pk} = (d - |\bar{x} - m|)/(3s) = 1.6925$, and $\delta = |\bar{x} - m|/s = 0.103$. By solving the posterior probability (9), the critical value is found to be $C^*(p) = 1.4869$ based on $w = 1.33$, $p = 0.95$, and $n = 150$. Note that the computer program for calculating $C^*(p)$ is available from authors. Since $\hat{C}_{pk} = 1.6925$ is greater than the critical value $C^*(p) = 1.4869$ in this case, it is therefore concluded with 95% confidence ($\alpha = 0.05$) that the grooves of the piston manufacturing process satisfies the requirement '$C_{pk} > 1.33$'. Thus, at least 99.9934% of the

Table 2. A total of 150 observations collected from factory

13.207	13.194	13.186	13.204	13.202	13.222	13.172	13.200	13.197	13.192
13.178	13.190	13.215	13.199	13.196	13.205	13.203	13.195	13.194	13.206
13.184	13.215	13.199	13.182	13.207	13.203	13.206	13.184	13.184	13.194
13.208	13.212	13.207	13.200	13.191	13.206	13.195	13.203	13.194	13.200
13.215	13.211	13.187	13.211	13.207	13.189	13.215	13.203	13.198	13.206
13.184	13.218	13.201	13.198	13.207	13.214	13.199	13.197	13.206	13.208
13.192	13.203	13.207	13.193	13.209	13.201	13.196	13.213	13.198	13.211
13.194	13.207	13.190	13.207	13.202	13.209	13.206	13.192	13.209	13.208
13.204	13.218	13.191	13.209	13.191	13.187	13.200	13.190	13.209	13.212
13.198	13.186	13.197	13.187	13.205	13.193	13.196	13.210	13.199	13.199
13.207	13.184	13.208	13.202	13.199	13.203	13.190	13.195	13.189	13.199
13.206	13.212	13.207	13.210	13.205	13.208	13.222	13.203	13.196	13.203
13.205	13.218	13.208	13.196	13.208	13.199	13.190	13.189	13.218	13.193
13.181	13.194	13.197	13.213	13.187	13.212	13.212	13.189	13.206	13.198
13.205	13.190	13.211	13.217	13.190	13.196	13.214	13.207	13.200	13.190

produced oil-hydraulic cylinders are conformed to the manufacturing specifications, which are considered satisfactory and reliable in terms of product quality (originally set by the product designers or the manufacturing engineers).

7 Conclusions

In the last decade, numerous process capability indices have been proposed to provide measure on whether a process is capable of reproducing items meeting the quality requirement preset by the product designer. Those indices are effective tools for process capability analysis and quality assurance. In process capability analysis, the usual practice for estimating the capability indices from sample data are based on the traditional distribution frequency approach. An alternative is to use the Bayesian approach. The Bayesian approach specifies a prior distribution for the parameter of interest, to obtain the posterior distribution for the parameter, then infer about the parameter using it posterior distribution given the observations. This paper considers estimating and testing capability index C_{pk} using Bayesian approach. The posterior distribution of C_{pk} is derived and an accordingly Bayesian procedure for capability testing is proposed. For users' convenience in applying our Bayesian procedure, we tabulate the minimum values of $\hat{C}_{pk}$ required to ensure the posterior probability p reaching various pre-specified capability levels.

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