Diagnosability of t-Connected Networks and Product Networks under the Comparison Diagnosis Model

Chien-Ping Chang, Pao-Lien Lai, Jimmy Jiann-Mean Tan, and Lih-Hsing Hsu

Abstract—Diagnosability is an important factor in measuring the reliability of an interconnection network, while the (node) connectivity is used to measure the fault tolerance of an interconnection network. We observe that there is a close relationship between the connectivity and the diagnosability. According to our results, a *t*-regular and *t*-connected network with at least 2t + 3 nodes is *t*-diagnosable. Furthermore, the diagnosability of the product networks is also investigated in this work. The product networks, including hypercube, mesh, and tori, comprise very important classes of interconnection networks. Herein, different combinations of *t*-diagnosable and *t*-connected are employed to study the diagnosability of the product networks.

Index Terms—Diagnosability, comparison diagnosis model, *t*-diagnosable, connectivity, order graph, product networks.

1 Introduction

Many studies have proposed and examined the feasibility topologies of multiprocessor interconnection networks. Such a topology is usually modeled as an undirected graph where the set of nodes represents the processors and the set of edges represents the communication links between the processors. Desirable features of an interconnection network include topological properties such as symmetry, regularity, large connectivity, and others. Related studies have investigated a class of graphs called *Cayley* graphs, with their desirable features. In [1], Cayley graphs are based on permutation groups and are a very useful framework for the design and analysis of interconnection networks. All Cayley graphs are regular, explaining why this study considers regular graphs throughout.

The reliability of an interconnection network system is essential to system design and system maintenance. The reliability of a system is maintained by ensuring that it can discriminate the faulty nodes from the fault-free ones. Then, fault-free nodes must replace the faulty nodes. Identifying the faulty nodes is called the *diagnosis* of the system. The *diagnosability* of the system refers to the maximum number of faulty nodes that can be identified by the system. The fault tolerance is another important issue related to interconnection networks. The fault tolerance of an interconnection network can be measured from the connectivity of the underlying graph. In an interconnection network with connectivity t, the fault-free node is guaranteed to communicate with any other fault-free node even if $(t-1)$ nodes are faulty. Hence, diagnosability and connectivity are important properties of interconnection networks. A t-regular and t-connected interconnection network may not be t-diagnosable, accounting for why the condition under which a given t-regular and t-connected interconnection network is t-diagnosable is of interest. This study will prove that, given a t-regular and t-connected interconnection network with at least $2t+3$ nodes, the interconnection network is t-diagnosable according to the comparison diagnosis model. Therefore, many well-known interconnection networks are found to be t-diagnosable under the comparison diagnosis model.

A product network is generated by applying the graph *Cartesian product* operation to factor networks. *Combining* two known topologies with established properties into a new one with the properties of both would be valuable. The *Cartesian product* can be used to perform this *combining*. Product networks are very important classes of interconnection networks. Some well-known interconnection networks, e.g., hypercubes, meshes, tori, k-ary n-cubes, and generalized hypercubes, are product networks [2], [3], [4]. Motivated by this observation, this work addresses the diagnosability of product networks by applying the comparison diagnosis model. Although related studies have investigated various characteristics of product networks (e.g., connectivity, diameter, shortest path routing, and embedding) [5], [6], [8], [10], [11], [16], [17], [19], [23], this paper studies some topological properties different from those investigated elsewhere. The diagnosability of hypercubes and enhanced hypercubes was studied in [12], [21], [22] and that of crossed cubes was considered in [9]. The diagnosability of the product networks under the PMC model was investigated in [2]. Lai et al. [13] addressed the diagnosability of matching composition networks. In

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[13], the matching composition network can be viewed as a particular product network of G and K_2, where G is a t-connected network. This study examines the diagnosability of the product network of G_1 and G_2, where G_i is t_i-diagnosable or t_i-connected for $i = 1, 2$. Moreover, we apply different combinations of t_i-diagnosability and t_i-connectivity to investigate the diagnosability of the product networks.

Previous studies have proposed various models for diagnosis [14], [15], [18]. An important approach, first proposed by Malek and Maeng [14], [15], is called the comparison diagnosis model (MM model). In the MM model, the number of faulty nodes is limited and all faults are permanent. The MM model deals with the faulty diagnosis by sending the same input (or task) from a node w to each pair of distinct neighbors, u and v, and then comparing their responses. The node w is called the *comparator* of the two nodes u and v. Different comparators may examine the same pair of nodes. The result of the comparison is that either the two responses are consistent or two responses disagree. The goal is to use the comparison results to identify the faulty/fault-free status of the nodes in the system. Using the comparison diagnosis model, Sengupta and Dahbura characterized the diagnosable system and presented a polynomial algorithm to determine the set of all faulty nodes [20].

The rest of this paper is organized as follows: Section 2 summarizes some known results on product networks and provides necessary background and notation used herein. Section 3 shows that, under certain conditions, a t-connected network is also t-diagnosable. Section 4 presents the diagnosability of the product networks under the comparison diagnosis model. Conclusions are finally made in Section 5.

2 Preliminaries and Notation

Let $G = (V, E)$ be a graph. V and E represent the set of nodes and the set of edges of G, respectively. The topology of an interconnection network is usually denoted by a graph $G = (V, E)$, where nodes represent processors and edges represent links between processors. Let V' be a subset of V; $G - V'$ represents the subgraph of G induced by $V - V'$. The (node) connectivity of G is defined as

$$\kappa(G) = min\{|V'| |V' \subseteq V \text{ and } G - V' \text{ is not connected}\}.$$

A graph G is t-connected if $\kappa(G) \geq t$. Given a t-connected graph, Menger's theorem states there exist t *internally node-disjoint* (abbreviated as *disjoint*) paths between any two distinct nodes.

The comparison scheme of the system can be modeled as a multigraph $M = (V, C)$, where V represents the node set and C the labeled-edge set. Let $(u, v)_w$ denote an edge labeled by w. In M, an edge $(u, v)_w \in C$ represents the nodes u and v, which are to be compared by w. The same pair of nodes may be compared by various comparators, so M is a multigraph. For $(u, v)_w \in C$, $r((u, v)_w)$ denotes the results of comparing nodes u and v by w such that $r((u, v)_w) = 0$ if the outputs of u and v agree and $r((u, v)_w) = 1$ if the outputs of u and v disagree. If $r((u, v)_w) = 0$ and w is fault-free, then both u and v are fault-free. If $r((u, v)_w) = 1$, then at least one of u, v, and w must be faulty. If w is faulty, then the result of comparison is unreliable and the exact status of u and v are unknown. The complete result of all comparisons, defined as a function s: $C \rightarrow \{0, 1\}$ is called the *syndrome* of the diagnosis.

A subset $F \subseteq V$ is said to be consistent with a syndrome s if s can arise from the circumstance that all the nodes in F are faulty and all the nodes in $V - F$ are fault-free. A system is said to be *diagnosable* if a unique $F \subseteq V$ is consistent with s for every syndrome s. In [20], a system is called a *t-diagnosable* system if the system is diagnosable as long as the number of faulty nodes therein does not exceed t. Let $\sigma(F)$ represent the set of syndromes which could be generated if F is the set of faulty nodes. Two distinct sets $S_1, S_2 \subset V$ are said to be indistinguishable if and only if $\sigma(S_1) \cap \sigma(S_2) \neq \emptyset$; otherwise, S_1 and S_2 are said to be *distinguishable*. Clearly, a system is *t-diagnosable* if and only if each pair of sets $S_1, S_2 \subset V$ are distinguishable and $|S_1| \leq t$ and $|S_2| \leq t$.

Consider two interconnection networks modeled by two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The Cartesian product, $G_1 \times G_2$, of two *factor* networks is an interconnection network, defined as follows:

Definition 1. *The Cartesian product $G = G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$, where the set of nodes V and the set of edges E are given by:*

1. $V = \{\langle x,\ y\rangle | x \in V_1 \text{ and } y \in V_2\}$, *and*
2. *for* $u = \langle x_u,\ y_u\rangle$ *and* $v = \langle x_v,\ y_v\rangle$ *in* V, $(u, v) \in E$ *if and only if* $(x_u,\ x_v) \in E_1$ *and* $y_u = y_v$, *or* $(y_u,\ y_v) \in E_2$ *and* $x_u = x_v$.

Let y be a fixed node of G_2. The subgraph G_1^y-component of $G_1 \times G_2$ has node set $V_1^y = \{(x, y) | x \in V_1\}$ and edge set $E_1^y = \{(u, v) | < u = x_u, y >, v =< x_v, y >, (x_u, x_v) \in E_1\}$. Similarly, let x be a fixed node of G_1; the subgraph G_2^x-component of $G_1 \times G_2$ has node set $V_2^x = \{(x, y) | y \in V_2\}$ and edge set

$$E_2^x = \{(u, v) |\ u =< x, y_u >, v =< x, y_v >, (y_u, y_v) \in E_2\}.$$

Clearly, the G_1^y-component (abbreviated as G_1^y) and the G_2^x-component (abbreviated as G_2^x) are isomorphic with G_1 and G_2, respectively (as illustrated in Fig. 1). The following lemma lists a set of known results [5], [6], [8], [10], [23] related to the topological properties of the Cartesian product of $G_1 \times G_2$ of two graphs G_1 and G_2.

Lemma 1. *Let $u = \langle x_u, y_u\rangle$ and $v = \langle x_v, y_v\rangle$ be two nodes in $G_1 \times G_2$. The following properties hold:*

1. $G_1 \times G_2$ *is isomorphic to* $G_2 \times G_1$,
2. $|G_1 \times G_2| = |G_1| \bullet |G_2|$, *where* $|G|$ *is the number of nodes in* G,
3. $\deg_{G1 \times G2}(u) = \deg_{G1}(x_u) + \deg_{G2}(y_u)$,
4. $\text{dist}_{G1 \times G2}(u, v) = \text{dist}_{G1}(x_u, x_v) + \text{dist}_{G2}(y_u, y_v)$, *where* $\text{dist}_G(u, v)$ *is the distance between* u *and* v *in* G,
5. $D(G_1 \times G2) = D(G_1) + D(G_2)$, *where* $D(G)$ *is the diameter of* G,
6. $\kappa(G_1 \times G_2) \geq \kappa(G_1) + \kappa(G_2)$, *where* $\kappa(G)$ *is the connectivity of* G.

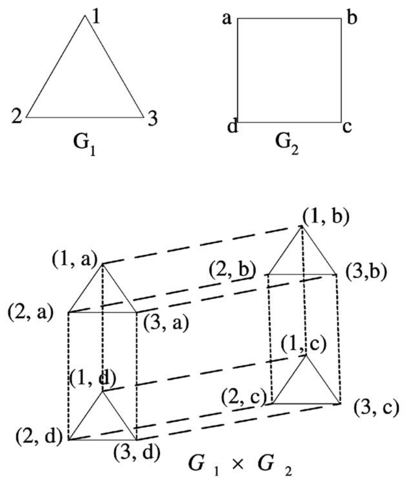

Fig. 1. The product network $G_1 \times G_2$.

Let $G = (V, E)$ be a graph. A node *cover* of G is a subset $Q \subseteq V$ such that every edge of E has at least one end node in Q. A node cover with the minimum cardinality is called a *minimum node cover*. Let $N(u)$ be the set of neighbors of u, $N(u) = \{v|(u, v) \in E\}$. Let V_1, V_2 be two subsets of nodes, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$; the neighbor set of V_1 in V_2 is defined as $N(V_2, V_1)\{x|x \in V_2, x \notin V_1, \text{a node } y \in V_1 \text{ and } (x, y) \in E\}$.

Given an interconnection network G, let $M = (V, C)$ represent the comparison scheme of G. For a node $u \in V$, let

$$X_u = \{v|(u, v) \in E \text{ or } (u, v)_w \in C, \text{for some } w\},$$
$$Y_u = \{(v, w)|v, w \in X_u \text{ and } (u, v)_w \in C\}.$$

In [20], the *order graph* of u is defined as $G(u) = (X_u, Y_u)$ and the *order* of the node u is defined as the cardinality of a minimum node cover of $G(u)$. Given a network G and a comparison scheme M, for a subset of nodes $V' \subset V$, $T(G, V')$ denotes the set of all nodes in $V - V'$, which are compared to some node of V' by the other nodes of V'. Therefore,

$$T(G, V') = \{v|(u, v)_w \in C \text{ and } u, w \in V' \text{ and } v \in V - V'\}.$$

For $V' \subset V$and $|V'| > 1$, if the subgraph induced by V' is connected, then $T(G, V') = N(V - V', V')$, where $N(V - V', V')$ is the neighbor set of V' in $V - V'$. Fig. 2 shows an example of $T(G, V')$. For $V' = \{0, 1, 5\}$, $T(G, V') = \{2, 3, 4, 6\}$; for $V' = \{0, 2, 6\}$, $T(G, V') = \emptyset$.

Five theorems presented by Sengupta and Dahbura [20] must be applied to characterize whether a system is t-diagnosable. The results of these theorems are as follows:

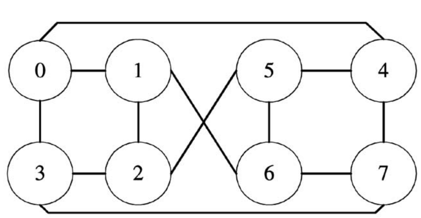

Fig. 2. Example of $T(G, V')$ with V'.

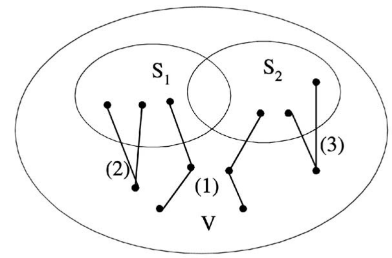

Fig. 3. Three conditions of distinguishable.

Lemma 2 [20]. *For any S_1, S_2 where $S_1, S_2 \subset V$ and $S_1 \neq S_2$, (S_1, S_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied (as shown in Fig. 3):*

1. $\exists\, u, w \in V - S_1 - S_2$ *and* $\exists\, v \in (S_1 - S_2) \cup (S_2 - S_1)$ *such that* $(u, v)_w \in C$,
2. $\exists\, u, v \in S_1 - S_2$ *and* $\exists\, w \in V - S_1 - S_2$ *such that* $(u, v)_w \in C$,
3. $\exists\, u, v \in S_2 - S_1$ *and* $\exists\, w \in V - S_1 - S_2$ *such that* $(u, v)_w \in C$.

Lemma 3 [20]. *If a system with N nodes is t-diagnosable, then $N \geq 2t + 1$.*

Lemma 4 [20]. *If, in a system, each node has order at least t, then, for each $S_1, S_2 \subset V$ such that $|S_1 \cup S_2| \leq t$, (S_1, S_2) is a distinguishable pair.*

Lemma 5 [20]. *A system is t-diagnosable if and only if each node has order at least t and for each distinct pair of sets $S_1, S_2 \subset V$ such that $|S_1| = |S_2| = t$ and at least one of the conditions of Lemma 2 is satisfied.*

From condition 1 of Lemma 2 and Lemma 5, the following is a sufficient condition for a system to be a t-diagnosable.

Lemma 6 [20]. *A system with N nodes is t-diagnosable if 1) $N \geq 2t + 1$, 2) each node has order at least t, 3) for each $V' \subset V$ such that $|V'| = N - 2t + p$ and $0 \leq p \leq t - 1$, $|T(G, V')| > p$.*

3 Diagnosability of *t*-Connected Networks

This section considers the problem that, under suitable conditions, a t-regular and t-connected interconnection network is also t-diagnosable. A t-regular and t-connected interconnection network with at least $2t + 3$ nodes is first proven also to be t-diagnosable. Moreover, the product network of G_1 and G_2 is shown to be $(t_1 + t_2)$-diagnosable, where G_i is t_i-connected with regularity t_i for $i = 1, 2$.

Lemma 7. *Let G be a t-regular and t-connected network with $N \geq 2t + 1$ nodes and $t > 2$. Then, each node v of G has order t.*

Proof. Let v be a node of G and let $G(v)$ be the order graph of v in G. Assume that node v has order $k < t$. Since G contains $N \geq 2t + 1$ nodes and the order of v is $k < t$, there exists at least one node $y \in V$, $y \neq v$, $y \notin N(v)$, and $y \notin Q$. The distance between v and y is at least 2. Each edge of $G(v)$ has at least one endpoint in Q, so all paths from v to y in G must be from v via z, which is a node in

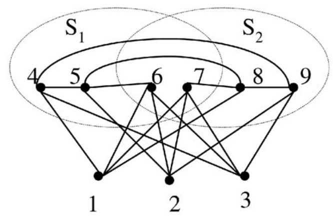

Fig. 4. An example of 4-connected and 3-diagnosable.

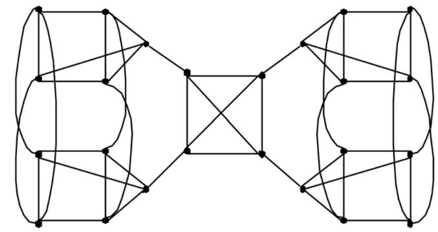

Fig. 5. An example of 2-connected and 4-diagnosable.

Q. Deleting all the nodes of Q in G ensures that no path exists from v to y. However, exactly k nodes are deleted, contradicting the assumption that G is a t-connected network, so $k \geq t$. $N(v)$ is a node cover of $G(v)$, so the node v must have order $k = t$. □

Given a t-diagnosable system, by Lemma 3, the number of nodes must exceed or be equal to $2t+1$. However, a t-regular and t-connected network with $N = 2t+1$ nodes is not necessarily t-diagnosable. The graph shown in Fig. 4 is a 4-regular and 4-connected network with $N = 9$ nodes since any two arbitrarily distinct nodes in Fig. 4 are contained in two disjoint cycles. For example, two distinct nodes 4 and 5 are present in cycles <4, 9, 8, 5> and <4, 1, 6, 5, 2, 7, 3>.This graph can be easily seen to be not 4-diagnosable, since {4, 5, 6, 7} and {6, 7, 8, 9} constitute an indistinguishable pair. With regard to $N = 2t+2$, the three-dimensional crossed cube CQ_3 and the three-dimensional hypercube Q_3 are 3-regular, 3-connected networks and each node has order $t = 3$. However, [9], [22] demonstrated that CQ_3 and Q_3 are not 3-diagnosable under the comparison diagnosis model. The t-regular and t-connected network G with $N \geq 2t+3$ nodes is thus considered in the following theorem.

Theorem 1. *Let $G = (V, E)$ be a t-regular and t-connected network with N nodes and $t > 2$. G is t-diagnosable if $N \geq 2t+3$.*

Proof. Let S_1 and S_2 be two distinct subsets of V with $|S_1| = |S_2| = t$, $|S_1 \cap S_2| = p$, and $0 \leq p \leq t-1$. By Lemmas 5 and 7, G can be shown to be t-diagnosable by showing that (S_1, S_2) is a distinguishable pair. Let $V'' = S_1 \cup S_2$ and $V' = V - V''$. Then, $|V''| = 2t - p > t$. Notably, V' may not be connected.

The case in which all connected components of the subgraph induced by V' are isolative nodes is considered first. For $0 \leq p \leq t-1$, the following cases are considered:

Case 1. $0 \leq p \leq t-3$. Since $0 \leq p \leq t-3$ and G is a t-regular graph, each node of V' has at least two neighbors in $S_1 - S_2$ or $S_2 - S_1$ for $t > 2$. Thus, either condition 2 or condition 3 in Lemma 2 is satisfied.

Case 2. $p = t-2$. In this case, $|V''| = t+2$, $N \geq 2t+3$, and $|V'| = N - (t+2) \geq t+1$. Assume that the pair S_1, S_2 are indistinguishable. Therefore, conditions 2 and 3 in Lemma 2 cannot be satisfied, implying that each node of V' must be connected to $t-2$ nodes in $S_1 \cap S_2$, one node in $S_1 - S_2$, and one node in $S_2 - S_1$. Therefore, at most t nodes in V' satisfy this assumption, contradicting the condition $|V'| \geq t+1$. Hence, either condition 2 or condition 3 in Lemma 2 must be satisfied.

Case 3. $p = t-1$. $|V''| = t+1$ and $|V'| = N - t - 1$. The subgraph induced by V' consists of isolative nodes and G is a t-regular graph, so $(N-t-1)\ t$ edges are adjacent to the nodes of V' and V''. However, G has exactly $Nt/2$ edges. For $N \geq 2t+3$, we have $(N-t-1)\ t > Nt/2$, which is a contradiction, so $p = t-1$ is impossible.

Now, consider that the subgraph induced by V' contains a connected component R with cardinality of at least 2. Let $u \in R$ and $v \in (S_1 - S_2) \cup (S_2 - S_1)$. G is t-connected, so there exist t disjoint paths from u to v. However, at most p disjoint paths exist from u to v via the nodes of $S_1 \cap S_2$. Therefore, there exists at least one path from u to v such that no node of the path belongs to $S_1 \cap S_2$. Since u is a node in R, there exists another node w adjacent to u. Hence, condition 1 in Lemma 2 is satisfied, completing the proof of the theorem. □

Corollary 1. *For $t_1, t_2 > 2$, let G_1 and G_2 be two t_1-connected and t_2-connected networks, with regularity $t_1$and t_2, respectively. Let $G = (V, E)$ be the product network of G_1 and G_2. Then, the product network $G = G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable with regularity $t_1 + t_2$.*

Proof. G_1 is t_1-regular and t_1-connected, so at least t_1+1 nodes exist in G_1. Similarly, the number of nodes in G_2 is at least t_2+1. Therefore, G contains at least $(t_1+1)(t_2+1)$ nodes. Moreover, by Lemma 1, the degree of every node in G is t_1+t_2 (regularity t_1+t_2). $\delta(G)$ is used to denote the minimum degree of G. That [7] $\kappa(G) \leq \delta(G)$ is well-known. However, by Lemma 1, $\kappa(G) \geq \kappa(G_1) + \kappa(G_2) = t_1 + t_2$. Since

$$t_1 + t_2 \leq \kappa(G) \leq \delta(G) = t_1 + t_2,$$

$\kappa(G) = t_1 + t_2$. Since $(t_1+1)(t_2+1) > 2(t_1+t_2)+3$ for $t_1, t_2 > 2$, Theorem 1 implies that G is (t_1+t_2)-diagnosable. Therefore, the corollary follows. □

Notice that the number N_i of nodes is greater than or equal to $n_i + 1$ for n_i-connected $i = 1, 2$ in Corollary 1. The following corollary is immediately obtained from Corollary 1 and by induction.

Corollary 2. *Let G be a product network of $G_1, G_2, \ldots,$ and G_k. Each G_i is t_i-regular and t_i-connected and $t_i > 2$ for $1 \leq i \leq k$, where $k > 2$. Then, the product network G is $(t_1 + t_2 + \cdots + t_k)$-regular and $(t_1 + t_2 + \cdots + t_k)$-diagnosable.*

Theorem 1 indicates that a t-connected network with $N \geq 2t+3$ nodes is also t-diagnosable. However, a t-diagnosable network is not necessarily a t-connected network (as depicted in Fig. 5). The example shown in Fig. 5 is 4-regular and 4-diagnosable, but not 4-connected. The t-diagnosability and t-connectivity are not equivalent terms,

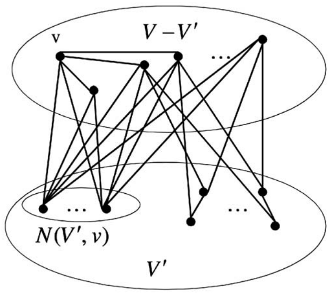

Fig. 6. Illustration in Lemma 8 for the example of Case 2.

but these two concepts are closely related; Theorem 1 provides an example.

4 Diagnosability of Product Networks

The product networks are distinguished into homogeneous product networks and heterogeneous product networks. Homogeneous product networks refer to every factor network of the product that is t-diagnosable and t-regular (or being t-connected and t-regular, respectively), while heterogeneous products are of factor networks one of which is t-diagnosable and the other is t-connected. Section 4.1 addresses the diagnosability of homogeneous product networks. Section 4.2 presents the diagnosability of heterogeneous product networks.

4.1 Diagnosability of Homogeneous Product Networks

By Corollary 1, the homogeneous product network $G_1 \times G_2$ is $(t_1 + t_2)$-diagnosable, where G_i is t_i-connected and t_i-regular, $t_i > 2$ $i = 1, 2$. The homogeneous product network $G_1 \times G_2$ is also $(t_1 + t_2)$-diagnosable, where G_i is t_i-diagnosable and t_i-regular, $t_i > 2$ $i = 1, 2$. Several lemmas must be proven first.

Lemma 8. *Let $G = (V, E)$ be a t-regular network with $N \geq 2t + 1$ nodes. Suppose each node of G has order t, $t > 2$. If $V' \subset V$ and $|V - V'| \leq t$, then $T(G, V') = V - V'$.*

Proof. Let v be an arbitrary node in $V - V'$, and let $G(v)$ be the order graph of v in G. The following two cases are considered:

Case 1. $|V - V'| < t$. For $|V - V'| < t$, the degree of each node is t, so each node in V' has at least one neighbor in V'. Therefore, no isolated node exists in V'. Similarly, every node in $V - V'$ has at least one neighbor in V'. Hence, $T(G, V') = V - V'$.

Case 2. $|V - V'| = t$. For $|V - V'| = t$, each node in $V - V'$ has at least one neighbor in V'. $N(V', v)$ is used to denote the neighbor set of v in V'. Assume that no node in $N(V', v)$ is adjacent to any other node in V'. Then, every node in $N(V', v)$ is adjacent only to $V - V'$ (as shown in Fig. 6). Thus, $V - V' - \{v\}$ is a node cover of $G(v)$ because every node in $N(V', v)$ is an isolated node in V'.The cardinality of a minimum node cover of the order graph $G(v)$ can be easily determined to be at most $t - 1$. However, this contradicts the hypothesis that each node has order t. Therefore, $N(V', v)$ contains at least one neighbor u of v such that the node u is adjacent to another node w in V'. Hence, $T(G, V') = V - V'$. □

Lemma 9. *Let H be a t-regular network, $t > 2$, and let K_2 be the complete network with two nodes. Suppose that the order of each node in H is t. Then, each node of the product network $G = H \times K_2$ has order $t + 1$.*

Proof. Let G^0 and G^1 be two copies of H in G. $M = (V, C)$ represents the comparison scheme of G. Let v be a node of G and let $G(v)$ be the order graph of v in G. Without loss of generality, assume that v is a node in G^0 and that u is a neighbor of v in G^1. There exists at least one node w in G^1 such that $(v, w)_u \in C$. Then, let $G^0(v)$ be the order graph of v in G^0. Since $G^0(v)$ is a proper subgraph of $G(v)$, every node cover of $G(v)$ must contain a node cover of $G^0(v)$. However, (w, u) is an edge in $G(v)$ rather than in $G^0(v)$. Therefore, a node cover of $G(v)$ must include at least either u or w. The order of v in G therefore exceeds that of v in G^0 by one. Thus, the lemma is proven. □

Theorem 2. *For $t > 2$, let H be a t-regular and t-diagnosable network with N nodes. Then, the product network $G = H \times K_2$ is $(t + 1)$-diagnosable.*

Proof. Let $G^0 = (V^0, E^0)$ and $G^1 = (V^1, E^1)$ be two copies of H in $G = (V, E)$. Let S_1 and S_2 be two distinct subsets of V and let $V'' = S_1 \cup S_2$ with $|S_1| = |S_2| = t + 1$, $|S_1 \cap S_2| = p$, and $0 \leq p \leq t$. Then, let $V' = V - V''$ with $|V'| = 2N - 2(t + 1) + p$. Since G has $2N$ nodes, $2N \geq 2(2t + 1) > 2(t + 1) + 1$. Lemma 9 implies that each node of G has order $t + 1$. Hence, the theorem is proven if one of the conditions of Lemma 2 is satisfied. Now, let $V^{0'} = V' \cap V^0$ and $V^{1'} = V' \cap V^1$. G^0 and G^1 are isomorphic to H, so, without loss of generality, assume that $|V^{0'}| \geq |V^{1'}|$. Let $|V^0 - V^{0'}| = k$ and $|V^1 - V^{1'}| = 2(t + 1) - p - k$. Since $|V^{0'}| \geq |V^{1'}|$, $k \leq 2(t + 1) - p - k$. Thus, the proof is divided into the following cases:

Case 1. $2(t + 1) - p - k \leq t$ and $k < t$. From Lemma 8,

$$\begin{aligned}|T(G,\ V')| &\geq |T(G^0,\ V^{0'})| + |T(G^1,\ V^{1'})| \\ &= k + 2(t + 1) - p - k = 2(t + 1) - p.\end{aligned}$$

Since $p \leq t$, $|T(G,\ V')| \geq 2(t + 1) - p > p$. By Lemma 6, this case holds.

Case 2.1. $2(t + 1) - p - k > t$ and $k < t$. From Lemma 8, $|T(G^0,\ V^{0'})| = k$. Since $V^0 - V^{0'}$ contains $k < t$ nodes, each node in $V^{0'}$ has at least one neighbor in $V^{0'}$. Therefore, no isolated node is present in $V^{0'}$. Notably, at least $2(t + 1) - p - 2k$ nodes in $V^1 - V^{1'}$ are adjacent to some $2(t + 1) - p - 2k$ nodes in $V^{0'}$. Thus,

$$\begin{aligned}|T(G,\ V')| &\geq |T(G^0,\ V^{0'})| + N(V^1 - V^{1'}, V^{0'}) \\ &\geq k + 2(t + 1) - p - 2k = 2(t + 1) - p - k.\end{aligned}$$

Since $2(t + 1) - p - k > t \geq p$, by Lemma 6, the case holds.

Case 2.2. $2(t + 1) - p - k > t$ and $k = t$. Since $2(t + 1) - p - k > t$ and $k = t$, $(t + 2) - p > t$, implying $p < 2$. From Lemma 8,

$$|T(G,\ V')| \geq |T(G^0,\ V^{0'})| = t > 2 > p.$$

Then, the case follows.

Case 2.3. $2(t+1)-p-k>t$ and $k>t$. Since $2(t+1)-p-k>t$ and $k>t$, the number of nodes in $V-V'$ is $2(t+1)$, indicating $p=0$. Condition 1 in Lemma 2 is first supposed to be satisfied in G^0. Then, the subgraph induced by $V^{0'}$ includes at least one connected component R with a cardinality of at least 2. Given $|V^0-V^{0'}|=t+1$, Lemma 6 implies $|T(G^0, V^{0'})|\geq t>2$ since G^0 is t-diagnosable. Therefore,

$$|T(G, V')|\geq|T(G^0, V^{0'})|>2>p.$$

This result implies that condition 1 in Lemma 2 is also satisfied in G.

Next, consider that condition 1 in Lemma 2 is violated in G^0. Then, either condition 2 or condition 3 in Lemma 2 is satisfied in G^0. Since G^0 is t-regular and $t>2$, one node v in $V^{0'}$ is adjacent to at least three nodes in $V^0-V^{0'}$. Now, let u, w, and x be three nodes in $V^0-V^{0'}$ such that $u,w\in S_1$, and $x\in S_2$. Since $u,w\in S_1-S_2$, $v\in V-S_1-S_2$, and $p=0$, condition 2 in Lemma 2 is also satisfied in G. The theorem follows. □

Let G_i be a t_i-regular interconnection network $i=1,2$ and let $G=G_1\times G_2$ be the product network of G_1 and G_2. Then, the order of each node v in G is estimated from the following lemma.

Lemma 10. *Let $G_i=(V_i,E_i)$ be a t_i-regular network with $t_i>2$. Suppose each node of G_i has order at least t_i, $i=1,2$. Then, each node of the product network $G=G_1\times G_2$ has ordert t_1+t_2.*

Proof. Let $v=\langle x,y\rangle$ be an arbitrary node of G and let $G(v)$ be the order graph of v in G. According to the definition of product networks, x is a node of V_1 and y is a node of V_2. Therefore, the order of x is at least t_1 and the order of y is at least t_2. Let $G_1(x)$ be the order graph of x in G_1 and let $G_2(y)$ be the order graph of y in G_2. $N(x)$ is a node cover of $G_1(x)$, so the order of node x is exactly t_1. Similarly, the order of node y is t_2. Let $G_1^y(v)$ be the order graph of v in the subgraph G_1^y of G and let $G_2^x(v)$ be the order graph of v in the subgraph G_2^x of G. Since $V_1^y\cap V_2^x=v$, $V(G_1^y(v))\cap V(G_2^x(v))=\emptyset$, where $V(G_1^y(v))$ and $V(G_2^x(v))$ are the node sets of $G_1^y(v)$ and $G_2^x(v)$, respectively. $G_1^y(v)$ and $G_2^x(v)$ are observed to be subgraphs of $G(v)$. Thus, every node cover of $G(v)$ must contain a node cover of both $G_1^y(v)$ and $G_2^x(v)$. Since the subgraphs G_1^y and G_2^x of G are isomorphic to G_1 and G_2, respectively, $G_1^y(v)$ is isomorphic to $G_1(x)$ and $G_2^x(v)$ is isomorphic to $G_2(y)$. Therefore, the order of v in $G_1^y(v)$ is t_1 and the order of v in $G_2^x(v)$ is t_2. Since $V(G_1^y(v))\cap V(G_2^x(v))=\emptyset$, the order of v in $G(v)$ is t_1+t_2. Hence, the lemma follows. □

Corollary 1 was proven; it states that the product network $G_1\times G_2$ is (t_1+t_2)-diagnosable, in which G_i is ti-connectedfor $t_i>2$ $i=1,2$. The previous section also established that a t_i-diagnosable network is not equivalent to a t_i-connected network. The following theorem states that the product network $G_1\times G_2$ is (t_1+t_2)-diagnosable, where G_i is t_i-diagnosable for $t_i>2$ $i=1,2$. Theorem 3 is proven in Appendix A.

Theorem 3. *For $t_i>2$, let $G_i=(V_i,E_i)$ be a t_i-diagnosable and t_i-regular network with N_i nodes $i=1,2$. Let $G=(V,E)$ be the product network of G_1 and G_2. Then, the product network $G=G_1\times G_2$ is (t_1+t_2)-diagnosable with regularity t_1+t_2.*

Notice that, in Theorem 3, the number of nodes N_i is greater than or equal to $2t_i+1$ for t_i-diagnosable $i=1,2$. From Theorem 3 and by induction, the following corollary is obtained.

Corollary 3. *Let G be the product network of $G_1,G_2,\ldots,$ and G_k, where each G_i is t_i-diagnosable with regularity t_i and $t_i>2$ for $1\leq i\leq k$. Then, the product network G is $(t_1+t_2+\cdots+t_k)$-diagnosable with regularity $(t_1+t_2+\cdots+t_k)$.*

4.2 Diagnosability of Heterogeneous Product Networks

This section considers different combinations of t_i-diagnosability and t_i-connectivity to study the diagnosability of the product networks. The diagnosability of the heterogeneous product network G of G_1 and G_2 is considered in which G_1 is t_1-diagnosable and G_2 is t_2-connected. Although the heterogeneous product network differs from the homogeneous product network, a similar result is obtained as that obtained for the homogeneous product network. Lemmas 7 and 10 immediately yield the following lemma.

Lemma 11. *Let G_1 be a t_1-regular and t_1-diagnosable network with $t_1>2$ and let G_2 be a t_2-regular and t_2-connected network with $N_2\geq 2t_2+1$ nodes and $t_2>2$. Then, each node of the product network $G=G_1\times G_2$ has order t_1+t_2.*

Section 3 presents some examples to show that a t-diagnosable network is not equivalent to a t-connected network. Therefore, the following theorem is not implied by Theorem 3, but it can be proven by a similar technique. Theorem 4 is proven in Appendix B.

Theorem 4. *For $t_1,t_2>2$, let $G_1=(V_1,E_1)$ be a t_1-regular and t_1-diagnosable network with N_1 nodes and let $G_2=(V_2,E_2)$ be a t_2-regular and t_2-connected network with $N_2\geq 2t_2+1$ nodes. Then, the product network $G=G_1\times G_2$ is (t_1+t_2)-diagnosable with regularity t_1+t_2.*

In the above theorem, the factor network G_2 must have at least $2t_2+1$ nodes. Therefore, by Corollary 3 and Theorem 4, the following corollary holds.

Corollary 4. *Let G be the product network of $G_1,G_2,\ldots,$ and G_k. Suppose that G_1 is t_1-regular and t_1-connected with $N_1\geq 2t_1+1$ nodes and suppose that G_i is t_i-regular and t_i-diagnosable, $t_i>2$ for $2\leq i\leq k$. Then, the product network G is $(t_1+t_2+\cdots+t_k)$-diagnosable with regularity $(t_1+t_2+\cdots+t_k)$.*

However, Corollaries 2 and 3 yield the following corollary.

Corollary 5. *Let G be the product network of $G_1,G_2,\ldots,$ and G_k. Suppose that G_i is t_i-regular and t_i-connected, $t_i>2$ for $1\leq i\leq m$, where $m>2$, and suppose that G_j is t_j-regular and t_j-diagnosable, $t_j>2$ for $m+1\leq j\leq k$. Then, the* *product network G is $(t_1 + t_2 + \cdots + t_k)$-diagnosable with regularity $(t_1 + t_2 + \cdots + t_k)$.*

5 Conclusions

The reliability of an interconnection network is an important issue. The diagnosability is also an important factor in measuring the reliability of an interconnection network. The connectivity is used to measure the fault tolerance of an interconnection network. This study addresses how the connectivity and the diagnosability of an interconnection network are related. Given an n-regular and n-connected interconnection network with at least $2n+3$ nodes, the interconnection network is n-diagnosable. Illustrative examples reveal that the result may not hold if the condition of more than $2n+3$ nodes is replaced by $2n+1$ or $2n+2$. This finding suggests that fault diagnosis improves with fault tolerance. Among the many well-known interconnection networks that meet this condition are hypercubes, tori, crossed cubes, k-ary n-cubes, and generalized hypercubes, among others, and such interconnection networks are n-diagnosable. Besides, the Cartesian product has defined several interconnection networks and the product networks construct important classes of interconnection networks. This work also investigates the diagnosability of product networks. The homogeneous product network G of G_1 and G_2 is proven to be (n_1+n_2)-diagnosable, given that G_i is either n_i-diagnosable or n_i-connected with regularity n_i for $i = 1, 2$. Furthermore, different combinations of n_i-diagnosability and n_i-connectivity are considered to study the diagnosability of the product networks. The heterogeneous product network G of G_1 and G_2 is shown to be (n_1+n_2)-diagnosable, given that G_1 is n_1-diagnosable with regularity n_1, and G_2 is n_2-regular and n_2-connected with $2n_2+1$ nodes. Similarly, the product network G is also generalized in terms of the k factor networks, $G_1, G_2, \ldots,$ and G_k, all with regularity n_i, such that each G_i is either n_i-diagnosable or n_i-connected for $1 \le i \le k$. The product network G is shown to be $(n_1 + n_2 + \cdots + n_k)$-diagnosable.

Appendix A

Proof. Let S_1 and S_2 be two distinct subsets of V and let $V'' = S_1 \cup S_2$ with $|S_1| = |S_2| = (t_1+t_2)$, $|S_1 \cap S_2| = p$, and $0 \le p \le (t_1+t_2)-1$. Then, let $V' = V - V''$ with $|V'| = N_1N_2 - 2(t_1+t_2)+p$. Since G_1 is t_1-diagnosable and G_2 is t_2-diagnosable,

$$|V| = N_1N_2 \ge (2t_1+1)(2t_2+1) \ge 2(t_1+t_2)+1.$$

By Lemma 10, each node of the product network G has order t_1+t_2. Therefore, the proof is complete if Lemma 6 can be shown to be satisfied. G_1 and G_2 are t_1-diagnosable and t_2-diagnosable, respectively, so, without loss of generality, assume that $t_1 \ge t_2$. Hence, $G_1^y = (V_1^y, E_1^y)$ is isomorphic to G_1 for all $1 \le y \le N_2$. Let $V_1'^y = V' \cap V_1^y$, $V_1''^y = V'' \cap V_1^y$ for all $1 \le y \le N_2$. A set K is defined as $K = \{y : |V_1''^y| > 0\}$. Let S be a subset of K with $|V_1''^y| > t_1$ and let J be $V_2 - K$. The following cases are discussed:

Case 1. $|K| \le t_2$. For all $1 \le y \le N_2$, since G_1^y is isomorphic to G_1, G_1^y must also be t_1-diagnosable. Two cases for $V_1''^y$ are thus distinguished.

Case 1.1. $|V_1''^y| \le t_1$ for all $y \in K$. In this case, the set S is empty. From Lemma 8,

$$|T(G, V')| \ge \sum_{y \in k} |T(G_1^y, V_1'^y)| = \sum_{y \in K} |V_1''^y| = 2(t_1+t_2) - p > p.$$

By Lemma 6, the case holds.

Case 1.2. At least one $V_1''^y$ exists for $y \in K$ such that $|V_1''^y| > t_1$. G_2 is t_2-diagnosable and, in this case, $|K| \le t_2$, so Lemma 8 implies that $T(G_2, J) = K$. This lemma also implies that each $y \in K$ is adjacent to at least one connected component in J. Hence, for $y \in K$ and $r \in J$, each V_1^y is adjacent to at least one V_1^r. Since r is in J and $r \notin K$, $|V_1''^r| = 0$ such that V_1^r is also $V_1'^r$. $G_1^s - V_1'^s$ represents the subgraph of G induced by $V_1^s - V_1'^s$ for all $s \in S$. Let $V_1^s - V_1'^s$ be $V_1''^s$. For $r \in J$, $s \in S$, one such $V_1'^r$ is always adjacent to one specific V_1^s with $|V_1''^s| > t_1$. $|T(G_1^s - V_1'^s,\ V_1'^r)| = N(V_1''^s,\ V_1'^r) = |V_1''^s|$ is thus obtained for such s and r. Therefore,

$$\begin{aligned}|T(G, V')| &\ge \sum_{y \in K, y \notin S} |T(G_1^y, V_1'^y)| + \sum_{y \in S, r \in J} |T(G_1^s - V_1'^s, V_1'^r)| \\ &= \sum_{y \in K, y \notin S} |V_1''^y| + \sum_{s \in S} |V_1''^s| = 2(t_1+t_2) - p > p.\end{aligned}$$

Case 2. $|K| > t_2$. Similarly, two cases for $V_1''^y$ are considered.

Case 2.1. $|V_1''^y| \le t_1$ for all $y \in K$. The proof is similar to that of Case 1.1.

Case 2.2. At least one $V_1''^y$ exists for $y \in K$ such that $|V_1''^y| > t_1$. First, $0 \le p < t_1$ is considered. Now, S is nonempty and $|V''| \le 2(t_1+t_2)$, so $|V'' - V_1''^s| < t_1 + 2t_2$ for $s \in S$. Since G_2 is t_2-regular, s has exactly t_2 neighbors in V_2. Let $z_1, z_2, \ldots, z_{t_2}$ be these neighbors of s. Thus, all $V_1^{z_1}, V_1^{z_2}, \ldots,$ and $V_1^{z_{t_2}}$ are adjacent to V_1^s. Since $|V'' - V_1''^s| < t_1 + 2t_2$, s has at least one neighbor z_i for $1 \le i \le t_2$ such that $|V_1''^{z_i}| < (t_1+2t_2)/t_2 \le t_1$ for $t_1 \ge t_2 \ge 2$. For such z_i and s, the subgraph induced by $V_1^{z_i} \cup V_1^s$ is isomorphic to $G_1 \times K_2$. Since $|V_1''^s| > t_1$ and $|V_1''^{z_i}| < t_1$, it follows from Case 2.1 of Theorem 2 that

$$|(T(G, V')| \ge |T(G_1^s \cup G_1^{z_i}, V_1'^{z_i})| \ge |V_1''^s| > t_1 > p.$$

The other cases, with other relative positions of z_i and s, can be treated similarly.

Next, $t_1 \le p < t_1 + t_2$ is considered. Then, $|V''| \le t_1 + 2t_2$ and $|V'' - V_1''^s| < 2t_2$ for $s \in S$. If another $V_1''^t$ exists with $t \in S$, then $|V'' - V_1''^s - V_1''^t| \le 2t_2 - 1 - t_1 - 1 = 2t_2 - t_1 - 2 \le t_2 - 2$ for $t_1 \ge t_2$. Therefore, the number of $V_1''^y$ is at most t_2 for $y \in K$, violating the assumption that $|K| > t_2$. Hence, exactly one $V_1''^s$ satisfies $|V_1''^s| > t_1$ and $|K| - 1$ $V_1''^y$s have $|V_1''^y| \le t_1$. Since G_2 is t_2-regular and $|V'' - V_1''^s| < 2t_2$, this situation is similar to the case $0 \le p \le t_1$ in which, for at least one neighbor z of s, $|V_1''^z| < 2t_2/t_2 = 2$. Similarly, the subgraph induced by $V_1^z \cup V_1^s$ is isomorphic to $G_1 \times K_2$. Following the same argument as in case $0 \le p < t_1$ with $|V_1''^s| > t_1$ and $|V_1''^z| < 2$, $|T(G_1^s \cup G_1^z,\ V_1'^z)| \ge |V_1''^s|$. Lemma 8 implies

$$|T(G,V')| \geq \left(\sum_{y\in K, y\neq s,z} |T(G_1^y, W_1'^y)|\right) + |T(G_1^s \cup G_1^z, V_1'^z)|$$
$$\geq \left(\sum_{y\in K, y\neq s,z} |V_1''^y|\right) + |V_1''^s|$$
$$= (2(t_1+t_2) - p - |V_1''^s|) + |V_1''^z|) + |V_1''^s|$$
$$= 2(t_1+t_2) - p - |V_1''^z|.$$

Since $|V_1''^z| < 2$,

$$2(t_1+t_2) - p - |V_1''^z| \geq 2(t_1+t_2) - p - 1 \geq t_1 + t_2 > p.$$

In other cases, $|T(G, V')| > p$ can be similarly proven. Furthermore, by Lemma 1, the degree of every node in G is $t_1 + t_2$ (regularity $t_1 + t_2$). Hence, the proof is completed. □

APPENDIX B

Proof. Let $G = (V, E)$ be the product network of G_1 and G_2, $G = G_1 \times G_2$. Now, G_1 is t_1-diagnosable and the number of nodes in G_2 is N_2, so

$$|V| = N_1 N_2 \geq (2t_1+1)(2t_2+1) > 2(t_1+t_2)+1.$$

By Lemma 11, each node of the product network G has order $t_1 + t_2$. Thus, the proof is complete if we can show Lemma 6 is satisfied. Let S_1, S_2 be two distinct subsets of V and let $V'' = S_1 \cup S_2$ with $|S_1| = |S_2| = (t_1 + t_2)$, $|S_1 \cap S_2| = p$, $0 \leq p \leq (t_1+t_2) - 1$. Now, let $V' = V - V''$ with $|V'| = N_1 N_2 - 2(t_1+t_2) + p$. For $t_1 \geq t_2$, the proof is similar to that of Theorem 3.

G_2 is not t_2-diagnosable, so considering only $t_1 \geq t_2$ does not suffice. The case $t_1 < t_2$ must therefore also be considered in the following proof. By definition of the product networks, $G_2^x = (V_2^x, E_2^x)$ is isomorphic to G_2 for all $1 \leq x \leq N_1$. Let $V_2'^x = V' \cap V_2^x$, $V_2''^x = V'' \cap V_2^x$ for all $1 \leq x \leq N_1$. A set K is defined as $K = \{x : |V_2''^x| > 0\}$. Let S be a subset of K with $|V_2''^x| > t_2$ and let J be $V_1 - K$. The following cases are discussed:

Case 1. $|K| \leq t_1$. G_1 is t_1-diagnosable and $|V_1 - J| = |K| \leq t_1$, so Lemma 8 implies that $T(G_1, J) = K$. The lemma also implies that each $x \in K$ is adjacent to at least one connected component in J. Thus, for $x \in K$ and $r \in J$, each V_2^x is adjacent to at least one V_2^r. Since r is in J and $r \notin K$, $|V_2''^r| = 0$ such that V_2^r is also $V_2'^r$. For $S = \emptyset$, the proof is similar to that of Case 1.1 in Theorem 3. Therefore, the case $S \neq \emptyset$ is considered as follows: $G_2^s - V_2'^s$ represents the subgraph of G induced by $V_2^s - V_2'^s$ for all $s \in S$. Let $V_2^s - V_2'^s$ be $V_2''^s$. For $r \in J$, $s \in S$, there always exists one such $V_2'^r$ that is adjacent to one specific V_2^s with $|V_2'^s| > t_2$. The following are thus obtained:

$$|(T(G_2^s - V_2'^s, V_2'^r)| = N(V_2''^s, V_2'^r) = |V_2''^s|$$

for such s and r. Therefore,

$$|T(G,V')| \geq \sum_{x\in K, x\notin S} |T(G_2^x, V_2'^x)| + \sum_{s\in S, r\in J} |T(G_2^s, V_2'^s, V_2'^r)|$$
$$= \sum_{x\in K, x\notin S} |V_2''^x| + \sum_{s\in S} |V_2''^s| = 2(t_1+t_2) - p > p.$$

Case 2. $|K| > t_1$. Since G_2^x is isomorphic to G_2, it follows that G_2^x is t_2-connected. Thus, two cases for $V_2''^x$ are distinguished.

Case 2.1. $|V_2''^x| \leq t_2$ for all $x \in K$. In this case, the set S is empty. From Lemma 8,

$$|T(G,V')| \geq \sum_{x\in K} |T(G_2^x, V_2'^x)| = \sum_{x\in K} |V_2''^x|$$
$$= 2(t_1+t_2) - p > p.$$

Lemma 6 implies that the case holds.

Case 2.2. At least one $V_2''^x$ exists for $x \in K$ such that $|V_2''^x| > t_2$. First, $0 \leq p < t_2$ is considered. Since S is nonempty and $|V''| \leq 2(t_1+t_2)$, $|V'' - V_2''^s| < 2t_1 + t_2$ for $s \in S$. G_1 is t_1-regular, so exactly t_1 neighbors of s are present in V_1. Let $z_1, z_2, \ldots, z_{t_1}$ be such neighbors of s. Hence, all $V_2^{z_1}, V_2^{z_2}, \ldots,$ and $V_2^{z_{t_1}}$ are adjacent to V_2^s. Since $|V'' - V_2''^s| < 2t_1 + t_2$, there exists at least one neighbor z_i of s for $1 \leq i \leq t_1$ such that $|V_2''^{z_i}| < (2t_1+t_2)/t_1 < t_2$ for $t_2 > t_1$. For such z_i, the subgraph induced by $V_2'^{z_i}$ is connected. Furthermore, $T(G_2^{z_i}, V_2'^{z_i}) = N(V_2''^{z_i}, V_2'^{z_i}) = V_2''^{z_i}$. Notably, at least $|V_2''^s| - |V_2''^{z_i}|$ nodes in $V_2''^s$ that are adjacent to some $|V_2''^s| - |V_2''^{z_i}|$ nodes in $V_2'^{z_i}$. Therefore,

$$|T(G,V')| \geq |T(G_2^s, V_2'^{z_i})| + |T(G_2^{z_i}, V_2'^{z_i})|$$
$$\geq (|V_2''^s| - |V_2''^{z_i}|) + |V_2''^{z_i}|$$
$$= |V_2''^s| > t_2 > p.$$

The other cases with different relative positions of z_i and s are similarly treated.

Now, consider $t_2 \leq p < t_1 + t_2$. Then, $|V''| \leq 2t_1 + t_2$ and $|V'' - V_2''^s| < 2t_1$ for $s \in S$. Another $V_2''^t$ exists with $t \in S$, then $|V'' - V_2''^s - V_2''^t| \leq 2t_1 - 1 - t_2 - 1 < t_1 - 2$ for $t_2 > t_1$. Thus, the number of $V_2''^x$ is less than t_1 for $x \in K$, contradicting the assumption that $|K| > t_1$. Therefore, exactly one $V_2''^s$ has $|V_2''^s| > t_2$ and $|K| - 1$ $V_2''^x$s have $|V_2''^x| \leq t_2$. G_1 is t_1-regular and $|V'' - V_2''^s| < 2t_1$, so this case is similar to that of $0 \leq p < t_2$, in which at least one neighbor z of s has $|V_2''^z| < 2t_1/t_1 = 2$. Obviously, the subgraph induced by $V_2'^z$ is connected. For such z and s with $|V_2''^s| > t_2$ and $|V_2''^z| < 2$, $|T(G_2^s \cup G_2^z, V_2'^z)| \geq |V_2''^s|$ can be obtained. By Lemma 8, $|T(G_2^x, V_2'^x)| = |V_2''^x|$ for $x \in K$ and $x \neq s$. Therefore,

$$|T(G,V')| \geq \left(\sum_{x\in K, x\neq s,z} |T(G_2^x, V_2'^x)|\right) + |T(G_2^s \cup G_2^z, V_2'^z)|$$
$$\geq \left(\sum_{x\in K, x\neq s,z} |V_2''^x|\right) + |V_2''^s|$$
$$= (2(t_2+t_2) - p - |V_2''^s| - |V_2''^z|) + |V_2''^s|$$
$$= 2(t_1+t_2) - p - |V_2''^z|.$$

Since $|V_2''^z| < 2$,

$$2(t_1+t_2) - p - |V_1''^z| \geq 2(t_1+t_2) - p - 1 \geq t_1 + t_2 > p.$$

In other cases, $|T(G, V')| > p$ can be similarly proven. Furthermore, Lemma 1 implies that the degree of every node in G is $t_1 + t_2$ (regularity $t_1 + t_2$). Therefore, the proof is completed. □

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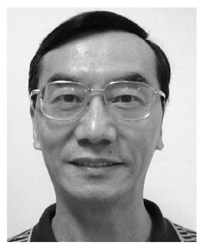

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