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On prime labellings

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Abstract

Let $G=(V,E)$ be a graph. A bijection $f: V \to \{1,2,\ldots,|V|\}$ is called a *prime labelling* if for each $e=\{u,v\}$ in E, we have $\mathrm{GCD}(f(u),f(v))=1$. A graph admits a prime labelling is called a *prime graph*. Around ten years ago, Roger Entringer conjectured that every tree is prime. So far, this conjecture is still unsolved. In this paper, we show that the conjecture is true for trees of order up to 15, and also show that a few other classes of graphs are prime.

1. Introduction

Let $G=(V,E)$ be a graph. A bijection $f: V \to \{1,2,\ldots,|V|\}$ is called a *prime labelling* if for each $e=\{u,v\}\in E$, we have $\mathrm{GCD}(f(u),f(v))=1$. A graph that admits a prime labelling is called a *prime graph*. Prime graphs have been considered in [1, 3–5, 7]. The most interesting problem is to prove the prime tree conjecture: 'every tree is prime' which was proposed by Roger Entringer around ten years ago. So far, the conjecture has been verified for some classes of trees such as complete binary trees, caterpillars, star-like trees, spider trees, etc.

In this paper, we show a few classes of graphs are prime. Furthermore, we also show that the conjecture is true for the trees of small orders.

2. The general results

Let G be a prime graph of order v. It is easy to see that the set of vertices which are labelled with even numbers less than or equal to v is an independent set. Thus we have the following lemma.

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Lemma 2.1. *If G is a prime graph of order v, then the independence number* $\alpha(G) \geqslant \lfloor v/2 \rfloor$.

The above lemma can be slightly extended. A *clique* of a graph G is a maximal complete subgraph of G. The *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G. Since no two vertices in the same clique of G can be independent, $\alpha(G) \leqslant |V(K(G))|$. The following corollary is easy to see.

Corollary 2.2. *For each graph G, if* $|V(K(G))| < \lfloor v/2 \rfloor$, *then G is not a prime graph.*

With the above two results, we can conclude that there are many graphs that are not prime. We will not go any further in this direction.

Let us consider a complete bipartite graph and let $P(t, v)$ be the set of all prime x such that $t < x \leqslant v$. Then we have the following proposition.

Proposition 2.3. *A complete bipartite graph of order v,* $G=(A, B)$, $|A| \leqslant |B|$, *is prime if and only if* $|A| \leqslant |P(v/2, v)| + 1$.

Proof. Since 1 and the elements of $P(v/2, v)$ are the only elements which are relatively prime to all the other elements in the set $\{1, 2, \ldots, v\}$, the result follows. □

As an immediate corollary we have the following corollary.

Corollary 2.4. *Let* $G=(A, B)$ *be a bipartite graph with* $|A| \leqslant |B|$ *and* $|A| \leqslant |P(v/2, v)| + 1$. *Then G is prime.*

Now we will focus on trees. Since every tree is bipartite, a tree $T(A, B)$ of order v with $|A| \leqslant |P(v/2, v)| + 1$ is prime. But, in general, the set $P(v/2, v)$ is of very small size. For example, $|P(11/2, 11)| = 2$. In order to show that any tree of order 11 is prime, we also have to consider $|A| = 4$ or 5.

Before we prove the next lemma, we need a definition. Suppose that $\mathscr{F} = \{S_1, S_2, \ldots, S_n\}$ is a family of sets. Let $X = (a_1, a_2, \ldots, a_n)$ be an n-tuple with $a_i \in S_i, i = 1, 2, \ldots, n$. Then X is called a *system of distinct representatives* (an SDR) for $\mathscr{F}$ provided that all the a_i's are distinct.

Lemma 2.5. *Let* $T=(A, B)$ *be a tree with* $|A| \leqslant |B|$ *and* $|A| \leqslant |P(v/3, v)| + 1$. *Then T is prime.*

Proof. By Proposition 2.3, it suffices to show that if $s = |P(v/2, v)| + 1 < |A| \leqslant |P(v/3, v)| + 1$, then T is prime. We note that if $P(v/3, v/2) = \phi$, then there is nothing to prove. Let $|A| = t$, the vertices in A be $u_1, u_2, \ldots, u_t$ such that $\deg(u_1) \geqslant \deg(u_2) \geqslant \ldots \geqslant \deg(u_t)$, and $|B| = q$, the vertices in B be $v_1, v_2, \ldots, v_q$. It is not difficult to see that $\deg(u_i) \leqslant \lfloor (q + i - 1)/i \rfloor$ for each $i = 1, 2, \ldots, t$. First, label the vertices $u_1, u_2, \ldots, u_s$ with

1 and the elements in $P(v/2, v)$ in any order. Let p_i be a prime in $P(v/3, v/2)$ which corresponds to the vertex u_{s+i} in A for each $i=1,2,\ldots,t-s$. (Distinct vertices should correspond to distinct primes.) Furthermore, let S_i be the set of vertices of B which are not adjacent with u_i. Now consider any collection of k sets, $S_{j_1}, S_{j_2}, \ldots, S_{j_k}$. If $k=1$, by the fact that $|S_{j_i}| \geqslant q - \lfloor (q+s)/(s+1) \rfloor \geqslant 1 (s \geqslant 1)$, we have $|\bigcup_{i=1}^{k} S_{j_i}| \geqslant k$. For the case $k \geqslant 2$, if $|\bigcup_{i=1}^{k} S_{j_i}| \leqslant k-1 \leqslant t-2 \leqslant q-2$ then there exist two vertices of B which are adjacent to each vertex of $\bigcup_{i=1}^{k} \{u_{j_i}\}$ which has at least two vertices. This contradicts to the fact that T contains no 4-cycle. Hence, by the Theorem of P. Hall [2], $\mathscr{F}$ contains an SDR $(v^{(1)}, v^{(2)}, \ldots, v^{(t-s)})$. By labelling the vertex $v^{(i)}$ with $2p_i$ and the other vertices of B with the integers in $\{1,2,\ldots, v=t+q\} \setminus (P(v/2, v) \cup \{1\} \cup \{2p_i \mid p_i \in P(v/3, v/2), i=1,2,\ldots,t-s\})$, it is easy to check that T is prime. □

As a consequence of Lemma 2.5, we have the following corollary.

Corollary 2.6. *If T is a tree of order less than 9, then T is prime.*

As the size of A gets larger, the set $P(v/3, v)$ is not large enough. More numbers are needed to label the vertices in A. It is easy to see that if an odd prime $x \notin P(v/3, v)$ with $x^2 \in \{1,2,\ldots,v\}$, then $\mathrm{GCD}(y,x) \neq 1$ if and only if $\mathrm{GCD}(y,x^2) \neq 1$. As an example, let T be a tree of order 9 and $|A|=4$. We can label the vertices which are arranged in the order of degrees in A with 1, 7, 9, and 3, then label a vertex in B which is not adjacent 9 and 3 with 6. (This is always possible.) The other vertices in B are labelled with the rest of elements. Trees of order 11 and 13 can be shown to be prime in the same fashion. In general, consider a subset S of $\{1,2,\ldots,v\}$ such that $S \cap P(v/2, v) = \phi$. Let $n(S) = \{y : y \in \{1,2,\ldots,v\} \setminus S$ and $\mathrm{GCD}(y,x) \neq 1$ for some $x \in S\}$ and for a subset W of A, $d(W) = \sum_{u_i \in W} \deg(u_i)$. We have the following result.

Lemma 2.7. *Let T be a tree of order v with $T=(A,B)$ and $|A| \leqslant |P(v/2, v)| + |S| + 1$, where S is a subset of $\{1,2,\ldots,v\}$. If there exists a subset W of A such that $|W| = |S|$ and $|B| - d(W) \geqslant |n(S)|$, then T is prime.*

Proof. Label the vertices in W with elements of S and those vertices which are not adjacent to W with the elements of $n(S)$. Then the other vertices are easy to label. □

We can modify the above lemma slightly. Let $T=(A,B)$. If there exists an S satisfying the conditions in Lemma 2.7, then T is prime. Otherwise, let S_0 be a subset of $\{1,2,\ldots,v\}$ with maximum size such that there exists a subset W_0 of A with $|W_0| = |S_0|$ and $|B| - d(W) \geqslant |n(S_0)|$. Now, $|P(v/2, v)| + 1 + |S_0|$ vertices of A can be labelled with the number in $\{1\} \cup P(v/2, v) \cup S_0$. To label the rest of vertices of A, we need another process. Find a number $z \notin \{1\} \cup P(v/2, v) \cup S_0$ such that $|n(\{z\})| \leqslant |B| - |n(S_0)| - \deg(u_k)$, $u_k \in A$ and u_k is not labelled. If this is possible, label u_k with z and label those vertices (not labelled) of B which are not adjacent with u_k with the elements of $n(\{z\})$. If this process can be continued until all the vertices of A are labelled, then T is prime. In general, we do not know whether the above process can be continued until all vertices of A are labelled. But, it is not difficult to show that all trees of small order are prime by the above process.

Theorem 2.8. *All trees of order less than 16 are prime.*

Proof. Since the proof is quite tedious, we only do one case. The other cases are handled by a similar argument. Let $T=(A, B)$ be a tree of order 15 with $|A|=7$ and $|B|=8$. Moreover, suppose $A=\{u_1,u_2,\ldots,u_7\}$, $\deg(u_1)\geqslant\deg(u_2)\geqslant\cdots\geqslant\deg(u_7)$, and $B=\{v_1,v_2,\ldots,v_8\}$. Now $v=15$ and $|P(v/2,v)|=2$. Label u_1,u_2, and u_3 with 1, 11, 13. Let $S=\{3,9\}$, then $n(S)=\{6,12,15\}$. Let $W=\{u_4,u_5\}$. By a direct counting, $d(W)\leqslant 4$. Suppose that $\{v_5,v_6,v_7,v_8\}\subseteq B\backslash N(W)$, where $N(W)=\{v_i\colon v_i\in B$ and v_i is adjacent to a vertex in $W\}$. Label u_4 with 3 and u_5 with 9. Since $\deg(u_6)\leqslant 2$, there exist at least two vertices of $\{v_5,v_6,v_7,v_8\}$ which are not adjacent to u_6. Let them be v_7 and v_8. Label u_6,v_5,v_6,v_7, and v_8 with 5, 6, 12, 10, and 15, respectively. Label u_7 with 7. By labelling a vertex in $\{v_1,v_2,v_3,v_4\}$ which is not adjacent to u_7, say v_1, with 14, then we can obtain a prime labelling for T. Thus we have the proof of this case. $\square$

3. Some special graphs

There are several classes of prime graphs that are worthy of mention. First, we give a short proof that a complete binary tree of level n, $T_2(n)$, is prime. See Fig. 1. (The authors were informed by Sin–Min Lee this is a known result, but we cannot find the reference.)

Proposition 3.1. *A complete binary tree of level $n, T_2(n)$, is prime for all $n\geqslant 1$.*

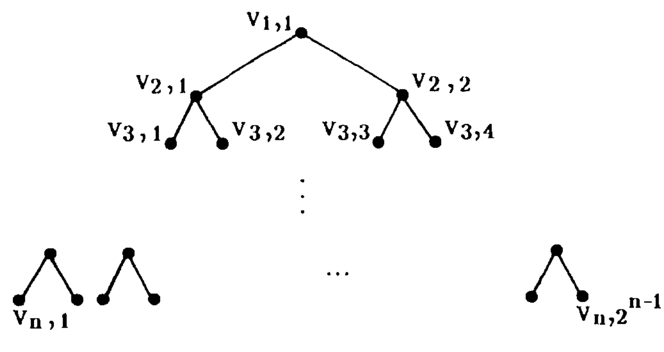

Fig. 1.

Proof. In $T_2(n)$, let $v_{p,q}$ be the qth vertex (from the left) of the pth level. Define a bijection f from $V(T_2(n))$ onto $\{1,2,\ldots,2^n-1\}$ as follows:

$$f(v_{p,q})=\begin{cases}1, & \text{if } p=n \text{ and } q=2^{n-1},\\ (2q-1)\cdot 2^{n-p}+1, & \text{otherwise.}\end{cases}$$

Then it is routine to check that f is a prime labelling of $T_2(n)$. This concludes the proof. □

In Section 2, we obtained a necessary and sufficient condition for a complete bipartite graph to be prime. We can extend that result to a complete multipartite graph.

Proposition 3.2. *A complete t-partite graph* $K(k_1,k_2,\ldots,k_t)$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_t$ *and* $k=\sum_{i=1}^{t} k_i$, *is prime if and only if* $k_1 \geqslant \lfloor k/2 \rfloor$ *and* $|P(k/2,k)| \geqslant k-k_1-1$.

Proof. (*Necessity*). That $k_1 \geqslant \lfloor k/2 \rfloor$ is a result of Lemma 2.1. Let $A_1, A_2, \ldots, A_t$ be the t-partite vertex sets. If $K(k_1,k_2,\ldots,k_t)$ is prime, then the complete bipartite graph induced by the edges from A_1 to $\bigcup_{i=2}^{t} A_i$ is also prime. Thus, by Proposition 2.3, $|P(k/2,k)| \geqslant k-k_1-1$.

The sufficiency is easy to see. □

The following corollary can be considered as a direct result of Proposition 3.2, and it applies to many graphs.

Corollary 3.3. *Let G be a graph of order v. If* $\alpha(G) \geqslant v-|P(v/2,v)|-1$, *then G is prime.*

Finally, we consider a class of graphs which was mentioned in [3]. A palm tree $P(n,k)$ is a tree as in Fig. 2. In [3], the authors proved that $P(n,k)$ is prime whenever $k \leqslant 5$. In what follows we will use a result from number theory to improve the above results.

Let S be a set of positive integers. An element x of S is called a *relprime* of S if $\mathrm{GCD}(x,y)=1$ for each $y \in S \setminus \{x\}$. Pillai [6] showed that if $t \leqslant 16$ and $I_t(x)$ is a set of t consecutive positive integers starting with x, then $I_t(x)$ contains a relprime. Actually,

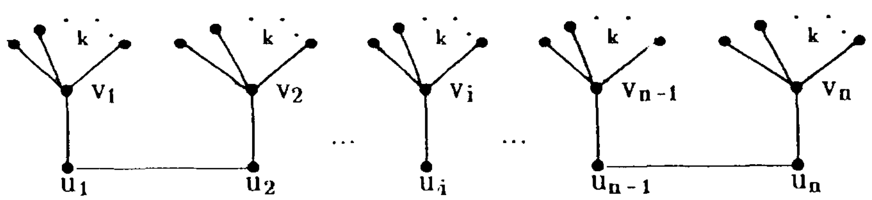

Fig. 2.

it is not difficult to show that for $t \leqslant 16$, $I_t(x)$ contains a relprime which is not x. Now we are ready to prove the following proposition.

Proposition 3.4. *For each* $k \leqslant 14$, $P(n, k)$ *is prime.*

Proof. As in Fig. 2, we label u_i with $(i-1)(k+2)+1$ and v_i with the relprime x_i of the set $\{(i-1)(k+2)+1, \ldots, i\cdot(k+2)\}$, where, $x_i \neq (i-1)(k+2)+1$ and $1 \leqslant i \leqslant n$. Then the other vertices are easy to label. This concludes the proof. □

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