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ORIGINAL ARTICLE

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Procedures for testing manufacturing precision C_p based on $(\bar{X}, R)$ or $(\bar{X}, S)$ control chart samples

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Abstract Process precision index C_p has been widely used in the manufacturing industry for measuring process potential and precision. Estimating and testing process precision based on one single sample have been investigated extensively. In this paper, we consider the problem of estimating and testing process precision based on multiple samples taken from $(\bar{X}, R)$or $(\bar{X}, S)$control chart. We first investigate the statistical properties of the natural estimator of C_p and implement the hypothesis testing procedure. We then develop efficient MAPLE programs to calculate the lower confidence bounds, critical values, and p-values based on m samples of size n. Based on the test, we develop a step-by-step procedure for practitioners to use in determining whether their manufacturing processes are capable of reproducing products satisfying the preset precision requirement.

Keywords Critical value · Lower confidence bound · Process precision index · p-value · Testing hypothesis

1 Introduction

Process precision index C_p has been proposed in the manufacturing industry for measuring process potential and precision ([1–4], among others). The precision index C_p is designed to provide numerical measures on process potential (product quality consistency) in meeting the preset process precision (process variation relative to the manufacturing tolerance) requirements. The precision index C_p is defined as

$$C_p = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma} \tag{1}$$

where USL and LSL are the upper and the lower specification limits, and σ is the process standard deviation of the quality characteristic. The precision index C_p is primarily designed to monitor process data that are taken independently, from a normal process under statistical control. Using the index under other process conditions, without proper modifications, would certainly give severely inaccurate measurement on process precision.

The use of the precision index C_p and other capability indices was first explored within the automotive industry. Ford Motor Company initially used C_p to keep track of the process performance. Recently, the manufacturing industries have been making an extensive effort to implement statistical process control (SPC) in their plants and supply bases. Capability indices derived from SPC have received increasing usage not only in capability assessments, but also in the evaluation of purchasing decisions. Capability indices are becoming the standard tools for quality reporting, particularly, at the management level around the world. Proper understanding and accurate estimating of capability indices are essential for the company to maintain a capable supplier.

2 Estimating C_p based on control chart samples

For applications where the data are collected as one single sample, Pearn et al. [3] considered an unbiased estimator of C_p. They showed that the unbiased estimator is the UMVUE (uniformly minimum variance unbiased estimator) of C_p. They also proposed an efficient test for C_p based on one single sample, and showed that the test is the UMP (uniformly most powerful) test. Kirmani et al. [5] considered the estimation of σ and the precision index C_p for cases where the data are collected as multiple samples. Pearn and Yang [6] proposed an unbiased estimator of C_p for multiple samples, and showed that the unbiased estimator is the UMVUE of C_p, which is asymptotically efficient. Pearn

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and Yang [6] also developed an efficient test for C_p for cases with multiple samples, and showed that the proposed test is indeed the UMP test.

For applications where routine-based data collection plans are implemented, a common practice on process control is to estimate the process precision by analyzing past "in control" data. Consider m preliminary multiple samples (subgroups) each of size n taken from the control chart samples. To estimate σ we typically use either the sample standard deviation or the sample range. The control chart can be used as a monitoring device or logbook to show the effect of changes in the process performance. We note that a process may be in control but not necessarily operating at an acceptance level. Thus, management intervention is required either to improve the process capability, or to change the manufacturing requirements to ensure that the products meet the minimum acceptable level. We remark that the process must be stable in order to produce a reliable estimate of process capability. If the process is out of control in the early stages of process capability analysis, it will be unreliable to estimate process capability. The top priority is to find and eliminate the assignable causes in order to bring the process into an in control state.

2.1 Estimating C_p based on $(\bar{X}, R)$ samples

If m samples each of size n from the $(\bar{X}, R)$ control chart are available, let $R_{1,n}$ be the range of a sample of size n and $\bar{R}_{m,n}$ be the average range in m samples of each size n. Then the mean and variance of the relative range $\bar{R}_{m,n}/\sigma$ are given by

$$E\left(\bar{R}_{m,n}/\sigma\right) = E\left(R_{1,n}/\sigma\right) = d_2 \tag{2}$$

$$Var\left(\bar{R}_{m,n}/\sigma\right) = \frac{Var(R_{1,n})}{m\sigma^2} = \frac{d_3^2}{m} \tag{3}$$

where d_2 and d_3 are functions of n, which are available in quality control books and literature (see Pearson's Table A [7]). Thus, the estimated process capability precision by the range method can be expressed as

$$\hat{C}_{p(R)} = \frac{\mathrm{USL}-\mathrm{LSL}}{6\hat{\sigma}_R}, \quad \hat{\sigma}_R = \frac{\bar{R}_{m,n}}{d_2}. \tag{4}$$

If $m = 1$, the cumulative distribution function of the range from a normal distribution is

$$F(x) = P\left(\frac{R_{1,n}}{\sigma} \le x\right) = n\int_{-\infty}^{\infty} [\Phi(x+t) - \Phi(t)]^{n-1}\,\phi(t)dt, \quad \text{for } t > 0 \tag{5}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution function and probability density function of the standard normal distribution N(0, 1). Furthermore, using the first two moments of the average range, Patnaik [8] has shown that $\bar{R}_{m,n}/\sigma$ is distributed approximately as $c\chi_v/\sqrt{v}$, where χ_v^2 is the chi-square distribution with v degree of freedom and c and v are constants which are functions of the first two moments of the range as follows:

$$E\left(\bar{R}_{m,n}/\sigma\right) = \frac{c}{\sqrt{v}}\sqrt{2}\Gamma\left(\frac{v+1}{2}\right)/\Gamma\left(\frac{v}{2}\right) \tag{6}$$

$$Var\left(\bar{R}_{m,n}/\sigma\right) = \frac{c^2}{v}\left\{v - 2\left[\Gamma\left(\frac{v+1}{2}\right)/\Gamma\left(\frac{v}{2}\right)\right]^2\right\}. \tag{7}$$

The values of the mean and variance in Eq. 2 and Eq. 3 are known from the coefficients of the mean and variance of the average range, d_2 and d_3. By letting Eq. 6 equal Eq. 2 and Eq. 7 equal Eq. 3, we obtain the values of c and v, which are solutions to the above system of equations. Table 1 displays the values of d_2 ,and d_3 and the corresponding c and v for multiple samples with $m = 5(5)25$ and $n = 2(1)10$.

In the early days of control chart usage, the range method of estimating σ was employed to simplify the arithmetic associated with control chart operation. With modern computer software and hand-held calculators for control chart operation, this is not a consideration, and other methods could be used. If the sample size is relatively small, the range method yields almost as good an estimator of variance σ^2 as does the usual sample variance S^2. The relative efficiency (RE) of the range method to S^2 is shown in Table 2 for various sample sizes [4]. For values of $n \ge 10$, the range method loses efficiency rapidly, since it ignores all the information in the sample between the maximum and minimum values. However, for the small sample sizes often employed on variables control charts ($n = 4$, 5, or 6), it is entirely satisfactory.

Table 1. Coefficients of distribution for multiple samples with $m = 5(5)25$, $n = 2(1)10$, and $\alpha = 0.01$, 0.025, 0.05

			$m=5$		$m=10$		$m=15$		$m=20$		$m=25$	
n	d_2	d_3	c	v	c	v	c	v	c	v	c	v
2	1.128	0.853	1.191	4.582	1.160	8.973	1.149	13.351	1.144	17.727	1.141	22.101
3	1.693	0.888	1.739	9.317	1.716	18.414	1.708	27.505	1.705	36.594	1.702	45.682
4	2.059	0.880	2.096	13.923	2.078	27.616	2.071	41.304	2.068	54.992	2.067	68.679
5	2.326	0.864	2.358	18.359	2.342	36.483	2.337	54.603	2.334	72.723	2.332	90.842
6	2.534	0.848	2.562	22.565	2.548	44.893	2.543	67.218	2.541	89.542	2.540	111.866
7	2.704	0.833	2.730	26.586	2.717	52.932	2.713	79.276	2.710	105.620	2.709	131.963
8	2.847	0.820	2.871	30.380	2.859	60.519	2.855	90.656	2.853	120.793	2.852	150.929
9	2.970	0.808	2.992	34.022	2.981	67.803	2.977	101.581	2.975	135.359	2.974	169.137
10	3.078	0.797	3.099	37.532	3.088	74.822	3.085	112.110	3.083	149.398	3.082	186.685

Table 2. The relative efficiency of the range method to S^2

n	2	3	4	5	6	10
RE	1.000	0.992	0.975	0.955	0.930	0.850

2.2 Estimating C_p based on $(\bar{X}, S)$ samples

If m samples each of size n from the $(\bar{X}, S)$ control chart are available, Kirmani et al. [5] considered measuring the process precision index C_p, and the natural estimator $\hat{C}_p$ defined as the following:

$$\hat{C}_{p(S)} = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}_S}, \quad \hat{\sigma}_S = \frac{1}{\varepsilon_{n-1}}\bar{S}, \quad \text{where} \tag{8}$$

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i, \quad S_i = \left[\frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2\right]^{1/2} \quad \text{and}$$

$$\varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\,\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}.$$

The $\bar{X}_i$ and S_i represent the sample mean and sample standard deviation of the ith sample, and ε_{n-1} is denoted by c_4 in the general quality control literature. Kirmani et al. [5] showed that under the normality assumption, the statistic $\bar{S}$ is approximately distributed as the normal distribution. That is,

$$\frac{\bar{S} - \sqrt{n-1}\varepsilon_{n-1}}{\sqrt{\frac{(n-1)\left(1-\varepsilon_{n-1}^2\right)}{m}}} \sim \text{N}(0, 1). \tag{9}$$

This is particularly true in situations where reasonable tight control of the process variability is needed so that moderately large subgroups ($n > 10$) are required. In this case, the S-chart is preferred to the R-chart. We note that the expressions for the distribution of $\hat{C}_{p(S)}$ obtained in Kirmani et al. [5], Kocherlakota [9], and Kotz and Lovelace [10] need to be modified. In fact, they addressed the distribution of $\hat{\sigma}_S$ as

$$\hat{\sigma}_S \sim \left(\sigma, \frac{\sigma^2}{m}\frac{1-\varepsilon_{n-1}^2}{\varepsilon_{n-1}^2}\right). \tag{10}$$

Hence, we have

$$\hat{C}_{p(S)} \sim \left[1 + N\left(0, \frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}\right)\right]^{-1} C_p : \tag{11}$$

The estimator $\hat{C}_{p(S)}$ is biased, and its probability density function (PDF) can be obtained, and expressed as the following, for $x > 0$, which is a function of C_p.

$$g(x) = \frac{C_p}{\sqrt{2\pi}k}x^{-2}\exp\left[-\frac{(C_p/x - 1)^2}{2k^2}\right] \tag{12}$$

$$k = \sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}}, \quad \text{and } \varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\,\frac{\Gamma(n/2)}{\Gamma\left[(n-1)/2\right]}.$$

Figures 1–4 display the PDF plots of $\hat{C}_{p(S)}$ for various sample sizes of $m = 10, 15, 20, 25$, $n = 5$, with $C_p = 1.00, 1.33, 1.50, 1.67$, and 2.00 (from left to right in plot). Figures 5–8 display the PDF plots of $\hat{C}_{p(S)}$ for various sample sizes of $m = 15$, $n = 3$, 6, 9, 12, with $C_p = 1.00, 1.33, 1.50, 1.67$, and 2.00 (from left to right in plot). We note that (i) for fixed sample size m and n, the variance of $\hat{C}_{p(S)}$ increases as C_p increases, (ii) for fixed n and

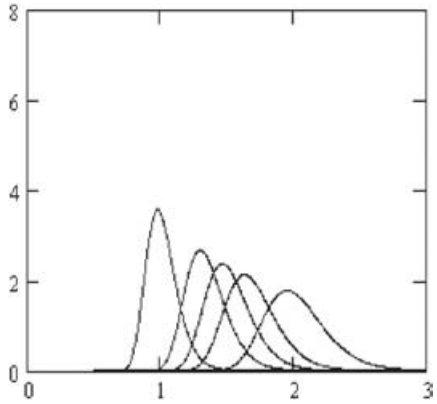

Fig. 1. PDF plot of $\hat{C}_{p(S)}$ with $m = 10$ and $n = 5$, for various values of C_p

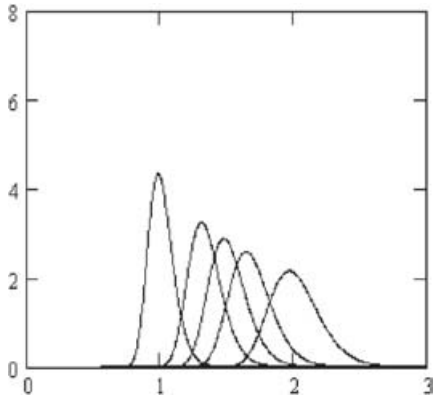

Fig. 2. PDF plot of $\hat{C}_{p(S)}$ with $m = 15$ and $n = 5$, for various values of C_p

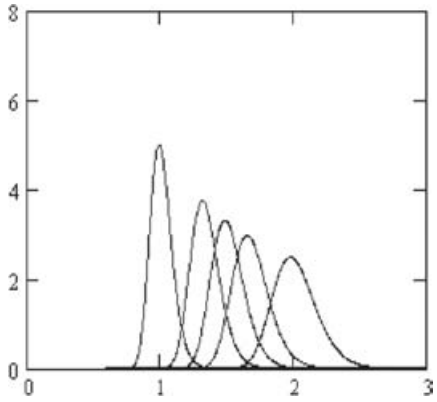

Fig. 3. PDF plot of $\hat{C}_{p(S)}$ with $m = 20$ and $n = 5$, for various values of C_p

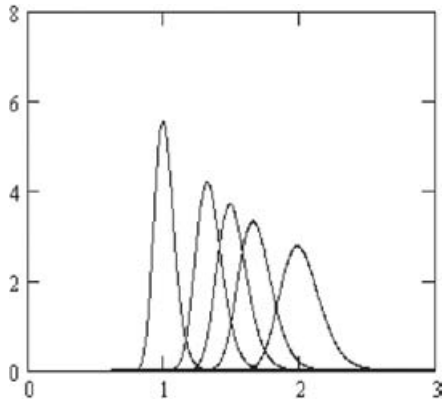

Fig. 4. PDF plot of $\hat{C}_{p(S)}$ with $m = 25$ and $n = 5$, for various values of C_p

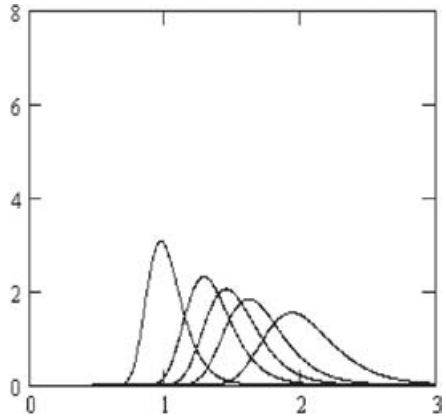

Fig. 5. PDF plot of $\hat{C}_{p(S)}$ with $m = 15$ and $n = 3$, for various values of C_p

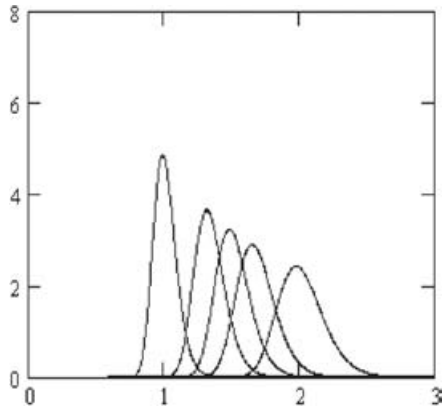

Fig. 6. PDF plot of $\hat{C}_{p(S)}$ with $m = 15$ and $n = 6$, for various values of C_p

C_p, the variance of $\hat{C}_{p(S)}$ decreases as m increases, and (iii) for fixed m and C_p, the variance of $\hat{C}_{p(S)}$ decreases as n increases.

Lower confidence bound on C_p. Since $\hat{C}_p$ (denote either $\hat{C}_{p(R)}$ or $\hat{C}_{p(S)}$) is subject to sampling error, it is desirable to construct a confidence interval to provide a range, which contains the true C_p with high probability. For cases where multiple samples taken

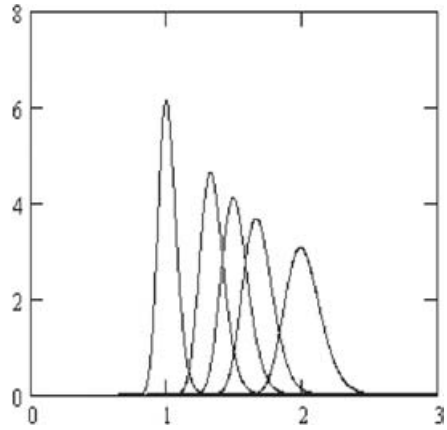

Fig. 7. PDF plot of $\hat{C}_{p(S)}$ with $m = 15$ and $n = 9$, for various values of C_p

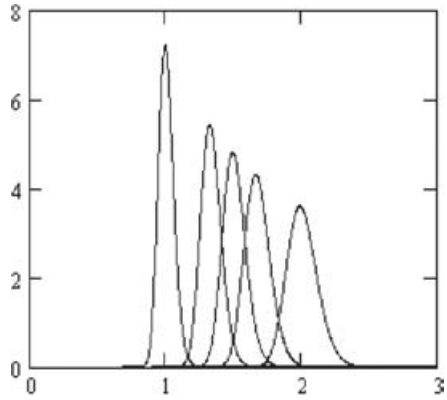

Fig. 8. PDF plot of $\hat{C}_{p(S)}$ with $m = 15$ and $n = 12$, for various values of C_p

from $(\bar{X}, R)$ control chart at various points in time are available, by Patnaik's approximate distribution of the average range, the $100(1-\alpha)\%$ lower confidence bound $C_{L(R)}$ can be constructed, which satisfies

$$\begin{aligned} P\left(C_p \geq C_{L(R)}\right) &= 1-\alpha \\ &= P\left(\frac{\hat{\sigma}_R}{\sigma} \geq \frac{C_{L(R)}}{\hat{C}_{p(R)}}\right) \\ &= P\left(\frac{\bar{R}_{m,n}}{\sigma} \geq \frac{d_2 C_{L(R)}}{\hat{C}_{p(R)}}\right) \\ &\simeq P\left(\chi_v \geq \frac{\sqrt{v} d_2 C_{L(R)}}{c\hat{C}_{p(R)}}\right). \end{aligned} \tag{13}$$

Thus, we can obtain that

$$\begin{aligned} &\frac{\sqrt{v} d_2 C_{L(R)}}{c\hat{C}_{p(R)}} = \chi_{v,\alpha}\,, \quad \text{or the ratio} \\ &\frac{C_{L(R)}}{\hat{C}_{p(R)}} = \frac{c}{\sqrt{v} d_2}\sqrt{\chi^2_{v,\alpha}}, \end{aligned} \tag{14}$$

where $\chi^2_{v,\alpha}$ is the lower α-th percentile of the chi-square distribution with v degree of freedom. We notice that the ratio values

of $C_{L(R)}/\hat{C}_{p(R)}$ depends on v, c, d_2, and α. The values of v, c, d_2 are determined from the number of samples m and sample size n. We refer to this ratio $C_{L(R)}/\hat{C}_{p(R)}$ as lower confidence factors. Table 3 displays the lower confidence factors for multiple samples with $m = 5(5)25$, $n = 2(1)10$, and $\alpha = 0.01$, 0.025, 0.05. From Table 3 we observe that for fixed sample size n and α, the lower confidence factors $C_{L(R)}/\hat{C}_{p(R)}$ increases as the number of samples m increases. As an example, for fixed $n = 5$, we have $m = 5$ with $C_{L(R)}/\hat{C}_{p(R)} = (0.636, 0.689, 0.735)$, $m = 15$ with $C_{L(R)}/\hat{C}_{p(R)} = (0.784, 0.817, 0.845)$, and $m = 25$ with $C_{L(R)}/\hat{C}_{p(R)} = (0.831, 0.857, 0.879)$. On the other hand, for a fixed number of samples m and α, the lower confidence factors increase as the sample size n increases. This phenomenon can be explained easily. Since the estimation is usually more accurate as the total collected sample increases, we need only a smaller penalty of $\hat{C}_{p(R)}$ to account for the smaller uncertainty in the estimation.

For cases where m multiple samples of size n are available due to sampling from $(\bar{X}, S)$ control chart at various point in time, Kirmani et al. [5] constructed the $100(1-\alpha)\%$ lower confidence bound as the following:

$$C_{L(S)} = \hat{C}_{p(S)} \left[1 + Z_\alpha \sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}} \right]. \tag{15}$$

Table 4 displays the lower confidence factors of $\hat{C}_{p(S)}$ for multiple samples with $m = 10(5)25$, $n = 2(1)15$, and $\alpha = 0.01$, 0.025, 0.05. For example, with input parameters $m = 10$, $n = 5$, risk $\alpha = 0.05$ and $\hat{C}_{p(S)} = 1.520$, the program gives the lower confidence bound of C_p as 1.233. Or, by simply checking Table 4, we obtain the lower confidence factor 0.811. Multiplying the lower confidence factor 0.811 by the $\hat{C}_{p(S)} = 1.520$ we obtain 1.233. Thus, it is ensured that with 95 percent confidence, the process precision is no less than 1.233, or $C_p \geq 1.233$. Other values of $\hat{C}_{p(S)} < C_{L(S)}$ will support the null hypothesis that the process is incapable. Hence, the process is capable for any value of the required C_p that is greater than the lower confidence bound. It is noted that for fixed sample size n and α, the lower confidence factors $C_{L(S)}/\hat{C}_{p(S)}$ increase as the number of samples m increases. And for a fixed number of samples m and α, the lower confidence factors increase as the sample size n increases. The results are consistent with the lower confidence bound obtained by the range method, and the explanation is the same as above.

An efficient MAPLE computer program for solving the corresponding Eq. 15 and calculating the lower confidence bound $C_{L(S)}$ is listed below. The input parameters are set to: the upper specification limit USL = 12, the lower specification limit LSL = 4, number of samples $m = 10$, the

Table 3. Lower confidence factors $C_{L(R)}/\hat{C}_{p(R)}$ for multiple samples with $m = 5(5)25$, $n = 2(1)10$, and $\alpha = 0.01$, 0.025, 0.05

m	5			10			15			20			25		
$n\,\alpha$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
2	0.327	0.406	0.482	0.495	0.563	0.624	0.578	0.637	0.690	0.630	0.683	0.730	0.667	0.715	0.758
3	0.503	0.570	0.631	0.637	0.689	0.735	0.700	0.744	0.783	0.738	0.777	0.811	0.765	0.800	0.831
4	0.586	0.645	0.697	0.700	0.744	0.783	0.753	0.790	0.822	0.785	0.817	0.845	0.807	0.836	0.862
5	0.636	0.689	0.735	0.738	0.777	0.811	0.784	0.817	0.845	0.812	0.841	0.865	0.831	0.857	0.879
6	0.670	0.718	0.760	0.763	0.798	0.829	0.805	0.834	0.860	0.830	0.856	0.879	0.848	0.871	0.891
7	0.695	0.740	0.779	0.781	0.814	0.843	0.820	0.847	0.871	0.843	0.867	0.888	0.860	0.881	0.900
8	0.714	0.756	0.793	0.795	0.826	0.853	0.831	0.857	0.879	0.853	0.876	0.895	0.869	0.889	0.906
9	0.729	0.769	0.804	0.706	0.835	0.861	0.840	0.865	0.886	0.861	0.883	0.901	0.876	0.895	0.911
10	0.741	0.780	0.813	0.815	0.843	0.867	0.848	0.871	0.891	0.868	0.888	0.906	0.882	0.900	0.916

Table 4. Lower confidence factors $C_{L(S)}/\hat{C}_{p(S)}$ for multiple samples with $m = 10(5)25$, $n = 2(1)15$, and $\alpha = 0.01$, 0.025, 0.05

m	10			15			20			25		
$\alpha\,n$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
2	0.443	0.532	0.607	0.545	0.618	0.679	0.606	0.669	0.722	0.648	0.704	0.751
3	0.615	0.676	0.728	0.686	0.735	0.778	0.728	0.771	0.808	0.756	0.795	0.828
4	0.689	0.738	0.780	0.746	0.786	0.821	0.780	0.815	0.845	0.803	0.835	0.861
5	0.733	0.775	0.811	0.782	0.816	0.846	0.811	0.841	0.866	0.831	0.858	0.881
6	0.762	0.800	0.832	0.806	0.836	0.863	0.832	0.858	0.881	0.849	0.873	0.894
7	0.783	0.818	0.847	0.823	0.851	0.875	0.847	0.871	0.892	0.863	0.885	0.903
8	0.800	0.832	0.859	0.837	0.863	0.885	0.858	0.881	0.900	0.873	0.894	0.911
9	0.813	0.843	0.868	0.847	0.872	0.892	0.868	0.889	0.907	0.882	0.901	0.917
10	0.824	0.852	0.876	0.856	0.879	0.899	0.876	0.895	0.912	0.889	0.906	0.921
11	0.833	0.860	0.882	0.864	0.886	0.904	0.882	0.901	0.917	0.895	0.911	0.926
12	0.841	0.866	0.888	0.870	0.891	0.908	0.888	0.906	0.921	0.900	0.916	0.929
13	0.848	0.872	0.893	0.876	0.896	0.912	0.893	0.910	0.924	0.904	0.919	0.932
14	0.854	0.877	0.897	0.881	0.900	0.916	0.897	0.913	0.927	0.908	0.922	0.935
15	0.860	0.882	0.901	0.885	0.904	0.919	0.900	0.916	0.930	0.911	0.925	0.937

sample size $n = 4$, the collected data d[data], and the risk $\alpha = 0.01$. The program calculates the process precision $\hat{C}_{p(S)}$ as 0.742, and finds the lower confidence bound as $C_{L(S)} = 0.511$.

Maple program for lower confidence bounds $C_{L(S)}$.

```
#input values of parameter U(USL), L(LSL), d[data], m, n, z;
#Shat:= Sigmahat; Cphat:= Ĉ_p(S); CL:= lower confidence bound
>U:=12; L:=4; d[1]:=[10,5,7,9];
d[2]:=[5,8,7,7]; d[3]:=[7,6,6,8]; d[4]:=[5,7,9,5]; d[5]:=[6,5,8,9]; d[6]:=[10,9,11,4];
d[7]:=[4,5,10,6]; d[8]:=[6,6,7,9]; d[9]:=[8,10,6,7]; d[10]:=[9,6,11,8];
   n:=4; m:=10; alpha:=0.01; z:=icdf[normal[0,1](alpha)];
   v:=matrix([seq(d[i],i=1..10)]);
   sample:=[seq([seq(v[i,j],j=1..4)],i=1..10)];
   s:=evalf(sum('describe[standarddeviation](t[i])',i=1..10)/10):
   epsilon:=evalf(( ((2/(n−1))^(1/2))*(GAMMA(n/2))/(GAMMA((n−1)/2)) )):
   Shat:=evalf(s/epsilon);
   Cphat:=evalf((U−L)/(6*Shat));
   CL:=evalf(Cp*(1+z*((1−epsilon^2)/(m*epsilon^2))^(1/2)));
```

The output is:
U:=12
L:=4
n:=4
m:=10
alpha:=0.01
Shat:=1.79692
Cphat:=0.74201
CL:=0.51129

3 Testing C_p based on multiple control chart samples

Cases where the data are collected as one single sample of size n have been discussed by Kane [1]. In this case, Chou et al. [11] give tables for lower confidence limit on C_p when σ is estimated by the sample standard deviation S. When σ is estimated by range divided by d_2, Li et al. [12] also give tables for the lower confidence limit on C_p. Now, we are interested in the test of process precision C_p based on $(\bar{X}, R)$ or $(\bar{X}, S)$ control chart samples. To test whether the process meets the precision requirement, we consider the following testing hypothesis with H_0: $C_p \le C$ (the process is incapable), versus the alternative H_1: $C_p > C$ (the process is capable). Thus, we may consider the test $\phi^*(x) = 1$ if $\hat{C}_p > C_0$, and $\phi^*(x) = 0$, otherwise. The test ϕ^* rejects the null hypothesis if $\hat{C}_p > C_0$, with type I error $\alpha(C_0) = \alpha$, the chance of incorrectly judging an incapable process as a capable one.

3.1 Testing C_p based on $(\bar{X}, R)$ samples

When the estimated process capability precision by the range method from the $(\bar{X}, R)$ control chart samples, the critical value $C_{0(R)}$ can be obtained by finding the appropriate value satisfying the following equation:

$$
\begin{aligned}
P\left(\hat{C}_{p(R)} \ge C_{0(R)} | C_p = C\right) &= \alpha \\
&= P\left(\frac{d}{3\hat{\sigma}_R} \ge C_{0(R)}\right) \\
&= P\left(\frac{\bar{R}_{m,n}}{\sigma} \ge \frac{d_2}{C_{0(R)}}\frac{d}{3\sigma}\right) \\
&\simeq P\left(\chi_v \le \frac{\sqrt{v}d_2}{cC_{0(R)}}C\right). \qquad (16)
\end{aligned}
$$

In fact, the critical value $C_{0(R)}$ can be found and expressed as the following:

$$
C_{0(R)} = \frac{\sqrt{v}d_2}{c\sqrt{\chi^2_{v,\alpha}}}C\,. \qquad (17)
$$

Table 5 displays the critical values $C_{0(R)}$ for precision requirement $C_p = 1.00$ with m subgroups of size n, and various risks $\alpha = 0.01$, 0.025, and 0.05. We see that the critical value $C_{0(R)}$ is proportional to the precision requirement C. Hence, we need only to calculate $C_{0(R)}$ for the $C_p = 1.00$ case. For general $C_p = C$ (common requirements as 1.33, 1.67, 2.00), we obtain the corresponding critical values by multiplying C to the critical values $C_{0(R)}$ with $C_p = 1.00$. For instance, if the required precision requirement C, is set to 1.33, with $m = 10$ subgroups of size $n = 5$, and various risks $\alpha = 0.01$, 0.025, and 0.05, then the corresponding critical values are $C^*_{0(R)} = (1.355, 1.287, 1.233) \times 1.33 = (1.802, 1.712, 1.640)$. It is noted that for fixed α and sample size n, the critical value $C_{0(R)}$ increases as the number of samples m decreases, and for fixed α and m the critical value $C_{0(R)}$ increases as the n decreases. It can be understood intuitively, since the estimation error is potentially larger as the total sample size $m \times n$ is smaller. It is reasonable that we need a larger $C_{0(R)}$ to claim that the process is capable. Under the same conditions, the p-value corresponding to $\hat{c}_{p(R)}$, a specific

Table 5. Critical values $C_{0(R)}$ for $C_p = 1.00$, with $m = 5(5)25$, $n = 2(1)10$, and $\alpha = 0.01$, 0.025, 0.05

m	5			10			15			20			25		
$n\,\alpha$	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
2	3.058	2.463	2.075	2.020	1.776	1.603	1.730	1.570	1.449	1.587	1.464	1.370	1.499	1.399	1.319
3	1.988	1.754	1.585	1.570	1.451	1.361	1.429	1.344	1.277	1.355	1.287	1.233	1.307	1.250	1.203
4	1.706	1.550	1.435	1.429	1.344	1.277	1.328	1.266	1.217	1.274	1.224	1.183	1.239	1.196	1.160
5	1.572	1.451	1.361	1.355	1.287	1.233	1.276	1.224	1.183	1.232	1.189	1.156	1.203	1.167	1.138
6	1.493	1.393	1.316	1.311	1.253	1.206	1.242	1.199	1.163	1.205	1.168	1.138	1.179	1.148	1.122
7	1.439	1.351	1.284	1.280	1.229	1.186	1.220	1.181	1.148	1.186	1.153	1.126	1.163	1.135	1.111
8	1.401	1.323	1.261	1.258	1.211	1.172	1.203	1.167	1.138	1.172	1.142	1.117	1.151	1.125	1.104
9	1.372	1.300	1.244	1.416	1.198	1.161	1.190	1.156	1.129	1.161	1.133	1.110	1.142	1.117	1.098
10	1.350	1.282	1.230	1.227	1.186	1.153	1.179	1.148	1.122	1.152	1.126	1.104	1.134	1.111	1.092

value calculated from the sample data, can be calculated as

$$
\begin{aligned}
p\text{-value} &= P\left(\hat{C}_{p(R)} \geq \hat{c}_{p(R)} | C_p = C\right) \\
&= P\left(\frac{\bar{R}_{m,n}}{\sigma} \geq \frac{d_2}{\hat{c}_{p(R)}}\frac{d}{3\sigma}\right) \simeq P\left(\chi_v \leq \frac{\sqrt{v}d_2}{c\hat{c}_{p(R)}}C\right) \\
&= G\left(\left[\frac{\sqrt{v}d_2}{c\hat{c}_{p(R)}}C\right]^2\right), \qquad (18)
\end{aligned}
$$

where $G(\cdot)$ is the cumulative distribution of the chi-square distribution with v degree of freedom.

3.2 Testing C_p based on $(\bar{X}, S)$ samples

If the $(\bar{X}, S)$ control chart is available, then the critical value $C_{0(S)}$ can be obtained by finding the appropriate value satisfying the following equation:

$$
\begin{aligned}
&P\left(\hat{C}_{p(S)} \geq C_{0(S)} | C_p = C\right) = \alpha \\
&= 1 - \int_0^{c_0} \frac{C_p}{\sqrt{2\pi}k} x^{-2} \exp\left[-\frac{(C_p/x - 1)^2}{2k^2}\right] dx\,. \qquad (19)
\end{aligned}
$$

In fact, the critical value $C_{0(S)}$ can be found and expressed as the following, with Z_α representing the lower $100\alpha\%$ percentage point of the standard normal distribution, N(0, 1),

$$
\begin{aligned}
C_{0(S)} &= \frac{C}{1 + Z_\alpha\sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}}}, \quad \text{where} \\
\varepsilon_{n-1} &= \sqrt{\frac{2}{n-1}}\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\,. \qquad (20)
\end{aligned}
$$

An efficient MAPLE computer program is developed to calculate Eq. 20 , thereby obtaining the critical value $C_{0(S)}$ for given m, n, α. The program is listed below, with input parameters set to: $C = 1.00$, $m = 10$, $n = 10$, and $\alpha = 0.01$. The program gives the critical value $C_{0(S)} = 1.213$.

Maple program for critical value $C_{0(S)}$.

```
> #input parameter values C, n, m, z, alpha.
#C0:= critical value;
with (statevalf):
C:=1.00;
n:=10;
m:=10;
alpha:=0.01;
z:=icdf[normal[0,1](alpha)];
epsilon:=((2/(n−1))^(1/2))*(GAMMA(n/2))/(GAMMA((n−1)/2));
C0:=evalf(C/(1+z*((1−epsilon^2)/(m*epsilon^2))^(1/2)));
```

```
The output is:
C:=1.00
n:=10
m:=10
alpha:=0.01
z:=−2.326
epsilon:=0.972659
C0:=1.213075
```

It can be seen that the critical value $C_{0(S)}$ is proportional to the precision requirement C. Hence, we tabulate the critical values $C_{0(S)}$ with $m = 10(5)25$, subgroups of size $n = 2(1)15$, and various risks $\alpha = 0.05$, 0.025, and 0.01 for the precision requirement $C_p = 1.00$ case, displayed in Table 6. For general $C_p = C$ (common requirements as 1.33, 1.67, 2.00), we obtain the corresponding critical values by multiplying C to the critical values $C_{0(S)}$ with $C_p = 1.00$. For instance, if the required precision requirement C, is set to 1.33, with $m = 15$ subgroups of size $n = 10$, and various risks $\alpha = 0.01$, 0.025, and 0.05, then the corresponding critical values are $C^*_{0(S)} = (1.168, 1.137, 1.113) \times 1.33 = (1.553, 1.512, 1.480)$. It is noted that for fixed α and sample size n, the critical value $C_{0(S)}$ increases as the number of samples m decreases, and for fixed α and m the critical value $C_{0(S)}$ increases as n decreases. Again, it can be explained that since the estimation error is potentially larger based on the smaller total sample size, we need a larger $C_{0(S)}$ to claim that the process is capable. Under the same conditions, the p-value corresponding to $\hat{c}_{p(S)}$,

Table 6. Critical values $C_{0(S)}$ for $C_p = 1.00$, $m = 10(5)25$, $n = 2(1)15$, and $\alpha = 0.01$, 0.025, 0.05

m		10			15			20			25	
α n	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05	0.01	0.025	0.05
2	2.256	1.881	1.647	1.833	1.619	1.473	1.649	1.495	1.385	1.543	1.421	1.331
3	1.626	1.472	1.373	1.459	1.360	1.285	1.374	1.297	1.238	1.322	1.258	1.208
4	1.451	1.354	1.281	1.340	1.272	1.218	1.282	1.227	1.184	1.245	1.198	1.161
5	1.365	1.290	1.233	1.279	1.225	1.182	1.233	1.189	1.154	1.204	1.166	1.136
6	1.313	1.251	1.202	1.241	1.196	1.159	1.202	1.165	1.135	1.177	1.145	1.119
7	1.277	1.223	1.181	1.215	1.175	1.143	1.181	1.148	1.121	1.159	1.130	1.107
8	1.250	1.202	1.165	1.195	1.159	1.130	1.165	1.135	1.111	1.145	1.119	1.098
9	1.230	1.187	1.152	1.180	1.147	1.121	1.152	1.125	1.103	1.134	1.110	1.091
10	1.213	1.174	1.142	1.168	1.137	1.113	1.142	1.117	1.096	1.125	1.103	1.085
11	1.200	1.163	1.133	1.158	1.129	1.106	1.134	1.110	1.091	1.118	1.097	1.080
12	1.189	1.154	1.126	1.149	1.122	1.101	1.126	1.104	1.086	1.112	1.092	1.076
13	1.179	1.146	1.120	1.142	1.116	1.096	1.120	1.099	1.082	1.106	1.088	1.073
14	1.171	1.140	1.115	1.135	1.111	1.092	1.115	1.095	1.079	1.102	1.084	1.070
15	1.163	1.134	1.110	1.130	1.107	1.088	1.110	1.091	1.075	1.097	1.081	1.067

a specific value calculated from the sample data, can be expressed as

$$p\text{-value} = P\left(\hat{C}_{p(S)} > \hat{c}_{p(S)} | C_p = C\right) \tag{21}$$
$$= 1 - \int_0^{\hat{c}_{p(S)}} \frac{C_p}{\sqrt{2\pi}k} x^{-2} \exp\left[-\frac{(C_p/x - 1)^2}{2k^2}\right] dx\,,$$

where

$$k = \sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}}\,, \quad \varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\,.$$

An efficient MAPLE computer program is developed based on Eq. 21 , to calculate the p-value corresponding to the hypothesis test, H_0: $C_p \leqq C$ versus H_1: $C_p > C$. For given sample data of m subgroups of size n, and a specific value of $\hat{C}_p$ calculated from the sample data, the program reads the input (an example) with $m = 15$, $n = 8$, $C = 1.00$, and $\hat{C}_{p(S)} = 1.204$; the program gives the p-value $= 0.00785$.

Maple program for the p-value.

```
> #input parameter values Cphat, C, n, m;
> # Cphat=Ĉ_p(S);
Cphat:=1.204; C:=1.00; m:=15; n:=8;
epsilon:=((2/(n−1))^(1/2))*(GAMMA(n/2))/(GAMMA((n−1)/2));
b:=((1−epsilon^2)/(m*epsilon^2))^0.5;
f(x):=C*(exp(−((C/x−1)^2)/(2*b^2)))/((2*Pi)^0.5*b*(x^2));
p_value:=1−int(f(x),x=0..Cp);
```

```
The output is:
p_value:=0.00785
```

A procedure for testing process precision. To judge if a given process meets the preset precision requirement we first determine the value of C, the preset precision requirement, and the α-risk (the chance of wrongly concluding an incapable process as capable). Checking the appropriate table (or running the program), we may obtain the critical value C_0 based on given values of α-risk, C, and m samples of size n. If the estimated value $\hat{C}_p$ is greater than the critical value C_0 ($\hat{C}_p > C_0$), then we may conclude that the process meets the precision requirement $C_p > C$. Otherwise, we do not have sufficient information to conclude that the process meets the present precision requirement. In this case, we would believe that $C_p \leq C$. In the following, we develop a practical step-by-step procedure for testing process precision. The practitioners (engineers) can use the procedure in their in-plant applications to obtain reliable decisions.

Step 1: Decide the definition of "capable" (common requirement values of C include 1.00, 1.33, 1.50, 1.67, and 2.00), and the α-risk (normally set to 0.01, 0.025, or 0.05), the chance of wrongly concluding an incapable process as capable.

Step 2: Estimate the process precision C_p from the past "in control" data by using either the sample range defined in Eq. 4 or the sample standard deviation method defined in Eq. 8.

Step 3: Check the appropriate table (or run the attached computer program) to find the critical value C_0 based on the specified α-risk, C, and m samples of size n.

Step 4: Conclude that the process is capable ($C_p > C$) if $\hat{C}_p$ value is greater than the critical value C_0 (i.e., $\hat{C}_p > C_0$). Otherwise, we do not have enough information to conclude that the process is capable.

4 An application to chip resistors

Consider a resistor manufacturing process making certain types of chip resistors. The chip resistor is developed applying the surface mount technology, which impels the electronic component to be made like a chip. Designed for surface mount applications, this style is generally mounted with the resistor element face up. Attachment may be made by use of conductive epoxy or solder. For a solder attachment, pre-tinned chips with nickel barriers are recommended. For an epoxy attachment, terminations are generally Pd/Ag or Pt/Ag alloys and Au terminations are available upon request. The additional surface area provided by the wraparound style offers improved mechanical performance, as well as better thermal efficiency.

The chip resistor is made with a metal glaze layer screened on a high ceramic body. Its miniature size can be made compact on printed circuit board, and it has excellent mechanical strength and electrical stability. We investigate a specific chip resistor process taken from a factory located in Taiwan, with manufacturing specifications of USL = 12.0 Ω, and LSL = 11.5 Ω. Suppose the minimal precision requirement for this process is set to $C_p = 1.33$. The collected data of 15 subgroups of size 10 are displayed in Table 7. Figure 9 shows the individual observation plot of each sample with respect to the two-sided specifications.

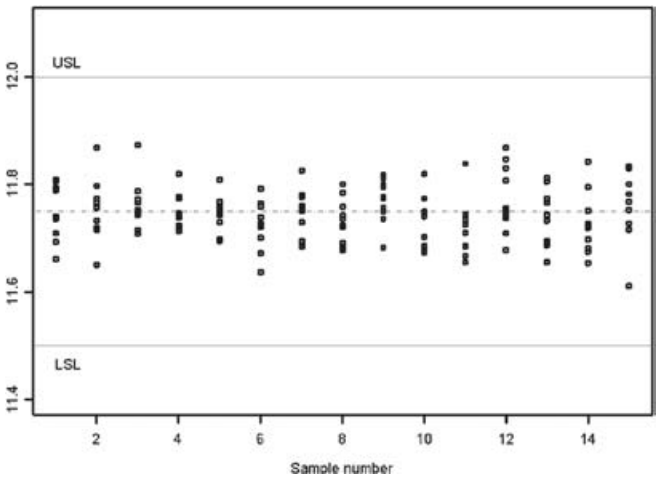

Fig. 9. Individual observation plot of each sample

Table 7. The collected 15 samples each of 10 observations

Sample 1	11.709	11.809	11.693	11.806	11.740	11.806	11.736	11.789	11.661	11.793
Sample 2	11.715	11.650	11.719	11.774	11.797	11.766	11.867	11.757	11.733	11.774
Sample 3	11.873	11.714	11.745	11.765	11.743	11.788	11.753	11.772	11.708	11.749
Sample 4	11.713	11.743	11.746	11.773	11.819	11.741	11.719	11.724	11.777	11.739
Sample 5	11.730	11.695	11.752	11.767	11.809	11.748	11.760	11.743	11.698	11.756
Sample 6	11.672	11.720	11.724	11.700	11.765	11.739	11.792	11.759	11.727	11.637
Sample 7	11.780	11.684	11.825	11.750	11.756	11.685	11.695	11.760	11.777	11.729
Sample 8	11.741	11.784	11.682	11.721	11.800	11.759	11.691	11.678	11.724	11.736
Sample 9	11.800	11.757	11.794	11.812	11.736	11.683	11.776	11.774	11.817	11.749
Sample 10	11.673	11.819	11.680	11.774	11.740	11.685	11.749	11.748	11.685	11.702
Sample 11	11.710	11.684	11.744	11.655	11.725	11.667	11.734	11.838	11.686	11.743
Sample 12	11.741	11.709	11.807	11.829	11.678	11.847	11.737	11.751	11.868	11.755
Sample 13	11.765	11.732	11.694	11.743	11.805	11.655	11.774	11.687	11.811	11.692
Sample 14	11.726	11.719	11.842	11.681	11.751	11.697	11.675	11.723	11.794	11.654
Sample 15	11.752	11.767	11.727	11.782	11.768	11.829	11.611	11.833	11.800	11.716

As mentioned earlier, in order to make the estimation of these capability indices meaningful, it is necessary to check whether the manufacturing process is under statistical control. For these collected 15 samples with each of sample size $n = 10$, the relative efficiency of the range method is about 85%. Therefore, we suggested using the $(\bar{X}, S)$ chart for retrospectively testing whether the process is in control, which are displayed in Fig. 10a,b. The $(\bar{X}, S)$ control charts show that all the sample points are within the control limits without any special pattern, and the process is justified to be well in control. Therefore, we consider the process stable and so we proceed with the capability measurement.

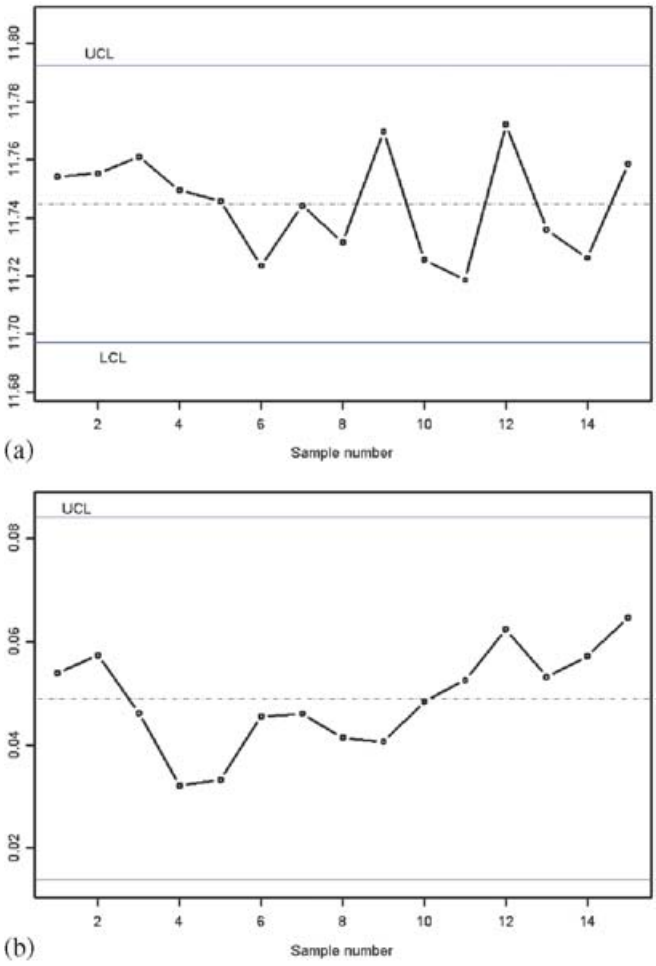

Fig. 10. a $\bar{X}$ control chart of the process, **b** S control chart of the process

The overall sample mean $\bar{\bar{X}} = 11.7448$ and the sample standard deviation $\bar{S} = 0.0490$ are first calculated. Then, the value of the estimated $\hat{\sigma}_S = \bar{S}/c_4 = 0.0504$ and $\hat{C}_{p(S)} = (\text{USL} - \text{LSL})/\left(6 \times \hat{\sigma}_S\right) = 1.6534$ are calculated from the 15 samples of each size 10. With risk $\alpha = 0.01$, the minimal precision requirement for this process is set to $C_p = 1.33$. We check Table 6 and find the corresponding critical value as $1.168 \times 1.33 = 1.553$. Therefore, in this case the estimated value $\hat{C}_{p(S)}$, based on the S-chart sample data, is 1.6534, which is greater than the obtained critical value, 1.553. The corresponding p-value is also found to be 0.00075. We therefore conclude with 99% confidence that the chip resistors manufacturing process satisfies the requirement of $C_p > 1.33$, which is considered satisfactory and reliable in terms of product quality (originally set by the product designers or the manufacturing engineers).

5 Conclusions

Process precision index C_p has been widely used in the manufacturing industry for measuring process potential and product precision. Estimating and testing process precision based on one single sample has been investigated extensively. In this paper, we considered the problem of estimating and testing process precision based on multiple samples taken from the $(\bar{X}, R)$ or $(\bar{X}, S)$ control charts. We investigated the statistical properties of the natural estimator of C_p (use either the sample standard deviation or the sample range method), and implemented the statistical hypothesis testing. We also developed efficient MAPLE programs to calculate the lower confidence bounds, the critical values, and the p-values based on m samples of size n. Based on the test, we developed a practical procedure for the practitioners to use for

their applications. Engineers can use the proposed testing procedure to determine whether their manufacturing processes are capable of reproducing products satisfying the preset precision requirement.

References

1. Kane, VE (1986) Process capability indices. J Qual Technol 18:41–52
2. Rado LE (1989) Enhance product development by using capability indexes. Qual Prog 22(4):38–41
3. Pearn WL, Lin GH, Chen KS (1998) Distributional and inferential properties of the process accuracy and process precision indices. Commun Stat: Theory & Methods 27(4): 985–1000
4. Montgomery DC (1985) Introduction to statistical quality Control. Wiley, New York
5. Kirmani SNUA, Kocherlakota K, Kocherlakota S (1991) Estimation of σ and the process capability index based on subsamples. Commun Stat: Theory & Methods 20:275–291
6. Pearn WL, Yang YS (2003) Distributional and inferential properties of the estimated precision index C_p based on multiple samples. Qual Quant 37:443–453
7. Pearson ES (1932) The percentage limits for the distribution of range in samples from a normal population. Biometrika 24: 404–417
8. Patnaik PB (1950) The use of mean range as an estimator of variance in statistical tests. Biometrika 37:78–87
9. Kocherlakota S (1992) Process capability index: recent developments. Sankhya Indian J Stat 54:352–369
10. Kotz S, Lovelace, CR (eds)(1998) Process capability indices in theory and practice. Arnold, London
11. Chou YM, Owen DB, Borrego SA (1990) Lower confidence limits on process capability indices. J Qual Technol 22:223–229
12. Li H, Owen DB, Borrego SA (1990) Lower confidence limits on process capability indices based on the range. Commun Stat: Simul and Comput 19(1):1–24