Journal of Combinatorial Optimization, 8, 295–306, 2004


# Vertex and Tree Arboricities of Graphs*

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*Received November 14, 2000; Revised November 22, 2002; Accepted November 22, 2002*

**Abstract.** This paper studies the following variations of arboricity of graphs. The vertex (respectively, tree) arboricity of a graph $G$ is the minimum number va($G$) (respectively, ta($G$)) of subsets into which the vertices of $G$ can be partitioned so that each subset induces a forest (respectively, tree). This paper studies the vertex and the tree arboricities on various classes of graphs for exact values, algorithms, bounds, hamiltonicity and NP-completeness. The graphs investigated in this paper include block-cactus graphs, series-parallel graphs, cographs and planar graphs.

**Keywords:** arboricity, acyclic, tree, block-cactus graph, series-parallel graph, cograph, girth, planar graph, hamiltonian cycle

## 1. Introduction

Our graph terminology and notation are standard, see Chartrand and Lensniak (1981) and West (1996) except as indicated. For convience, parallel edges are allowed as in West (1996). In particular, parallel eges may occur in the definition of series-parallel graphs in Section 4, and the planar graphs in Theorem 12. This is in fact does not affect as the main objects we are dealing with are vertices.

The *arboricity* of a graph $G$ is the minimum number $a(G)$ of edge subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic graph. The well-known theorem by Nash-Williams (1964) says that

$$a(G) = \max\lfloor |E(H)|/(|V(H)| - 1)\rfloor,$$

where the maximum is taken over all nontrivial induced subgraphs $H$ of $G$. This theorem can also be viewed in terms of matroids (see Welsh's book, 1976). Variations of arboricity have been studied extensively in the literature. Typical examples are linear arboricity, linear $k$-arboricity and star arboricity, whose definitions are the same as arboricity except that each subset of $E(G)$ induces a graph whose components are paths, paths of length at most $k$ and stars, respectively.

*This research was partially supported by the National Science Council under grants NSC89-2115-M-009-037 and NSC89-2121-M-009-026.

The *vertex arboricity* of a graph $G$ is the minimum number va($G$) of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic graph; such a partition is called an *acyclic partition* of $V(G)$. This vertex version of arboricity was first introduced by Chartrand et al. (1968), who called it *point-arboricity*. They proved that va($G$) $\leq \lceil(\triangle(G)+1)/2\rceil$ for any graph $G$, where $\triangle(G)$ is the maximum degree of vertices in $G$; and va($G$) $\leq 3$ for any planar graph $G$. Since then, many results for vertex arboricity along these two directions have been established in the literature.

For the upper bound in terms of degrees, Chartrand and Kronk (1969) proved

$$\mathrm{va}(G) \leq \rho(G) \equiv 1 + \max\lfloor\delta(H)/2\rfloor,$$

where the maximum is taken over all induced subgraphs $H$ of $G$ and $\delta(H)$ is the minimum degree of vertices of $H$ Chen (2000) gave the first linear-time algorithm for finding an acyclic partition of size $\rho(G)$. He also discussed parallel algorithms. On the other hand, the improvements on the upper bounds from $\lceil(\triangle(G)+1)/2\rceil$ to the Brook-type bound $\lceil\triangle(G)/2\rceil$ were studied by Catlin (1979), Catlin and Lai (1995), Kronk and Mitchem (1974/75), and Mitchem (1978).

The upper bound 3 for va($G$) on planar graphs has also been studied by Chartrand and Kronk (1969), Goddard (1991), Grünbaum (1973), Hakimi and Schmeichel (1989), Stein (1971), and Wegner (1973). Among them, Goddard (1991) proved a stronger result that the vertex set of any planar graph can be partitioned into three sets such that each set induces a linear forest. The path version of vertex arboricity, called *linear vertex arboricity*, has also been studied (see Alavi et al., 1991, 1994; Matsumoto, 1990; Poh, 1990). Another interesting result is that, for a maximal planar graph $G$ with at least 4 vertices, va($G$) $= 2$ if and only if its dual graph $G^*$ is hamiltonian; see Stein (1971) and Hakimi and Schmeichel (1989). It was known (Garey and Johnson, 1979, p. 193) that determining the vertex arboricity of a graph is NP-hard. Hakimi and Schmeichel (1989) showed that determining whether va($G$) $\leq 2$ is NP-complete for maximal planar graphs $G$. Roychoudbury and Sur-Kolay (1995) gave an $O(n^2)$-time algorithm for finding an acyclic partition of size 2 of a planar graph of order $n$, if such an acyclic partition exists (i.e., if the condition va($G$) $\leq 2$ is known).

In this paper we introduce the following variation of the vertex arboricity. The *tree arboricity* of a graph $G$ is the minimum number ta($G$) of subsets into which $V(G)$ can be partitioned so that each subset induces a tree; such a partition is called a *tree partition* of $V(G)$. The purpose of this paper is to study the vertex and the tree arboricities on various classes of graphs for exact values, algorithms, bounds, hamiltonicity and NP-completeness. The graphs investigated in this paper include block-cactus graphs, series-parallel graphs, cographs and planar graphs. For a good reference on graph classes (see Brandstädt et al., 1999).

## 2. Block-cactus graphs

This section gives exact formulas for the vertex and the tree arboricities of block-cactus graphs.

A *block* is a maximal connected subgraph containing no cut-vertices. *Block-cactus graphs* are graphs whose blocks are complete graphs or cycles. These graphs include two interesting subclasses, which are frequently studied in the literature:

(1) *block graphs*, which are graphs whose blocks are complete graphs;
(2) *cactus graphs*, which are graphs whose blocks are cycles or complete graphs of order 2.

The following lemma is obvious.

**Lemma 1.** *If $G$ is the disjoint union of graphs $G_1, G_2, \ldots, G_r$, then*

$$\mathrm{va}(G) = \max_{1\le i\le r} \mathrm{va}(G_i) \quad \textit{and} \quad \mathrm{ta}(G) = \sum_{i=1}^{r} \mathrm{ta}(G_i).$$

**Lemma 2.** *If $v_i$ is a specified vertex in graph $G_i$ $(1 \le i \le r)$ and $G$ is the graph obtained from the disjoint union of these $r$ graphs by identifying $v_1, v_2, \ldots, v_r$ as a vertex $v$, then*

$$\mathrm{va}(G) = \max_{1\le i\le r} \mathrm{va}(G_i) \quad \textit{and} \quad \mathrm{ta}(G) = \left(\sum_{i=1}^{r} \mathrm{ta}(G_i)\right) - r + 1.$$

**Proof:** The first equality is obvious. The second equality follows from the fact that $\mathcal{P}$ is a tree partition of $V(G)$ if and only if it is the disjoint union of $\mathcal{P}_i - \{A_{i,1}\}$ $(1 \le i \le r)$ and $\{\bigcup_{1\le i\le r} A_{i,1}\}$, where $\mathcal{P}_i = \{A_{i,1}, A_{i,2}, \ldots, A_{i,\mathrm{ta}(G_i)}\}$ is a tree partition of $V(G_i)$ with $v = v_i \in A_{i,1}$. $\square$

**Theorem 3.** *If $G$ is a connected graph with $b$ blocks $G_1, G_2, \ldots, G_b$, then*

$$\mathrm{va}(G) = \max_{1\le i\le b} \mathrm{va}(G_i) \quad \textit{and} \quad \mathrm{ta}(G) = \left(\sum_{i=1}^{b} \mathrm{ta}(G_i)\right) - b + 1.$$

**Proof:** By induction on $b$ using Lemma 2 with $r = 2$. $\square$

As $\mathrm{va}(K_n) = \mathrm{ta}(K_n) = \lceil n/2 \rceil$ and $\mathrm{va}(C_m) = \mathrm{ta}(C_m) = 2$, we have

**Corollary 4.** *If $G$ is a connected block-cactus graph with $r$ blocks of complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$ and $s$ blocks of cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_s}$, then*

$$\mathrm{va}(G) = \max\left\{\max_{1\le i\le r} \lceil n_i/2 \rceil, s'\right\} \quad \textit{and} \quad \mathrm{ta}(G) = \left(\sum_{i=1}^{r} \lceil n_i/2 \rceil\right) - r + s + 1,$$

*where $s' = 0$ for $s = 0$ and $s' = 2$ for $s \ge 1$.*

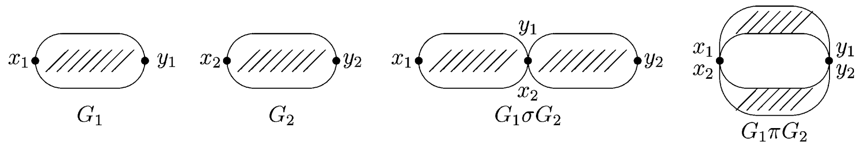


*Figure 1.* Series and parallel compositions.

## 3. Series-parallel graphs

This section establishes a linear algorithm for the tree arboricities of *series-parallel graphs* (*with two terminals*) which are defined recursively as follows:

(SP1) The complete graph $K_2$ is a series-parallel graph.
(SP2) Suppose $G_i$ is a series-parallel graph with terminals $x_i$ and $y_i$ for $i = 1, 2$. Then so are the *series composition* $G_1 \sigma G_2$, which is obtained from $G_1 \cup G_2$ by identifying $y_1$ with $x_2$; and the *parallel composition* $G_1 \pi G_2$, which is obtained from $G_1 \cup G_2$ by identifying $x_1$ with $x_2$ and $y_1$ with $y_2$ (see figure 1).

Suppose $G$ is a graph with two terminals $x$ and $y$. A tree partition of $V(G)$ is of *Type*-1 (respectively, *Type*-2) if $x$ and $y$ are in the same subset (respectively, two different subsets). A tree partition of Type-2 is of *Type*-2c (respectively, *Type*-2d) if the union of the subset containing $x$ and the subset containing $y$ induces a connected (respectively, disconnected) graph. Denote by $\mathrm{ta}_1(G)$, $\mathrm{ta}_2(G)$, $\mathrm{ta}_{2c}(G)$, $\mathrm{ta}_{2d}(G)$ the minimum size of a tree partition of Type-1, Type-2, Type-2c, Type-2d, respectively. We have the following results.

**Lemma 5.** *For any graph $G$ with two terminals $x$ and $y$,*

$$\mathrm{ta}(G) = \min\{\mathrm{ta}_1(G), \mathrm{ta}_2(G)\} \quad \textit{and} \quad \mathrm{ta}_2(G) = \min\{\mathrm{ta}_{2c}(G), \mathrm{ta}_{2d}(G)\}.$$

**Lemma 6.** $\mathrm{ta}(K_2) = \mathrm{ta}_1(K_2) = 1$, $\mathrm{ta}_2(K_2) = \mathrm{ta}_{2c}(K_2) = 2$ *and* $\mathrm{ta}_{2d}(K_2) = \infty$.

**Theorem 7.** *Suppose $G_i$ is a series-parallel graph with terminals $x_i$ and $y_i$ for $i = 1, 2$. Then the following formulas hold.*
(1) $\mathrm{ta}_1(G_1\sigma G_2) = \mathrm{ta}_1(G_1) + \mathrm{ta}_1(G_2) - 1$.
(2) $\mathrm{ta}_{2c}(G_1\sigma G_2) = \min\{\mathrm{ta}_1(G_1) + \mathrm{ta}_{2c}(G_2) - 1, \mathrm{ta}_{2c}(G_1) + \mathrm{ta}_1(G_2) - 1\}$.
(3) $\mathrm{ta}_{2d}(G_1\sigma G_2) = \min\{\mathrm{ta}_1(G_1) + \mathrm{ta}_{2d}(G_2) - 1, \mathrm{ta}_{2d}(G_1) + \mathrm{ta}_1(G_2) - 1, \mathrm{ta}_2(G_1) + \mathrm{ta}_2(G_2) - 1\}$.
(4) $\mathrm{ta}_1(G_1\pi G_2) = \min\{\mathrm{ta}_1(G_1) + \mathrm{ta}_{2d}(G_2) - 2, \mathrm{ta}_{2d}(G_1) + \mathrm{ta}_1(G_2) - 2\}$.
(5) $\mathrm{ta}_{2c}(G_1\pi G_2) = \min\{\mathrm{ta}_{2c}(G_1) + \mathrm{ta}_2(G_2) - 2, \mathrm{ta}_2(G_1) + \mathrm{ta}_{2c}(G_2) - 2\}$.
(6) $\mathrm{ta}_{2d}(G_1\pi G_2) = \mathrm{ta}_{2d}(G_1) + \mathrm{ta}_{2d}(G_2) - 2$.

**Proof:** We only prove (4) and omit the proofs of others as they are similar.

Suppose $\mathcal{P} = \{A_1, A_2, \ldots, A_r\}$ is a minimum tree partition of Type-1 of $V(G_1\pi G_2)$. Assume that $x_1, y_1, x_2, y_2 \in A_1$. Then, for $2 \leq j \leq r$, either $A_j \subseteq V(G_1)$ or $A_j \subseteq V(G_2)$. Let $\mathcal{P}_i = \{A_j : 2 \leq j \leq r \text{ and } A_j \subseteq V(G_i)\}$ for $i = 1, 2$. Let $A_{1,1} = A_1 \cap V(G_1)$ and $A_{1,2} = A_1 \cap V(G_2)$. Since $G[A_1]$ is a tree, either $G_1[A_{1,1}]$ is a tree and $G_2[A_{1,2}]$ is a forest containing exactly two trees or $G_1[A_{1,1}]$ is a forest containing exactly two trees and $G_2[A_{1,2}]$ is a tree. In the former case, we may write $A_1 = A_{1,1} \cup A'_{1,2} \cup A''_{1,2}$ such that $G_2[A'_{1,2}]$ is the tree containing $x_2$ and $G_2[A''_{1,2}]$ is the tree containing $y_2$. Then, $\mathcal{P}_1 \cup \{A_{1,1}\}$ is a tree partition of Type-1 of $V(G_1)$ and $\mathcal{P}_2 \cup \{A'_{1,2}, A''_{1,2}\}$ is a tree partition of Type-2d of $V(G_2)$. Thus, $\mathrm{ta}_1(G_1\pi G_2) \geq |\mathcal{P}_1| + |\mathcal{P}_2| + 1 \geq \mathrm{ta}_1(G_1) + \mathrm{ta}_{2\mathrm{d}}(G_2) - 2$. In the latter case, we may write $A_1 = A'_{1,1} \cup A''_{1,1} \cup A_{1,2}$ such that $G_1[A'_{1,1}]$ is the tree containing $x_1$ and $G_1[A''_{1,1}]$ is the tree containing $y_1$. Then, $\mathcal{P}_1 \cup \{A'_{1,1}, A''_{1,1}\}$ is a tree partition of Type-2d of $V(G_1)$ and $\mathcal{P}_2 \cup \{A_{1,2}\}$ is a tree partition of Type-1 of $V(G_2)$. Again, $\mathrm{ta}_1(G_1\pi G_2) \geq |\mathcal{P}_1| + |\mathcal{P}_2| + 1 \geq \mathrm{ta}_1(G_1) + \mathrm{ta}_{2\mathrm{d}}(G_2) - 2$.

On the other hand, a minimum tree partition of Type-1 (respectively, Type-2d) of $V(G_1)$ together with a minimum tree partition of Type-2d (respectively, Type-1) of $V(G_2)$ can be combined as a tree partition of Type-1 of $V(G_1\pi G_2)$. This gives another side of the inequality of the formula. □

Based on Lemmas 5 and 6 and Theorem 7, we may design a linear-time algorithm for finding the tree arboricity of a series-parallel graph.

## 4. Cographs

This section establishes polynomial-time algorithms for the vertex and the tree arboricities of cographs which are defined recursively as follows:

(C1) The trivial graph $K_1$ is a cograph.

(C2) If $G_1$ and $G_2$ are cographs, then so are their disjoint union $G_1 \cup G_2$ and their *join* $G_1 + G_2$ which is obtained from $G_1 \cup G_2$ by joining each vertex in $G_1$ to each vertex in $G_2$.

Note that a graph is a cograph if and only if it contains no induced paths of 4 vertices (Brandstädt et al., 1999). Also note that there is a linear recognition algorithm for cographs (Corneil et al.,1985).

To find the vertex and the tree arboricities of cographs, we introduce the following definitions. For nonnegative integers $p$ and $q$, a $(p, q)$-acyclic (respectively, $(p, q)$-tree) partition of $V(G)$ is a partition of $V(G)$ into

$$C_1, C_2, \ldots, C_p, \quad I_1, I_2, \ldots, I_q, \quad S_1, S_2, \ldots, S_r$$

such that each $C_i$ is of size 1, each $I_j$ is an independent set, and each $S_k$ induces a forest (respectively, tree). We notice that the sizes of $I_j$ and $S_k$ are possibly 0. We denote by $\mathrm{va}(G, p, q)$ (respectively, $\mathrm{ta}(G, p, q)$) the minimum value of $r$ of a $(p, q)$-acyclic (respectively, $(p, q)$-tree) partition of $V(G)$. Notice that as $\mathrm{va}(G, p, q) = \mathrm{ta}(G, p, q) = 0$ for $p + q \geq |V(G)|$, we only consider the case when $p + q \leq |V(G)|$.

**Lemma 8.** *For any graph $G$, we have* $\mathrm{va}(G) = \mathrm{va}(G, 0, 0)$ *and* $\mathrm{ta}(G) = \mathrm{ta}(G, 0, 0)$.

**Lemma 9.** $\mathrm{va}(K_1, 0, 0) = \mathrm{ta}(K_1, 0, 0) = 1$ *and* $\mathrm{va}(K_1, 1, 0) = \mathrm{va}(K_1, 0, 1) = \mathrm{ta}(K_1, 1, 0) = \mathrm{ta}(K_1, 0, 1) = 0$.

**Theorem 10.** *Suppose $G_1$ and $G_2$ are two graphs, $p$ and $q$ are two nonnegative integers such that $p + q \le |V(G_1)| + |V(G_2)|$. Then the following formulas hold.*

(1) $\mathrm{va}(G_1 \cup G_2, p, q) = \min\{\max\{\mathrm{va}(G_1, p_1, q), \mathrm{va}(G_2, p - p_1, q)\} : 0 \le p_1 \le p,$ $p_1 + q \le |V(G_1)|,$ *and* $p - p_1 + q \le |V(G_2)|\}$.

(2) $\mathrm{ta}(G_1 \cup G_2, p, q) = \min\{\mathrm{ta}(G_1, p_1, q) + \mathrm{ta}(G_2, p - p_1, q) : 0 \le p_1 \le p,\ p_1 + q \le |V(G_1)|,$ *and* $p - p_1 + q \le |V(G_2)|\}$.

(3) $\mathrm{va}(G_1 + G_2, p, q) = \min\{\mathrm{va}(G_1, p_1 + r, q_1 + s) + \mathrm{va}(G_2, p - p_1 + s, q - q_1 + r) + r + s : 0 \le p_1 \le p,\ 0 \le q_1 \le q,\ r \ge 0,\ s \ge 0,\ p_1 + r + q_1 + s \le |V(G_1)|,$ *and* $p - p_1 + s + q - q_1 + r \le |V(G_2)|\}$.

(4) $\mathrm{ta}(G_1 + G_2, p, q) = \min\{\mathrm{ta}(G_1, p_1 + r, q_1 + s) + \mathrm{ta}(G_2, p - p_1 + s, q - q_1 + r) + r + s : 0 \le p_1 \le p,\ 0 \le q_1 \le q,\ r \ge 0,\ s \ge 0,\ p_1 + r + q_1 + s \le |V(G_1)|,$ *and* $p - p_1 + s + q - q_1 + r \le |V(G_2)|\}$.

**Proof:** (1) Suppose

$$C_1, C_2, \ldots, C_{p_1}, \quad I_1, I_2, \ldots, I_q, \quad S_1, S_2, \ldots, S_r$$

is a $(p_1, q)$-acyclic partition of $G_1$, and

$$C'_1, C'_2, \ldots, C'_{p-p_1}, \quad I'_1, I'_2, \ldots, I'_q, \quad S'_1, S'_2, \ldots, S'_s$$

a $(p - p_1, q)$-acyclic partition of $G_2$. We may assume $r = s$ by properly adding empty sets into the partitions if necessary. Then

$$C_1, C_2, \ldots, C_{p_1}, C'_1, C'_2, \ldots, C'_{p-p_1}, \quad I_1 \cup I'_1, I_2 \cup I'_2, \ldots, I_q \cup I'_q,$$
$$S_1 \cup S'_1, S_2 \cup S'_2, \ldots, S_r \cup S'_r$$

is a $(p, q)$-acyclic partition of $G_1 \cup G_2$. This gives that the left-hand side of the equality is less than or equal to the right-hand side.

On the other hand, suppose

$$C_1, C_2, \ldots, C_p, \quad I_1, I_2, \ldots, I_q, \quad S_1, S_2, \ldots, S_r$$

is a $(p, q)$-partition of $G$. We may assume that $C_i$ are in $G_1$ for all $i \le p_1$ and in $G_2$ for all $i > p_1$. Then

$$C_1, C_2, \ldots, C_{p_1}, \quad I_1 \cap V(G_1), I_2 \cap V(G_1), \ldots, I_q \cap V(G_1),$$
$$S_1 \cap V(G_1), S_2 \cap V(G_1), \ldots, S_r \cap V(G_1)$$

is a $(p_1, q)$-partition of $G_1$; and

$$C_{p_1+1}, C_{p_1+2}, \ldots, C_p, \quad I_1 \cap V(G_2), I_2 \cap V(G_2), \ldots, I_q \cap V(G_2),$$
$$S_1 \cap V(G_2), S_2 \cap V(G_2), \ldots, S_r \cap V(G_2)$$

is a $(p - p_1, q)$-partition of $G_2$. This gives that the left-hand side of the equality is greater than or equal to the right-hand side.

(2) The proof for this case is similar to (1) except now we only have that a tree in $G_1 \cup G_2$ is either entirly in $G_1$ or in entirly $G_1$.

(3) Let $\mathcal{P}$ be a minimum $(p, q)$-acyclic partition of $V(G_1 + G_2)$ and suppose that $r + s$ subsets in $\mathcal{P}$ contain vertices in both $V(G_1)$ and $V(G_2)$. Then, each of the $r + s$ subsets induces a star. Moreover, each of the $r + s$ stars has its center in some $V(G_i)$ and the other vertices in $V(G_{3-i})$. Assume that $r$ of the $r + s$ stars have their centers in $G_1$ and $s$ of the $r + s$ stars have their centers in $G_2$. The $r$ centers in $G_1$ must match $r$ independent sets in $G_2$ to form the $r$ stars and the $s$ centers in $G_2$ must match $s$ independent sets in $G_1$ to form the $s$ stars. Then, formula (3) is easily seen.

The proof of formula (4) is similar to that of (3) and is therefore omitted. □

Based on Lemmas 8 and 9 and Theorem 10, we may design a polynomial-time algorithm for finding the vertex and the tree arboricities of cographs.

## 5. Tree arboricity and girth

This section establishes an upper bound for the tree arboricity in terms of girth. The *girth* $g(G)$ of a graph $G$ is the minimum length of a cycle in $G$; for an acyclic graph, its girth is defined to be $\infty$.

**Theorem 11.** *If $G$ is a connected graph of order $n$ and girth $g(G) \geq m + 1$, where $m$ is a positive integer, then* $\mathrm{ta}(G) \leq \lceil \frac{n}{m} \rceil$. *Moreover, for any integer $k$ with $2 \leq k \leq \lceil \frac{n}{m} \rceil$, there exists a connected planar graph $G$ of order $n$ and girth $g(G) = m + 1$ satisfying* $\mathrm{ta}(G) = k$.

**Proof:** The theorem is trivial for $n \leq m$, since a connected graph of order less than its girth is a tree. Assume that $n \geq m + 1$. We may assume that $G$ is not a tree and choose an $r$-cycle $C = (v_1, v_2, \ldots, v_r, v_1)$. Note that $r \geq g(G) \geq m + 1$. As $G$ is connected, a revised breadth first search using all vertices in $C$ as the first level will make it possible to partition $V(G)$ into disjoint union of $V_1, V_2, \ldots, V_r$ such that each $V_i$ contains $v_i$ and induces a connected subgraph of $G$. Let $n_i = |V_i|$ for $1 \leq i \leq r$. By the pigeonhole principle, there exist $1 \leq i < j \leq r$ such that $\sum_{1 \leq k \leq i} n_k \equiv \sum_{1 \leq k \leq j} n_k \pmod{m}$, or equivalently, $\sum_{i<k \leq j} n_k$ is a multiple of $m$. Therefore, $A = \bigcup_{i<k \leq j} V_k$ induces a connected subgraph of $G$ that is of order a multiple of $m$, and $G - A$ is also connected. Since $g(G[A]) \geq g(G) \geq m + 1$ and $g(G - A) \geq g(G) \geq m + 1$, by the induction hypothesis, $\mathrm{ta}(G[A]) \leq \lceil \frac{|A|}{m} \rceil = \frac{|A|}{m}$ and $\mathrm{ta}(G - A) \leq \lceil \frac{n-|A|}{m} \rceil$. Note that a tree partition of $V(G[A])$ together with a tree partition of

$V(G - A)$ form a tree partition of $V(G)$. Therefore,

$$\mathrm{ta}(G) \le \mathrm{ta}(G[A]) + \mathrm{ta}(G - A) \le \frac{|A|}{m} + \left\lceil \frac{n - |A|}{m} \right\rceil = \left\lceil \frac{n}{m} \right\rceil.$$

Finally, for any integer $k$ with $2 \le k \le \lceil \frac{n}{m} \rceil$, consider a connected cactus graph $G$ with $n - mk + m - 1$ blocks of $K_2$ and $k - 1$ blocks of $C_{m+1}$. Note that $G$ is also a connected planar graph of order $n$ and girth $m + 1$. By Corollary 4, we have $\mathrm{ta}(G) = k$. □

## 6. Hamiltonicity and NP-completeness

This section studies the relationship between tree arboricity and hamiltonicity of planar graphs. We first develop a necessary and sufficient condition for the dual of a planar graph to be hamiltonian in terms of its tree arboricity. NP-completeness of the tree arboricity problem then follows. This is similar to the result for the vertex arboricity given by Hakimi and Schmeichel (1989). For completeness we keep the proof in the paper.

**Theorem 12.** *For a connected planar graph $G$, $\mathrm{ta}(G) = 2$ if and only if its dual $G^*$ is hamiltonian.*

**Proof:** Suppose $\mathrm{ta}(G) = 2$. Choose a tree partition $\{V_1, V_2\}$ of $V(G)$. Denote $E(V_1, V_2)$ the set of edges in $G$ joining a vertex in $V_1$ to a vertex in $V_2$, and consider the corresponding set of edges $E'$ in $G^*$. Since $G$ is connected and $G[V_1]$ and $G[V_2]$ are trees, $E(V_1, V_2)$ is a nonempty minimal edge cut of $G$, and so, $E'$ is a cycle in $G^*$. Since every cycle of $G$ contains an edge of $E(V_1, V_2)$, we have that every facial cycle of $G$ contains an edge of $E(V_1, V_2)$ and then $G^*[E']$ spans $G^*$. Thus, $G^*$ is hamiltonian.

Conversely, suppose $G^*$ has a hamiltonian cycle $C$, whose edge set $E'$ corresponds to a minimal edge cut $E(V_1, V_2)$ of $G$. Since every edge cut in $G^*$ contains at least one edge of $E'$, every cycle in $G$ contains at least one edge of $E(V_1, V_2)$, i.e., $G[V_1]$ and $G[V_2]$ are acyclic. Suppose $G[V_1]$ is not a tree. We can then partition $V_1$ into nonempty subsets $A$ and $B$ such that $E(A, B) = \emptyset$. Thus, the edge cut $E(A, V_2)$ of $G$ is a proper subset of $E(V_1, V_2)$, a contradiction. Thus, $G[V_1]$ is a tree. Similarly, $G[V_2]$ is a tree. These prove that $\mathrm{ta}(G) = 2$. □

Note that for a planar graph $G$ with at least 4 vertices, $G$ is maximal planar if and only if its dual $G^*$ is cubic 3-connected.

**Theorem 13.** *It is NP-complete to determine whether $\mathrm{ta}(G) = 2$ for a maximal planar graph $G$ or for a cubic 3-connected planar graph $G$.*

**Proof:** The theorem follows from Theorem 12 and the fact that it is NP-complete to determine whether a cubic 3-connected planar graph (Garey et al., 1976) or a maximal planar graph (Chvátal, 1985) is hamiltonian. □

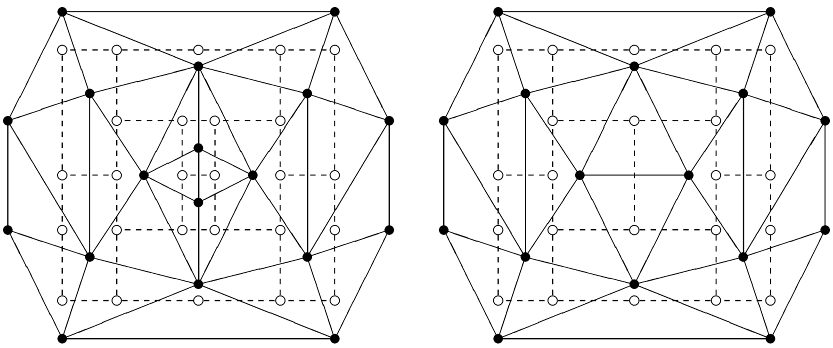

*Figure 2.* Simply-nested inner triangulations (solid lines) with 3 levels and their dual graphs (dashed lines) with the vertex corresponding to the exterior face omitted.

An *inner triangulation* is a 2-connected plane graph in which every interior face is a triangle. Examples of inner triangulations are maximal planar graphs, maximal outerplanar graphs and Euclidean Delaunay triangulations.

The vertices of a plane graph $G$ can be partitioned into levels according to their "distance" from the exterior face. Vertices on the exterior face are on level 1. In general, level-$i$ vertices are the vertices on the exterior face of the subgraph obtained from $G$ by deleting vertices on levels lower than $i$. A plane graph with $k$ levels is *simply-nested* if level 1 through $k-1$ are chordless cycles and level $k$ is either a cycle or a tree. Figure 2 shows examples of simply-nested inner triangulations with 3 levels and their dual graphs with the vertex corresponding to the exterior face omitted.

We shall use Theorem 12 to give an alternative proof of the following result obtained by Cimikowski (1990).

**Theorem 14.** *Every simply-nested inner triangulation $G$ is hamiltonian.*

**Proof:** Assume $G$ has $k'$ levels. All vertices of $G^*$, except the vertex $v_0$ corresponding to the exterior face of $G$, are of degree three. Note that $G^* - v_0$ is a simply-nested plane graph with $k = k'$ (respectively, $k = k' - 1$) levels when level $k'$ of $G$ is a cycle (respectively, tree). Let $L_1, L_2, \ldots, L_k$ be the levels of $G^* - v_0$, and consider $L_0 = \{v_0\}$ as "level 0" of $G^*$. Note that for $1 \leq i \leq k-1$, each vertex $x$ in $L_i$ is adjacent to exactly one vertex $P(x)$ in $L_{i-1} \cup L_{i+1}$ and adjacent to exactly two vertices in $L_i$. We call $P(x)$ the *partner* of $x$. For $x \in L_i$ and $y \in L_{i+1}$, $P(x) = y$ implies $P(y) = x$. Also,

$(*)$ if $1 \leq i \leq k-1$, then $L_i$ has at least two vertices whose partners are in $L_{i-1}$ and at least two vertices whose partners are in $L_{i+1}$.

Choose $x_1, y_1, \ldots, x_{k-1}, y_{k-1}$ as follows. According to $(*)$, we can choose two adjacent vertices $x_1, y_1 \in L_1$ such that $P(x_1) \in L_0$ and $P(y_1) \in L_2$. Once $x_1, y_1, \ldots, x_{i-1}, y_{i-1}$ have been chosen, according to $(*)$, we can choose two adjacent vertices $x_i, y_i \in L_i$ such that $P(x_i) \in L_{i-1} - \{y_{i-1}\}$ and $P(y_i) \in L_{i+1}$.

Let $A_0 = \emptyset$, $B_0 = L_0$, $A_i = \{x_i, y_i\}$ and $B_i = L_i - A_i$ for $1 \le i \le k-1$. When $G^*[L_k]$ is a tree, let $A_k = \emptyset$ and $B_k = L_k$. In case $G^*[L_k]$ is a cycle without crossing chords, we shall prove that $L_k$ can be partitioned into $A_k$ and $B_k$ such that $P(y_{k-1}) \in B_k$, $G^*[B_k]$ is a tree, and each component of $G^*[A_k]$ is a tree with exactly one vertex whose partner is in $B_{k-1}$. More precisely, it suffices to prove the following claim.

**Claim.** *Suppose $C = (v_1, v_2, \ldots, v_n, v_1)$ is a cycle without crossing chords and each vertex in $C$ has degree at most three. If $v_1$ has degree two in $C$, then $V(C)$ can be partitioned into $A$ and $B$ such that $v_1 \in B$, $C[B]$ is a tree, and each component of $C[A]$ is a tree with exactly one vertex of degree two in $C$.*

**Proof of Claim:** We shall prove the claim by induction on the number $p$ of noncrossing chords of $C$. When $p = 0$, $A = \{v_2\}$ and $B = V(C) - A$ satisfy the desired property. Assume $p \ge 1$ and the claim holds for $p' < p$. Since these $p$ chords of $C$ are noncrossing, there exists a chord $v_i v_j$ $(i + 1 < j)$ such that there is no chord incident to a vertex in $\{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\}$. Let $C'$ be the cycle $(v_1, \ldots, v_{i-1}, v_i, v_j, v_{j+1}, \ldots, v_n, v_1)$. Then $C'$ is a cycle with $p-1$ noncrossing chords such that $v_1$ has degree two in $C'$. By the induction hypothesis, $V(C')$ can be partitioned into $A'$ and $B'$ such that $v_1 \in B'$, $C'[B']$ is a tree, and each component of $C'[A']$ is a tree with exactly one vertex of degree two in $C'$. Since $v_i$ and $v_j$ are adjacent in $C'$ and they are of degree two in $C'$, at most one of them is in $A'$. For the case in which $v_i \in A'$, choose $A = A' \cup \{v_{i+1}\}$ and $B = B' \cup \{v_{i+2}, v_{i+3}, \ldots, v_{j-1}\}$. For the case in which $v_i \in B'$, choose $A = A' \cup \{v_{j-1}\}$ and $B = B' \cup \{v_{i+1}, v_{i+2}, \ldots, v_{j-2}\}$. It is straightforward to check that $A$ and $B$ satisfy the desired property. This proves the claim. □

From the above construction, we have that $G^*[B_i]$ is a tree for $0 \le i \le k$, $G^*[A_i]$ is a tree for $1 \le i \le k-1$, and $G^*[A_k]$ is acyclic. Also, each $B_i$ (respectively, $A_i$) has exactly one vertex adjacent to a vertex in $A_{i-1}$ (respectively, $B_{i-1}$) for $2 \le i \le k$ (respectively, $1 \le i \le k-1$), and each component of $G^*[A_k]$ has exactly one vertex adjacent to $B_{k-1}$ when $A_k \ne \emptyset$. Thus, $V_1 = (\cup_{i \text{ even}} B_i) \cup (\cup_{j \text{ odd}} A_j)$ and $V_2 = (\cup_{i \text{ odd}} B_i) \cup (\cup_{j \text{ even}} A_j)$ form a tree partition of $V(G^*)$. This proves that $\mathrm{ta}(G^*) \le 2$ and so $\mathrm{ta}(G^*) = 2$. The theorem then follows from Theorem 12. □

Cimikowski (1990) raised the problem of searching 3-connected, $K_4$-free, non-hamiltonian inner triangulations. We close this paper by constructing such triangulations.

**Lemma 15.** *Suppose $G$ and $H$ are two disjoint graphs that each contains a $K_n$, and $G'$ is the graph obtained from the disjoint union of $G$ and $H$ by identifying the $K_n$ in both graphs. If $G$ is non-hamiltonian, then so is $G'$.*

**Proof:** Suppose $G'$ has a hamiltonian cycle $C$. Suppose $\gamma$ is a run of vertices in $V(H) - V(G)$ preceded by $x$ and followed by $y$. Then $x$ and $y$ are two vertices in $K_n$. Thus the deletion of $\gamma$ from $C$ leaves a cycle. Continuing the same process, we have that the final cycle is a hamiltonian cycle of $G$, a contradiction. □

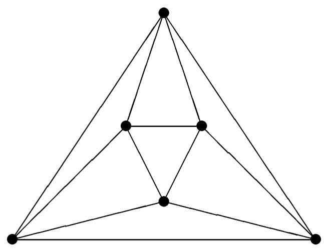

*Figure 3.* The graph $H$.

Suppose $G$ is an inner triangulation and $H$ is the graph in figure 3 whose exterior face can be viewed as a $K_3$. For each interior face of $G$, which is also a $K_3$, we repeatedly apply the operation in Lemma 15 to get a new inner triangulation $G\langle\langle H\rangle\rangle$. Let $H_1 = H$ and $H_{n+1} = H_n\langle\langle H\rangle\rangle$ for $n \geq 1$. All $H_n$ are in fact 3-connected, $K_4$-free, maximal planar graphs. $H_1$ is hamiltonian, but $H_2$ is not. This is because the deletion of the six vertices in the first copy of $H$ in $H\langle\langle H\rangle\rangle$ from $H\langle\langle H\rangle\rangle$ yields 7 $K_3$. However, the deletion of $k$ vertices from a hamiltonian graph yields a graph of at most $k$ components. By Lemma 15, $H_n$ is non-hamiltonian for $n \geq 2$.

## Acknowledgments

We thank the referees for many constructive suggestions.

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